

EXISTENCE OF SOLUTIONS FOR A NONHOMOGENEOUS $p(x)$ -BIHARMONIC PROBLEM

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ABSTRACT. This article is devoted to study the existence of solutions of non-linear problem of fourth order governed by $p(x)$ -biharmonic operator under certain conditions on f . Our technical approach is based on the Minty-Browder theorem applied to compact operators and operators of (S_+) type.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

In this article, we consider the fourth-order quasilinear elliptic equation with $p(x)$ -growth conditions

$$\begin{cases} \Delta(|\Delta u|^{p(x)-2}\Delta u) = f(x, u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is a bounded domain and regular of $\mathbb{R}^d$ ($d \geq 2$), $\Delta_{p(x)}^2 u = \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the $p(x)$ -biharmonic operator, which is a natural generalization of the p -biharmonic for p is a measurable function so called variable exponent and $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is Carathéodory function verifying the hypothesis below.

We prove that problems (1.1) have weak solutions in the the generalized Sobolev space

$$X := W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega),$$

under the norm $|\Delta \cdot|_{p(x)}$ (see [1, 6]).

In the constant case $p(x) = p$, by using a transformation of a problem to a known Poisson's problem (see [2]), the problem (1.1) has a non-decreasing sequence of positive eigenvalues. It is known that problems with $p(x)$ -growth conditions have more complicated nonlinearities than the constant case.

In the present paper, we study problem (1.1) that result was extended to the $p(x)$ -biharmonic operator in bounded domains. We were inspired by Drabek [2] in which problems involving the p -biharmonic operator is studied. Our technical approach is based on the Minty-Browder theorem applied to compact operators and operators of (S_+) type. This article is based on [9] where all the missing

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details can be found.

We assume that f satisfies the following conditions:

(H_1) $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Caratheodory decreasing function.

(H_2) For all $x \in \Omega$ there exists a real $C \in (0, 1)$ and a function $b \in L^{p'(x)}(\Omega)$ such that:

$$|f(x, s)| \leq C|s|^{p(x)-1} + b(x),$$

with $p \in C_+(\overline{\Omega})$, for all $s \in \mathbb{R}$.

Before we define the weak solution to (1.1) we recall some properties of the Dirichlet problem for Poisson equation:

$$\begin{cases} -\Delta w = f, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

It is well known that (1.2) is uniquely solvable in $L^{p(x)}(\Omega)$ for an $p(x) > 1$ for all $x \in \Omega$ and that the linear solution operator:

$$\begin{cases} \Lambda : L^{p(x)}(\Omega) \rightarrow X \\ f \mapsto \Lambda f = w \end{cases} \quad (1.3)$$

has the properties stated in the following lemma (see in [2],[3])

Lemma 1.1. *For $x \in \Omega$ and $p(x) \in C_+(\overline{\Omega})$ we have:*

(i) *For $f \in L^{p(x)}(\Omega)$ we have:*

$$\|f\|_{L^{p(x)}} < \infty \Rightarrow \|\Lambda f\|_{L^{p'(x)}} < \infty,$$

(ii) *There exists a constant $C = C(\text{diam}(\Omega), p(\cdot)) > 0$ such that*

$$\|\Lambda f\|_{W^{2,p(x)}} \leq C\|f\|_{L^{p(x)}},$$

holds for $f \in L^{p(x)}(\Omega)$ for all $x \in \Omega$.

(iii) *(Symmetry) The following identity*

$$\int_{\Omega} \Lambda u \cdot v dx = \int_{\Omega} u \cdot \Lambda v dx$$

holds for all $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$.

Remark 1.2. For $p(x) > 1$, we define the Nemytskii operator with variable exponents $\psi_{p(\cdot)} : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\begin{cases} \psi_{p(\cdot)}(s) = |s|^{p(\cdot)-2}s, & \text{if } s \neq 0 \\ \psi_{p(\cdot)}(0) = 0, \end{cases} \quad (1.4)$$

Denoting $p'(x) = \frac{p(x)}{p(x)-1}$ is the conjugate function of $p(x)$ we immediately obtain that:

$$\psi_{p(\cdot)}(s) = t \Leftrightarrow \psi_{p'(\cdot)}(t) = s$$

Let us denote $v = -\Delta u$ in (1.1). Then the problem (1.1) can be restated as an operator equation

$$\begin{cases} \psi_{p(\cdot)}(v) = \Lambda f(x, \Lambda v), & \text{in } \Omega, \\ v = \Delta v = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.5)$$

The operator Λ enables us to transform problems (1.1) to the problem (1.5) which we will study in the space $L^{p(x)}(\Omega)$.

Definition 1.3. The function $u \in X$ is called a solution of (1.1) if v solves (1.5) in $L^{p(x)}(\Omega)$.

The rest of this paper is organized as follows. In Section 2, we recall some basic facts about the variable exponent Lebesgue and Sobolev spaces. In the third section, we present some important preliminary lemmas. In section 4, contains the main result of this paper and its proof.

2. ON THE $L^{p(x)}(\Omega)$ AND $W^{k,p(x)}(\Omega)$ SPACES

To study $p(x)$ -biharmonic problems, we need some results on the spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$, and properties of $p(x)$ -biharmonic operator, which we will use later.

Define the generalized Lebesgue space by

$$L^{p(x)}(\Omega) := \left\{ u : \Omega \rightarrow \mathbb{R} \text{ measurable and } \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\},$$

where $p(x) \in C_+(\overline{\Omega})$ and

$$C_+(\overline{\Omega}) := \{ p \in C(\overline{\Omega}) : p(x) > 1 \}, \text{ for any } x \in \overline{\Omega}.$$

Denote

$$p^+ = \max_{x \in \overline{\Omega}} p(x), \quad p^- = \min_{x \in \overline{\Omega}} p(x),$$

and for any $x \in \overline{\Omega}$, $k \geq 1$,

$$p_k^*(x) := \begin{cases} \frac{Np(x)}{N-kp(x)} & \text{if } kp(x) < N, \\ +\infty & \text{if } kp(x) \geq N. \end{cases}$$

One introduces in $L^{p(x)}(\Omega)$ the norm

$$\|u\|_{p(x)} = \inf \left\{ \alpha > 0 : \int_{\Omega} \left| \frac{u(x)}{\alpha} \right|^{p(x)} dx \leq 1 \right\}.$$

The space $(L^{p(x)}(\Omega), \|\cdot\|_{p(x)})$ is a Banach space.

Proposition 2.1 ([5]). *The space $(L^{p(x)}(\Omega), \|\cdot\|_{p(x)})$ is separable, uniformly convex, reflexive and its conjugate space is $L^{q(x)}(\Omega)$ where $q(x)$ is the conjugate function of $p(x)$; i.e.,*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1,$$

for all $x \in \Omega$. For $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ we have

$$\left| \int_{\Omega} u(x)v(x) dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) \|u\|_{p(x)} \|v\|_{q(x)}.$$

The Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ is defined as

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq k\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_N^{\alpha_N}} u$ with $\alpha = (\alpha_1, \dots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$.

The space $W^{k,p(x)}(\Omega)$, equipped with the norm

$$\|u\|_{k,p(x)} := \sum_{|\alpha| \leq k} |D^\alpha u|_{p(x)},$$

also becomes a Banach, separable and reflexive space. For more details, we refer the reader to [4, 5].

Proposition 2.2 ([5]). *For $p, r \in C_+(\overline{\Omega})$ such that $r(x) \leq p_k^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous and compact embedding*

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega).$$

We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$.

Note that the weak solutions of (1.1) are considered in the generalized Sobolev space

$$X := W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega),$$

equipped with the norm

$$\|u\| = \inf \left\{ \alpha > 0 : \int_{\Omega} \left(\left| \frac{\Delta u(x)}{\alpha} \right|^{p(x)} + \left| \frac{u(x)}{\alpha} \right|^{p(x)} \right) dx \leq 1 \right\}.$$

Remark 2.3. (1) According to [8], the norm $\|\cdot\|_{2,p(x)}$, cited in the preliminaries, is equivalent to the norm $\|\Delta \cdot\|_{p(x)}$ in the space X . Consequently, the norms $\|\cdot\|_{2,p(x)}$, $\|\cdot\|$ and $\|\Delta \cdot\|_{p(x)}$ are equivalent.

(2) By the above remark and Proposition 2.2, there is a continuous and compact embedding of X into $L^{q(x)}(\Omega)$, where $q(x) < p_2^*(x)$ for all $x \in \overline{\Omega}$.

3. PRELIMINARY LEMMAS

Now we give the following fundamental proposition.

Proposition 3.1 (see [7]).

$$\begin{cases} (|x|^{p-2}x - |y|^{p-2}y)(x - y) \geq \frac{1}{2^p}|x - y|^p & \text{if } p \geq 2, \\ (|x|^{p-2}x - |y|^{p-2}y)(x - y) \geq (p-1)(|x| + |y|)^{p-2}|x - y|^2 & \text{if } 1 < p < 2, \end{cases} \quad (3.1)$$

The following lemma summarizes some properties of the function $\psi_{p(\cdot)}$.

Lemma 3.2. *For $p \in C_+(\overline{\Omega})$, it follows that*

- (i) $\psi_{p(\cdot)}$ is monotone and completely continuous,
- (ii) $\psi_{p(\cdot)}$ is of (S_+) type.

We recall that the operator T satisfying the condition (S_+) if $(v_n)_{n \in \mathbb{N}}$ is a sequence in $L^{p(x)}(\Omega)$ such that:

$$\left. \begin{array}{ll} (i) & v_n \rightharpoonup v, \text{ in } L^{p(x)}(\Omega) \\ (ii) & \limsup_{n \rightarrow +\infty} \langle T(v_n), v_n - v \rangle \leq 0 \end{array} \right\} \Rightarrow v_n \rightarrow v \text{ in } L^{p(x)}(\Omega)$$

Proof. (i) Let $p \in C_+(\bar{\Omega})$ and $u, v \in L^{p(x)}(\Omega)$ then using the elementary inequalities of the proposition 3.1, we obtain

$$\langle \psi_{p(x)}(u) - \psi_{p(x)}(v), u - v \rangle \geq 0, \text{ and for all } u \neq v \in L^{p(x)}(\Omega).$$

which means that $\psi_{p(\cdot)}$ is monotone.

Let $(u_n)_n$ be a sequence of $L^{p(x)}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $L^{p(x)}(\Omega)$. For any $\eta \in L^{p'(x)}(\Omega)$, by Hölder's inequality in $L^{p(x)}(\Omega)$ and continuous embedding of $L^{p(x)}(\Omega)$ into $L^{p'(x)}(\Omega)$, it follows that

$$\begin{aligned} |\langle \psi_{p(\cdot)}(u_n) - \psi_{p(\cdot)}(u), \eta \rangle| &= \int_{\Omega} (|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u)\eta dx \\ &\leq C \| |u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u \|_{L^{p'(x)}(\Omega)} \cdot \|\eta\|_{L^{p(x)}(\Omega)} \end{aligned}$$

On the other hand, using the compact embedding of $L^{p(x)}(\Omega)$ into $L^{p'(x)}(\Omega)$, we have

$$u_n \rightarrow u \text{ in } L^{p(x)}(\Omega)$$

Thus $\psi_{p(\cdot)}(u_n) \rightarrow \psi_{p(\cdot)}(u)$ in $L^{p'(x)}(\Omega)$. Therefore $\psi_{p(\cdot)}$ is completely continuous.

(ii) Let $(u_n)_n$ be a sequence of $L^{p(x)}(\Omega)$ such that

$$u_n \rightharpoonup u \text{ weakly in } L^{p(x)}(\Omega) \quad \text{and} \quad \limsup_{n \rightarrow +\infty} \langle \psi_{p(\cdot)}(u_n), u_n - u \rangle \leq 0$$

In view of the monotonicity of $\psi_{p(\cdot)}$, we have

$$\langle \psi_{p(\cdot)}(u) - \psi_{p(\cdot)}(v), u - v \rangle \geq 0,$$

and since $u_n \rightharpoonup u$ weakly in $L^{p(x)}(\Omega)$, it follows that

$$\limsup_{n \rightarrow +\infty} \langle \psi_{p(\cdot)}(u_n) - \psi_{p(\cdot)}(u), u_n - u \rangle = 0$$

By Hölder's inequality in $L^{p(x)}(\Omega)$, we obtain

$$\begin{aligned} 0 &\geq \limsup_{n \rightarrow +\infty} \int_{\Omega} \left(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u \right) (u_n - u) dx \\ &\geq \limsup_{n \rightarrow +\infty} \left[\left(\int_{\Omega} |u_n|^{p(x)} dx \right)^{\frac{1}{p'(x)}} - \left(\int_{\Omega} |u|^{p(x)} dx \right)^{\frac{1}{p'(x)}} \right] \times \\ &\quad \times \left[\left(\int_{\Omega} |u_n|^{p(x)} dx \right)^{\frac{1}{p(x)}} - \left(\int_{\Omega} |u|^{p(x)} dx \right)^{\frac{1}{p(x)}} \right] \\ &\geq 0 \end{aligned}$$

Hence $\|u_n\|_{L^{p(x)}} \rightarrow \|u\|_{L^{p(x)}}$

We deduce that $u_n \rightarrow u$ strongly in $L^{p(x)}(\Omega)$

□

Lemma 3.3. *Assume that (H_2) holds. Then the operator Λf is compact.*

Proof. Thanks to (H_2) , the function f satisfied

$$|f(x, u)| \leq C|u|^{p(x)-1} + b(x),$$

then for every $x \in \Omega$, we obtain

$$\|f(x, \Lambda u)\|_{L^{p'(x)}} \leq C\|\Lambda u\|_{L^{p(x)}}^{p(x)-1} + \|b\|_{L^{p'(x)}},$$

thus the operator $u \mapsto f(\cdot, \Lambda u)$ is bounded and the operator $u \mapsto \Lambda f(\cdot, \Lambda u)$ is completely continuous. Indeed, since the space $L^{p(x)}(\Omega)$ is reflexive, therefore the operator Λf is compact. $\square$

4. MAIN RESULTS

We consider the operator $T : L^{p(x)}(\Omega) \rightarrow L^{p'(x)}(\Omega)$ defined as

$$\langle T(v), \varphi \rangle = \int_{\Omega} (\psi_{p(\cdot)}(v) - \Lambda f(x, \Lambda v)) \varphi dx \quad \text{for any } v, \varphi \in L^{p(x)}(\Omega),$$

Now we have at our disposal everything what we need for the proof of the main result of this section.

Theorem 4.1. *If the operator T is of (S_+) type, coercive and homeomorphism. Then there exists a solution $v = -\Delta u$ of the equation (1.5) consequently u is solution to the (1.1) in the sense of definition (1.3).*

Proof. (1) Let $u, v \in L^{p(x)}(\Omega)$ such that $u \neq v$, we obtain

$$\langle T(u) - T(v), u - v \rangle = \int_{\Omega} ((\psi_{p(\cdot)}(u) - \psi_{p(\cdot)}(v))(u - v)) dx - \int_{\Omega} (\Lambda f(x, \Lambda u) - \Lambda f(x, \Lambda v))(u - v) dx,$$

From assertion (i) of the lemma 1.1, the operator is monotone

$$\int_{\Omega} ((\psi_{p(\cdot)}(u) - \psi_{p(\cdot)}(v))(u - v)) dx \geq 0,$$

Combining the two assumptions, f is decreasing function and symmetry condition (iii) to the lemma 1.1, we can easily have

$$\int_{\Omega} (\Lambda f(x, \Lambda u) - \Lambda f(x, \Lambda v))(u - v) dx = \int_{\Omega} (f(x, \Lambda u) - f(x, \Lambda v))(\Lambda u - \Lambda v) dx < 0$$

we obtain for all $u, v \in L^{p(x)}(\Omega)$ such that $u \neq v$,

$$\langle T(u) - T(v), u - v \rangle > 0,$$

which means that T is strictly monotone.

(2) Notice that, for showing T is of type (S_+) , it is sufficient to ensure $\psi_{p(\cdot)}$ to be a type (S_+) and Λf is compact.

From assertion (ii) of the lemma 3.2 the operator $\psi_{p(\cdot)}$ to be a type (S_+) and by the lemma 3.3 the operator Λf is compact.

Thus we deduce that T is a (S_+) type operator.

(3) Note that the strict monotonicity of T implies its injectivity. Moreover, T is a coercive operator. Indeed, since $C \in (0, 1)$ and $p^- - 1 > 0$, for each $v \in L^{p(x)}(\Omega)$ such that $\|v\|_{p(x)} \geq 1$, it follows that

$$\frac{\langle T(v), v \rangle}{\|v\|_{p(x)}} \geq \|v\|_{p(x)}^{p(x)-1} \left[1 - C - \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \frac{\|\Lambda b\|_{p'(x)}}{\|v\|_{p(x)}^{p(x)-1}} \right] \rightarrow \infty \quad \text{as } \|v\|_{p(x)} \rightarrow \infty.$$

Consequently, in view of Minty-Browder Theorem (see Theorem 26.A(d) in [9]), the operator T is an surjection and admits an inverse mapping.

It suffices then to show the continuity of T^{-1} , the proof of this assertion is similar to the proof in [1]. Let $(f_n)_n$ be a sequence of $L^{p'(x)}(\Omega)$ such that $f_n \rightarrow f$ in $L^{p'(x)}(\Omega)$. Let u_n and u in $L^{p(x)}(\Omega)$ such that

$$T^{-1}(f_n) = u_n \quad \text{and} \quad T^{-1}(f) = u.$$

By the coercivity of T , one deduces that the sequence (u_n) is bounded in the reflexive space $L^{p(x)}(\Omega)$. For a subsequence, we have $u_n \rightharpoonup \hat{u}$ in $L^{p(x)}(\Omega)$, which implies

$$\lim_{n \rightarrow +\infty} \langle T(u_n) - T(u), u_n - \hat{u} \rangle = \lim_{n \rightarrow +\infty} \langle f_n - f, u_n - \hat{u} \rangle = 0.$$

It follows by the second assertion and the continuity of T that

$$u_n \rightarrow \hat{u} \quad \text{in } L^{p(x)}(\Omega) \quad \text{and} \quad T(u_n) \rightarrow T(\hat{u}) = T(u) \quad \text{in } L^{p'(x)}(\Omega).$$

Moreover, since T is an injection, we conclude that $u = \hat{u}$. □

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