

# ON THE CHEBYSHEV POLYNOMIAL COEFFICIENT PROBLEM OF SOME SUBCLASSES OF BI-UNIVALENT FUNCTIONS

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**ABSTRACT.** We study a newly constructed subclass of bi-univalent functions. Furthermore, we establish Chebyshev polynomial bounds for the coefficients, and Fekete-Szegő inequality, for the class  $S_{\sigma}(\mu, t)$ .

## 1. INTRODUCTION AND PRELIMINARIES

Let  $E$  be the unit disk

$$\{z : z \in \mathbb{C} \text{ and } |z| < 1\},$$

and let  $A$  be the class of functions analytic in  $E$ , satisfying the conditions

$$f(0) = 0 \text{ and } f'(0) = 1.$$

Then each  $f \in A$  has the Taylor expansion

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m. \quad (1.1)$$

Moreover, by  $S$  we shall represent the class of all functions in  $A$  which are univalent in  $E$ .

An analytic function  $f$  is subordinate to an analytic function  $g$ , written

$$f(z) \prec g(z), \quad (z \in E)$$

provided there is an analytic function  $\phi$ , defined on  $E$ , with

$$\phi(0) = 0 \text{ and } |\phi(z)| < 1, \quad (z \in E)$$

such that

$$f(z) = g(\phi(z)) \quad (z \in E).$$

The Koebe One-Quarter Theorem (see [8]) ensures that the image of  $E$  under every  $f \in S$  contains a disk of radius  $\frac{1}{4}$ . So, every  $f \in S$  has an inverse  $f^{-1}$  which satisfies

$$f^{-1}(f(z)) = z, \quad (z \in E)$$

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and

$$f(f^{-1}(w)) = w, \quad \left( |w| < r_0(f), \quad r_0(f) \geq \frac{1}{4} \right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots$$

A function  $f \in A$  is said to be bi-univalent in  $E$  if both  $f$  and  $f^{-1}$  are univalent in  $E$ . Let  $\sigma$  denote the class of bi-univalent functions defined in the unit disk  $E$ . Historically, Lewin [14] investigated the class of bi-univalent functions, obtaining the bound 1.51 for modulus of the second coefficient  $|a_2|$ . Ever since then, motivated substantially by the aforementioned pioneering work on this subject by Srivastava et al. [18], authors discussed estimates on the initial Taylor-Maclaurin coefficients  $|a_2|$  and  $|a_3|$  for functions in these subclasses (see, for example, [1, 5, 6, 10, 15, 19, 20]). But, in the literature, there are only a few works determining the general coefficient bounds  $|a_n|$  for  $n \geq 4$  (see [2, 3, 11, 12, 13]). The coefficient estimate problem for each of  $|a_n|$  is still an open problem.

One of the important tools in numerical analysis, from both theoretical and practical points of view, is Chebyshev polynomials. There are four kinds of Chebyshev polynomials. The majority of research papers dealing with specific orthogonal polynomials of Chebyshev family, contain mainly results of Chebyshev polynomials of first and second kinds  $T_m(t)$  and  $U_m(t)$  and their numerous uses in different applications, see for example, Doha [9] and Mason [16].

The Chebyshev polynomials of the first and second kinds are well known. In the case of a real variable  $t$  on  $(-1, 1)$ , they are defined by

$$T_m(t) = \cos m\theta,$$

$$U_m(t) = \frac{\sin(m+1)\theta}{\sin \theta},$$

where the subscript  $m$  denotes the polynomial degree and where  $t = \cos \theta$ .

**Definition 1.1.** A function  $f \in \sigma$  is said to be in the class  $S_\sigma(\mu, t)$  ( $0 < \mu \leq 1$ ), for  $t \in (\frac{1}{2}, 1]$ , if the following subordinations are satisfied:

$$\Re \left\{ \frac{1}{2} \left( \frac{zf'(z)}{f(z)} + \left( \frac{zf'(z)}{f(z)} \right)^{\frac{1}{\mu}} \right) \right\} \prec H(z, t) = \frac{1}{1 - 2tz + z^2} \quad (z \in E) \quad (1.2)$$

$$\Re \left\{ \frac{1}{2} \left( \frac{wg'(w)}{g(w)} + \left( \frac{wg'(w)}{g(w)} \right)^{\frac{1}{\mu}} \right) \right\} \prec H(w, t) = \frac{1}{1 - 2tw + w^2} \quad (w \in E) \quad (1.3)$$

where  $g = f^{-1}$ .

*Remark 1.2.* We note that for  $\mu = 1$  the class  $S_\sigma(\mu, t)$  reduces to the class  $S_\sigma(t)$ .

The class  $S_\sigma(t)$  is defined as follows:

**Definition 1.3.** A function  $f \in \sigma$  is said to be in the class  $S_\sigma(t)$ , for  $t \in (\frac{1}{2}, 1]$ , if the following conditions are satisfied:

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} \prec H(z, t) = \frac{1}{1 - 2tz + z^2} \quad (z \in E)$$

and

$$\Re \left\{ \frac{wg'(w)}{g(w)} \right\} \prec H(w, t) = \frac{1}{1 - 2tw + w^2} \quad (w \in E)$$

where  $g = f^{-1}$ .

If we choose  $t = \cos \alpha$ ,  $\alpha \in (-\frac{\pi}{3}, \frac{\pi}{3})$ , then

$$\begin{aligned} H(z, t) &= \frac{1}{1 - 2tz + z^2} \\ &= 1 + \sum_{m=1}^{\infty} \frac{\sin(m+1)\alpha}{\sin \alpha} z^m \quad (z \in E). \end{aligned}$$

Thus

$$H(z, t) = 1 + 2 \cos \alpha z + (3 \cos^2 \alpha - \sin^2 \alpha) z^2 + \cdots \quad (z \in E).$$

Following see, we write

$$H(z, t) = 1 + U_1(t)z + U_2(t)z^2 + \cdots \quad (z \in E, \quad t \in (-1, 1)),$$

where  $U_{m-1} = \frac{\sin(m \arccos t)}{\sqrt{1-t^2}}$  ( $m \in \mathbb{N}$ ) are the Chebyshev polynomials of the second kind. Also it is known that

$$U_m(t) = 2tU_{m-1}(t) - U_{m-2}(t),$$

and

$$\begin{aligned} U_1(t) &= 2t, \\ U_2(t) &= 4t^2 - 1, \\ U_3(t) &= 8t^3 - 4t, \\ &\vdots \end{aligned} \tag{1.4}$$

The Chebyshev polynomials  $T_m(t)$ ,  $t \in (-1, 1)$ , of the first kind have the generating function of the form:

$$\sum_{m=0}^{\infty} T_m(t) z^m = \frac{1 - tz}{1 - 2tz + z^2} \quad (z \in E).$$

Note that, the Chebyshev polynomials of the first kind  $T_m(t)$  and the second kind  $U_m(t)$  are well connected by the following relationships

$$\frac{dT_m(t)}{dt} = mU_{m-1}(t),$$

$$T_m(t) = U_m(t) - tU_{m-1}(t),$$

$$2T_m(t) = U_m(t) - U_{m-2}(t).$$

Motivated by the earlier work of Dziok et al. [7], we study the Chebyshev polynomial expansions to provide estimates for the initial coefficients of some subclasses of bi-univalent functions (see, for example, [4]). The aim of this paper to discuss a newly constructed subclass of bi-univalent functions. Furthermore, we establish Chebyshev polynomial bounds for the coefficients and Fekete-Szegő inequality, for the class  $S_\sigma(\mu, t)$ .

## 2. COEFFICIENT BOUNDS FOR THE FUNCTION CLASS $S_\sigma(\mu, t)$

**Theorem 2.1.** *Let  $f$  given by (1.1) be in the class  $S_\sigma(\mu, t)$ . Then*

$$|a_2| \leq \frac{4\mu t \sqrt{2t}}{\sqrt{4(\mu^2 - \mu)t^2 + (\mu + 1)^2}}$$

and

$$|a_3| \leq \frac{16\mu^2 t^2}{(\mu + 1)^2} + \frac{2\mu t}{\mu + 1}.$$

*Proof.* Let  $f \in S_\sigma(\mu, t)$ . From (1.2) and (1.3), we can write

$$\frac{1}{2} \left( \frac{zf'(z)}{f(z)} + \left( \frac{zf'(z)}{f(z)} \right)^{\frac{1}{\mu}} \right) = 1 + U_1(t)\phi(z) + U_2(t)\phi^2(z) + \cdots, \quad (2.1)$$

and

$$\frac{1}{2} \left( \frac{wg'(w)}{g(w)} + \left( \frac{wg'(w)}{g(w)} \right)^{\frac{1}{\mu}} \right) = 1 + U_1(t)\varphi(w) + U_2(t)\varphi^2(w) + \cdots, \quad (2.2)$$

for some analytic functions  $\phi, \varphi$  such that  $\phi(0) = \varphi(0) = 0$  and  $|\phi(z)| < 1$ ,  $|\varphi(w)| < 1$  for all  $z, w \in E$ . From the equalities (2.1) and (2.2), we obtain that

$$\frac{1}{2} \left( \frac{zf'(z)}{f(z)} + \left( \frac{zf'(z)}{f(z)} \right)^{\frac{1}{\mu}} \right) = 1 + U_1(t)p_1z + [U_1(t)p_2 + U_2(t)p_1^2]z^2 + \cdots, \quad (2.3)$$

and

$$\frac{1}{2} \left( \frac{wg'(w)}{g(w)} + \left( \frac{wg'(w)}{g(w)} \right)^{\frac{1}{\mu}} \right) = 1 + U_1(t)q_1w + [U_1(t)q_2 + U_2(t)q_1^2]w^2 + \cdots. \quad (2.4)$$

It is fairly well-known that if  $|\phi(z)| = |p_1z + p_2z^2 + p_3z^3 + \cdots| < 1$  and  $|\varphi(w)| = |q_1w + q_2w^2 + q_3w^3 + \cdots| < 1$ ,  $z, w \in E$ , then

$$|p_k| \leq 1, \quad \forall k \in \mathbb{N}.$$

It follows from (2.3) and (2.4) that

$$\frac{\mu + 1}{2\mu} a_2 = U_1(t)p_1, \quad (2.5)$$

$$\frac{\mu + 1}{2\mu} (2a_3 - a_2^2) + \frac{1 - \mu}{4\mu^2} a_2^2 = U_1(t)p_2 + U_2(t)p_1^2, \quad (2.6)$$

and

$$-\frac{\mu + 1}{2\mu} a_2 = U_1(t)q_1, \quad (2.7)$$

$$\frac{\mu+1}{2\mu} (3a_2^2 - 2a_3) + \frac{1-\mu}{4\mu^2} a_2^2 = U_1(t)q_2 + U_2(t)q_1^2. \quad (2.8)$$

From the equations (2.5) and (2.7), we can easily see that

$$p_1 = -q_1, \quad (2.9)$$

$$\frac{(1+\mu)^2}{2\mu^2} a_2^2 = U_1^2(t) (p_1^2 + q_1^2). \quad (2.10)$$

If we add (2.6) to (2.8), we get

$$\left[ \frac{1+\mu}{\mu} + \frac{1-\mu}{2\mu^2} \right] a_2^2 = U_1(t) (p_2 + q_2) + U_2(t) (p_1^2 + q_1^2). \quad (2.11)$$

Using (2.10) in equality (2.11),

$$\left[ \frac{2\mu^2 + \mu + 1}{2\mu^2} - \frac{U_2(t)}{U_1^2(t)} \frac{(1+\mu)^2}{2\mu^2} \right] a_2^2 = U_1(t) (p_2 + q_2). \quad (2.12)$$

From (1.4) and (2.12) we get

$$|a_2| \leq \frac{4\mu t \sqrt{2t}}{\sqrt{4(\mu^2 - \mu)t^2 + (\mu + 1)^2}}.$$

Next, If we subtract (2.8) from (2.6), we obtain

$$\frac{2(1+\mu)}{\mu} a_3 - \frac{2(1+\mu)}{\mu} a_2^2 = U_1(t) (p_2 - q_2) + U_2(t) (p_1^2 - q_1^2). \quad (2.13)$$

Then, in view of (2.9) and (2.10), also (2.13)

$$a_3 = \frac{2U_1^2(t)\mu^2}{(1+\mu)^2} (p_1^2 + q_1^2) + \frac{U_1(t)\mu}{2(1+\mu)} (p_2 - q_2).$$

Notice that from (1.4)

$$|a_3| \leq \frac{16\mu^2 t^2}{(\mu + 1)^2} + \frac{2\mu t}{\mu + 1}.$$

□

**Corollary 2.2.** *Let  $f$  given by (1.1) be in the class  $S_\sigma(t)$ . Then*

$$|a_2| \leq 2t\sqrt{2t}$$

and

$$|a_3| \leq 4t^2 + t.$$

3. FEKETE-SZEGÖ INEQUALITIES FOR THE FUNCTION CLASS  $S_\sigma(\mu, t)$ 

**Theorem 3.1.** *Let  $f$  given by (1.1) be in the class  $S_\sigma(\mu, t)$  and  $\eta \in \mathbb{R}$ . Then*

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{2\mu t}{1+\mu}; \\ \quad \text{for } |\eta - 1| \leq \frac{1}{4\mu(\mu+1)} \left| \mu^2 - \mu + \frac{(1+\mu)^2}{4t^2} \right| \\ \frac{32\mu^2 |1-\eta| t^3}{4t^2(\mu^2 - \mu) + (1+\mu)^2}; \\ \quad \text{for } |\eta - 1| \geq \frac{1}{4\mu(\mu+1)} \left| \mu^2 - \mu + \frac{(1+\mu)^2}{4t^2} \right| \end{cases}.$$

*Proof.* From (2.12) and (2.13)

$$\begin{aligned} a_3 - \eta a_2^2 &= (1 - \eta) \frac{2\mu^2 U_1^3(t)(p_2 + q_2)}{(2\mu^2 + \mu + 1)U_1^2(t) - U_2(t)(1 + \mu)^2} + \frac{\mu U_1(t)(p_2 - q_2)}{2(\mu + 1)} \\ &= U_1(t) \left[ \left( h(\eta) + \frac{\mu}{2(\mu + 1)} \right) p_2 + \left( h(\eta) - \frac{\mu}{2(\mu + 1)} \right) q_2 \right] \end{aligned}$$

where

$$h(\eta) = \frac{2\mu^2 U_1^2(t)(1 - \eta)}{(2\mu^2 + \mu + 1)U_1^2(t) - U_2(t)(1 + \mu)^2}.$$

Then, in view of (1.4), we conclude that

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{2\mu t}{1+\mu}; & 0 \leq |h(\eta)| \leq \frac{\mu}{2(1+\mu)} \\ 4t|h(\eta)|; & |h(\eta)| \geq \frac{\mu}{2(1+\mu)}. \end{cases}$$

□

If we choose  $\eta = 1$ , we get the next corollary.

**Corollary 3.2.** *If  $f \in S_\sigma(\mu, t)$ , then*

$$|a_3 - a_2^2| \leq \frac{2\mu t}{1+\mu}.$$

**Corollary 3.3.** *Let  $f$  given by (1.1) be in the class  $S_\sigma(t)$ . Then*

$$|a_3 - a_2^2| \leq t.$$

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