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ANALYTICAL SOLUTION OF TIME FRACTIONAL BLACK-SCHOLES EQUATION WITH TWO ASSETS THROUGH NEW SUMUDU TRANSFORM ITERATIVE METHOD

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Abstract. There is a scopious rise in the study of financial derivatives over the past two or three decades. Mathematical model proposed by Black and Scholes expounds financial derivatives in a more momentous way. The Black-Scholes model on a single asset is a partial differential equation (PDE) characterizing the behavior of European options. In this article, we introduce the new Sumudu transform iterative method (NSTIM) as a new technique to obtain the analytical solution of time fractional Black-Scholes model involving European options with two assets. The proposed model is the advanced version of regular Black-Scholes model. Explicit solution of the problem has been obtained with the help of generalized Mittag-Leffer function. The numerical analysis prove that this method is efficacious in solving various problems of financial theory.

1. Introduction

Option problem has become a persuasive problem on both theoretical and practical grounds in distinct areas of financial industry since last two or three decades. Options are recognized as dominant and salient financial derivatives in the financial markets. Moreover, they contribute noteworthily in the continuous evolution of option trading. Charles Castley [13] in 1877 made the first attempt to study modern derivative pricing. Castley's work laid emphasis on certain concepts of option trading such as speculative trading and hedging. Later on, Osborne [35] put forward a new formula of option pricing utilizing arithmetic Brownian motion with drift to model the underlying asset. But, the pioneering work on option pricing was published by Fisher Black and Myron Scholes. Black and Scholes [6] derived an exceptionally important mathematical formula popularly known as the Black-Scholes (B-S) model to study and present the approximate statement of the performance of the underlying asset. The B-S model utilized geometrical Brownian motion and derived the option prices pretty close to the actual prices or observed prices to price the underlying asset. Since then continuing efforts

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of investigators have modified the basic Black-Scholes model because it was obtained under certain assumptions. This process of advancement of B-S model is still in progress and various mathematical methods are applied to make the B-S model more pragmatic [27, 31].

In recent years, researchers have employed different techniques to solve the Black-Scholes model and obtained various modified models to price options. Among these, the mentioned ones are, Levy processes [40], stochastic parameters or inclusion of jumps [26], Black-Scholes model with transaction costs [4], models including uncertain processes [32]. There are various other numerical and analytical methods available for the pricing of options. The numerical methods include finite element method [36], homotopy perturbation method [23], finite difference method [14], Monte Carlo simulations [8], binomial tree method [19] and Adomian decomposition method [7]. The analytical methods include Laplace transform method [17, 30], Fourier transform method [24] and Mellin transform method [41].

Investigation of movement of options through Black-Scholes equation is a novel area of interest. Since the financial markets are interdependent on each other, the one dimensional B-S equation is replaced by multi-dimensional B-S equation. The B-S model with multi assets is more effective in illiquid markets or markets with complex conditions than the B-S model with a single asset. Investigators have used diverse methods and perspectives to solve the multi dimensional B-S model such as finite difference method [14], modified Guass-Siedel method [39], radical basic function method [33], Mellin transform method [1, 28] and homotopy perturbation method [3].

Fractional differential equations have grown in an extra ordinary way to find their applications in various fields of science and engineering including finance [9, 25]. Fractional B-S PDE is a phenomenal outcome of the exceptional enhancement of fractional differential equations in financial theory. The notable fractional derivatives used to price options are time and space fractional derivatives. Under time fractional derivative, Wyss [42] solved the time fractional B-S model involving vanilla options. Kumar et al. [29] used homotopy analysis method (HAM) and homotopy perturbation method (HPM) to solve fractional B-S equation for European options. Akrami et al. [2] employed a new technique by reconstructing the variational iteration method to solve time fractional B-S equation for European options. Ghandehari and Ranjbar [22] solved time fractional B-S equation using decomposition method. Kanth and Aruna [38] solved time fractional B-S equation using fractional differential transform method (FDTM) and fractional modified differential transform method (MFDTM). Under the space fractional derivative, Cartea and Del-Castilo-Negrete [12] derived various space fractional B-S equations to price exotic options with jumps. Chen et al. [16] solved the finite moment log stable model (FMLS) for European vanilla options. Carr and Wu [11] defined the FMLS model that showed wider applications over other conventionally used financial models.

In this work, a novel technique called as new Sumudu transform iterative method (NSTIM) has been employed to solve the time fractional B-S model with two assets. The new method incorporates Sumudu transform with new iterative method (NIM) and obtains the explicit solution in a convergent series form. The proposed novel technique is simple in application and effectual enough to reduce the computational work. This paper is framed as; the basic definitions and preliminaries are given in section 2. The B-S model with two assets is presented in section 3. The NSTIM is given in section 4. The given problem is solved through NSTIM in section 5. Results and discussion part is given in section 6. The concluding remarks are given in section 7.

2. Definitions

In this section, some basic definitions used in this work are given [21, 37]:

Definition 2.1. The Caputo fractional derivative $D_z^\alpha(\alpha > 0)$ is given as [10]:

$$D_z^\alpha f(z) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_a^z (z-t)^{m-\alpha-1} f^{(m)}(t)dt, & m-1 < \alpha < m \in \mathbb{N} \\ \left(\frac{d}{dz}\right)^m f(z), & \alpha = m \in \mathbb{N} \end{cases} \tag{2.1}$$

where a defines the initial value of function f and α is the order of the derivative.

Definition 2.2. The time fractional derivative of order $\alpha > 0$ in Caputo's sense is given as:

$$D_t^\alpha v(x,t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\xi)^{(m-\alpha-1)} \frac{\partial^m v(x,\xi)}{\partial \xi^m} d\xi, & m-1 < \alpha < m \\ \frac{\partial^m v(x,t)}{\partial t^m}, & \alpha = m \in N \end{cases} \tag{2.2}$$

Some basic properties of Caputo's fractional derivative are listed as:

$$D_a^\alpha C = 0, \text{ where } C \text{ is constant.}$$

$$D_a^\alpha z^\gamma = \begin{cases} 0, & \gamma \leq \alpha - 1 \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} z^{\gamma-\alpha}, & \gamma > \alpha - 1 \end{cases}$$

Definition 2.3. The two parameter type Mittag-Leffler function is defined as [37]:

$$E_{\alpha,\beta}(z) = \sum_{p=0}^{\infty} \frac{z^p}{\Gamma(\alpha p + \beta)}, \ (\alpha > 0, \ \beta > 0). \tag{2.3}$$

Definition 2.4. The Sumudu transform of Caputo time-fractional derivative of $v(x,t)$ of order $\alpha > 0$ is given as [15]:

$$S\left[\frac{\partial^\alpha v(x,t)}{\partial t^\alpha}\right] = U^{-\alpha} S[v(x,t)] - \sum_{n=0}^{m-1} \left[U^{-\alpha+n} \frac{\partial^n v(x,0)}{\partial t^n}\right], m-1 < \alpha \leq m, m \in N. \tag{2.4}$$

3. The model

In this section, the general two dimensional Black-Scholes model for European options has been considered which is based on certain assumptions; the markets are efficient, there is no illiquidity in the markets and there are no dividends paid during the life time of the option. Moreover, it is assumed that σ_1, σ_2, ρ and r are constants. Thus, the resulting PDE to evaluate options is given as:

$$\frac{\partial f}{\partial \tau}+\frac{1}{2}\sigma_1^2 S_1^2\frac{\partial^2 f}{\partial S_1^2}+\frac{1}{2}\sigma_2^2 S_2^2\frac{\partial^2 f}{\partial S_2^2}+\rho\sigma_1\sigma_2\frac{\partial^2 f}{\partial S_1\partial S_2}+r\left[S_1\frac{\partial f}{\partial S_1}+S_2\frac{\partial f}{\partial S_2}\right]-rf=0, S_1,S_2\in[0,\infty),\tau\in[0,T], \tag{3.1}$$

subjected to initial and terminal conditions:

$$f(S_1,0,\tau)=0=f(0,S_2,\tau),\quad f(S_1,S_2,T)=\max(B_1S_1+B_2S_2-K,0),$$

and boundary conditions:

$$\begin{aligned} f(S_1,S_2,\tau)=&0 \text{ as } (S_1,S_2)\to 0\\ f(S_1,S_2,\tau)=&B_1S_1+B_2S_2-Ke^{-r(T-\tau)} \text{ as } S_1 \text{ or } S_2\to\infty, \end{aligned}$$

where $f(S_1,S_2,\tau)$ denotes the option price based on first asset's price S_1, second asset's price S_2 and time τ. σ_1, σ_2 denotes the volatilities of S_1 and S_2 respectively, T is the expiry time, ρ is the correlation between S_1 and S_2, r gives the risk free interest rate, K is the strike price. B_1 and B_2 are coefficients to maintain all risky asset prices at the same level.
By applying the transformations [18]:

$$x=\ln(S_1)-(r-\frac{1}{2}\sigma_1^2)\tau \text{ and } \quad y=\ln(S_2)-(r-\frac{1}{2}\sigma_2^2)\tau.$$

Equation (3.1) takes the form:

$$\frac{\partial f}{\partial \tau}+\frac{1}{2}\sigma_1^2\frac{\partial^2 f}{\partial x^2}+\frac{1}{2}\sigma_2^2\frac{\partial^2 f}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial^2 f}{\partial x\partial y}-rf=0,\ (x,y,\tau)\in\mathbb{R}\times\mathbb{R}\times[0,T], \tag{3.2}$$

subjected to the terminal condition:

$$f(x,y,T)=\max\left(B_1e^{x+(r-\frac{1}{2}\sigma_1^2)T}+B_2e^{y+(r-\frac{1}{2}\sigma_2^2)T}-K,0\right),$$

and boundary conditions:

$$\begin{aligned} f(x,y,\tau)=&0 \ \ as\ \ (x,y)\to-\infty \text{ and}\\ f(x,y,\tau)=&B_1e^{x+(r-\frac{1}{2}\sigma_1^2)\tau}+B_2e^{y+(r-\frac{1}{2}\sigma_2^2)\tau}-Ke^{-r(T-\tau)} \text{ as } x \text{ or } y\to\infty. \end{aligned}$$

In order to end the last term on LHS of (3.2), we make the transformations again by setting v as $f(x,y,\tau)=e^{-r(T-\tau)}v(x,y,\tau)$ and substitute into (3.2), we get:

$$\frac{\partial v}{\partial \tau}+\frac{1}{2}\sigma_1^2\frac{\partial^2 v}{\partial x^2}+\frac{1}{2}\sigma_2^2\frac{\partial^2 v}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial^2 v}{\partial x\partial y}=0,\quad (x,y,\tau)\in\mathbb{R}\times\mathbb{R}\times[0,T], \tag{3.3}$$

subjected to terminal condition:

$$v(x,y,T)=\max\left(B_1e^{x+(r-\frac{1}{2}\sigma_1^2)T}+B_2e^{y+(r-\frac{1}{2}\sigma_2^2)T}-K,0\right),$$

and boundary conditions:

$$\begin{aligned} v(x,y,\tau)=&0 \ \ as\ \ (x,y)\to-\infty \text{ and}\\ v(x,y,\tau)=&B_1e^{x+rT-\frac{1}{2}\sigma_1^2\tau}+B_2e^{y+rT-\frac{1}{2}\sigma_2^2\tau}-K \text{ as } x \text{ or } y\to\infty. \end{aligned}$$

A forward time co-ordinate is defined and used to find the solution of initial boundary value problem (3.3). Therefore, we have:

$$\frac{\partial v}{\partial t}=\frac{1}{2}\sigma_1^2\frac{\partial^2 v}{\partial x^2}+\frac{1}{2}\sigma_2^2\frac{\partial^2 v}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial^2 v}{\partial x\partial y}, \tag{3.4}$$

with initial condition:

$$v(x,y,0)=\max(\bar{B}_1e^x+\bar{B}_2e^y-K,0),$$

and boundary conditions:

$$\begin{aligned} v(x,y,t)=&0 \text{ as } \ (x,y)\to-\infty \ \text{ and}\\ v(x,y,t)=&\bar{B}_1e^{x+\frac{1}{2}\sigma_1^2t}+\bar{B}_2e^{y+\frac{1}{2}\sigma_2^2t}-K \quad \text{as } x \text{ or } y \ \to\infty, \end{aligned}$$

where $\bar{B}_1=B_1e^{(r-\frac{1}{2}\sigma_1^2)T}$ and $\bar{B}_2=B_2e^{(r-\frac{1}{2}\sigma_2^2)T}$.
The time derivative when replaced with Caputo fractional derivative gives the two dimensional Black-Scholes model along with initial and boundary conditions in the following form:

$$\frac{\partial^\alpha v}{\partial t^\alpha}=\frac{1}{2}\sigma_1^2\frac{\partial^2 v}{\partial x^2}+\frac{1}{2}\sigma_2^2\frac{\partial^2 v}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial^2 v}{\partial x\partial y},\ 0<\alpha\le 1, \tag{3.5}$$

with initial condition:

$$v(x,y,0)=\max(\bar{B}_1e^x+\bar{B}_2e^y-K,0), \tag{3.6}$$

and boundary conditions:

$$\begin{aligned} v(x,y,t)=&0 \quad \text{as } \ (x,y)\to-\infty \ \text{ and}\\ v(x,y,t)=&\bar{B}_1e^{x+\frac{1}{2}\sigma_1^2t}+\bar{B}_2e^{y+\frac{1}{2}\sigma_2^2t}-K \ \text{ as } x \text{ or } y\to\infty. \end{aligned} \tag{3.7}$$

4. Basic concept of NSTIM

To understand the fundamental concepts of NSTIM, We consider the equation:

$$\frac{\partial^{n\alpha}}{\partial t^{n\alpha}}v(x,y,t)+Nv(x,y,t)=g(x,y,t), \qquad n-1<n\alpha\le n, \tag{4.1}$$

subjected to initial condition:

$$v(x,y,0)=h(x,y),$$

and boundary condition:

$$B\left(x,\frac{\partial v}{\partial x},\frac{\partial v}{\partial y},\frac{\partial v}{\partial t}\right)=0,$$

where B represents boundary operator, $\frac{\partial^{n\alpha}}{\partial t^{n\alpha}}$ is the Caputo fractional derivative, N is the linear/nonlinear operator and $g(x,y,t)$ is the continuous function.
Operating with Sumudu transform on both sides of (4.1), we have:

$$S\left[\frac{\partial^{n\alpha}}{\partial t^{n\alpha}}v(x,y,t)\right]+S\left[Nv(x,y,t)\right]=S\left[g(x,y,t)\right]. \tag{4.2}$$

By Sumudu transform property, we have:

$$u^{-n\alpha}S\left[v(x,y,t)\right]-\sum_{k=0}^{n-1}u^{-n\alpha+k}v^{(k)}(x,y,0)+S\left[Nv(x,y,t)\right]=S\left[g(x,y,t)\right]. \quad (4.3)$$

After simplification, (4.3) can be written as:

$$S\left[v(x,y,t)\right]=\sum_{k=0}^{n-1}u^{k}v^{(k)}(x,y,0)-u^{n\alpha}S\left[Nv(x,y,t)\right]+u^{n\alpha}S\left[g(x,y,t)\right]. \quad (4.4)$$

The inverse Sumudu transform of (4.4) yields:

$$v(x,y,t)=S^{-1}\left\{\sum_{k=0}^{n-1}u^{k}v^{(k)}(x,y,0)\right\}-S^{-1}\left[u^{n\alpha}S[Nv(x,y,t)]-u^{n\alpha}S[g(x,y,t)]\right]. \quad (4.5)$$

(4.5) can be put in the form:

$$v(x,y,t)=S^{-1}\left\{\sum_{k=0}^{n-1}u^{k}v^{(k)}(x,y,0)+u^{n\alpha}S[g(x,y,t)]\right\}-S^{-1}\left[u^{n\alpha}S[Nv(x,y,t)]\right]. \quad (4.6)$$

Eq. (4.6) can be written as:

$$v(x,y,t)=f(x,y,t)+Mv(x,y,t), \quad (4.7)$$

where

$$f(x,y,t)=S^{-1}\left\{\sum_{k=0}^{n-1}u^{k}v^{(k)}(x,y,0)+u^{n\alpha}S[g(x,y,t)]\right\}$$
$$Mv(x,y,t)=-S^{-1}\left[u^{n\alpha}S[Nv(x,y,t)]\right],$$

where $Mv(x,y,t)$ is the linear/non-linear operator and $f(x,y,t)$ is the known function. The solution of equation (4.7) is obtained in an infinite series form [20]:

$$v(x,y,t)=\sum_{i=0}^{\infty}v_i(x,y,t), \quad (4.8)$$

where each term v_i can be computed recursively. The term M is dissolved as:

$$M\left(\sum_{i=0}^{\infty}v_i\right)=M(v_0)+\sum_{j=1}^{\infty}\left[M\left(\sum_{i=0}^{j}v_i\right)-M\left(\sum_{i=0}^{j-1}v_i\right)\right], \quad (4.9)$$

or

$$S^{-1}\left\{u^{n\alpha}S(N\sum_{i=0}^{\infty}v_i)\right\}=S^{-1}\left\{u^{n\alpha}S(N(v_0))\right\}+\sum_{j=1}^{\infty}S^{-1}\left\{u^{n\alpha}S(N\sum_{i=0}^{j}v_i)\right\}$$
$$-\sum_{j=1}^{\infty}S^{-1}\left\{u^{n\alpha}S(N\sum_{i=0}^{j-1}v_i)\right\}. \quad (4.10)$$

Using Equations (4.8) & (4.10) in (4.7), we get:

$$\sum_{i=0}^{\infty} v_i = S^{-1}\left\{\sum_{k=0}^{n-1} u^k v^{(k)}(x,y,0) + u^{n\alpha} S[g(x,y,t)]\right\} + S^{-1}\left\{u^{n\alpha} S(Nv_0)\right\} \\ + \sum_{j=1}^{\infty} S^{-1}\left\{u^{n\alpha} S(N\sum_{i=0}^{j} v_i)\right\} - \sum_{j=1}^{\infty} S^{-1}\left\{u^{n\alpha} S(N\sum_{i=0}^{j-1} v_i)\right\}. \tag{4.11}$$

Hence, each term can be defined recursively as:

$$\begin{aligned} v_0 &= S^{-1}\left\{\sum_{k=0}^{n-1} u^k v^{(k)}(x,y,0) + u^{n\alpha} S[g(x,y,t)]\right\} \\ v_1 &= S^{-1}\left\{u^{n\alpha} S(Nv_0)\right\} \\ v_{p+1} &= \sum_{j=1}^{\infty} S^{-1}\left\{u^{n\alpha} S(N\sum_{i=0}^{p} v_i)\right\} - \sum_{j=1}^{\infty} S^{-1}\left\{u^{n\alpha} S(N\sum_{i=0}^{p-1} v_i)\right\} \text{ for } p \geq 1. \end{aligned} \tag{4.12}$$

The approximate solution of (4.8) is obtained as $v(x,y,t) \approx v_0(x,y,t) + v_1(x,y,t) + ... + v_{p-1}(x,y,t)$. For the convergence of NSTIM we refer to [5].

5. APPLICATION OF NSTIM TO TIME FRACTIONAL BS MODEL

This section shows the application of proposed method (NSTIM) to solve time fractional Black-Scholes model (3.5) subjected to initial condition (3.6) and boundary condition (3.7).
We consider the model as:

$$N(v(x,y,t)) = \frac{1}{2}\sigma_1^2 \frac{\partial^2 v}{\partial x^2} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 v}{\partial y^2} + \rho\sigma_1\sigma_2 \frac{\partial^2 v}{\partial x \partial y}. \tag{5.1}$$

In general form, (3.5) can be written as:

$$\frac{\partial^\alpha v}{\partial t^\alpha} = N(v(x,y,t)). \tag{5.2}$$

The Sumudu transform of equation (5.2) gives:

$$S\left[\frac{\partial^\alpha v}{\partial t^\alpha}\right] = S[N(v(x,y,t))]. \tag{5.3}$$

The differential property of Sumudu transfom gives:

$$S[v(x,y,t)] = v(x,y,0) + u^\alpha S[N(v(x,y,t))]. \tag{5.4}$$

or

$$S[v(x,y,t)] = \max(\bar{B}_1 e^x + \bar{B}_2 e^y - K, 0) + u^\alpha S\left[\frac{1}{2}\sigma_1^2 \frac{\partial^2 v}{\partial x^2} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 v}{\partial y^2} + \rho\sigma_1\sigma_2 \frac{\partial^2 v}{\partial x \partial y}\right]. \tag{5.5}$$

The inverse Sumudu transform of (5.5) gives:

$$\begin{aligned} v(x,y,t) &= S^{-1}[\max(\bar{B}_1e^x+\bar{B}_2e^y-K,0)] + S^{-1}\left\{u^\alpha S\left[\frac{1}{2}\sigma_1^2\frac{\partial^2 v}{\partial x^2}+\frac{1}{2}\sigma_2^2\frac{\partial^2 v}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial^2 v}{\partial x\partial y}\right]\right\} \\ &= \max(\bar{B}_1e^x+\bar{B}_2e^y-K,0) + S^{-1}\left\{u^\alpha S\left[\frac{1}{2}\sigma_1^2\frac{\partial^2 v}{\partial x^2}+\frac{1}{2}\sigma_2^2\frac{\partial^2 v}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial^2 v}{\partial x\partial y}\right]\right\} \end{aligned} \tag{5.6}$$

For this model, we choose the initial condition as:

$$v(x,y,0) = \max(\bar{B}_1e^x+\bar{B}_2e^y-K,0) + e^{x+y}t^\alpha. \tag{5.7}$$

In view of initial condition (5.7), Equation (5.6) can be put in the form:

$$\begin{aligned} v(x,y,t) = {}& \max(\bar{B}_1e^x+\bar{B}_2e^y-K,0) + e^{x+y}t^\alpha - e^{x+y}t^\alpha \\ &+ S^{-1}\left\{u^\alpha S\left[\frac{1}{2}\sigma_1^2\frac{\partial^2 v}{\partial x^2}+\frac{1}{2}\sigma_2^2\frac{\partial^2 v}{\partial y^2}+\rho\sigma_1\sigma_2\frac{\partial^2 v}{\partial x\partial y}\right]\right\}. \end{aligned} \tag{5.8}$$

According to NSTIM and in view of recurrence relation (4.12) we have:

$$\begin{aligned} v_0(x,y,t) &= \max(\bar{B}_1e^x+\bar{B}_2e^y-K,0) + e^{x+y}t^\alpha \\ v_1(x,y,t) &= -\, e^{x+y}t^\alpha + S^{-1}\{u^\alpha S(N(v_0))\} \\ v_{p+1}(x,y,t) &= -\, e^{x+y}t^\alpha + \sum_{j=1}^{\infty} S^{-1}\left\{u^{n\alpha}S(N\sum_{i=0}^{p} v_i)\right\} \\ &\qquad - \sum_{j=1}^{\infty} S^{-1}\left\{u^{n\alpha}S(N\sum_{i=0}^{p-1} v_i)\right\}, \text{for } p \geq 1. \end{aligned} \tag{5.9}$$

Thus, $v_0, v_1, v_2, \ldots$ can be generally put as:

$$\begin{aligned} v_0(x,y,t) &= \max(\bar{B}_1e^x+\bar{B}_2e^y-K,0) + e^{x+y}t^\alpha \\ v_1(x,y,t) &= \frac{t^\alpha}{\Gamma(\alpha+1)}\left(\frac{1}{2}\sigma_1^2\max(\bar{B}_1e^x,0)+\frac{1}{2}\sigma_2^2\max(\bar{B}_2e^y,0)\right) \\ &\quad + e^{x+y}\frac{t^{2\alpha}\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\left(\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\rho\sigma_1\sigma_2\right) - e^{x+y}t^\alpha \\ v_{p+1}(x,y,t) &= \frac{t^{p\alpha}}{\Gamma(p\alpha+1)}\left(\frac{1}{2^p}\sigma_1^{2p}\max(\bar{B}_1e^x,0)+\frac{1}{2^p}\sigma_2^{2p}\max(\bar{B}_2e^y,0)\right) \\ &\quad + e^{x+y}\frac{t^{(p+1)\alpha}\Gamma(\alpha+1)}{\Gamma((p+1)\alpha+1)}\left(\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\rho\sigma_1\sigma_2\right)^p \\ &\quad - e^{x+y}t^{p\alpha}\frac{\Gamma(\alpha+1)}{\Gamma(p\alpha+1)}\left(\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\rho\sigma_1\sigma_2\right)^{(p-1)} \quad \text{for } p \geqslant 1. \end{aligned} \tag{5.10}$$

Thus the solution of the problem is given as:

$$\begin{aligned}
v(x,y,t) &= v_0(x,y,t)+v_1(x,y,t)+v_2(x,y,t)+...\\
&= \max(\bar{B}_1e^x+\bar{B}_2e^y-K,0)+e^{x+y}t^\alpha\\
&+\sum_{p=0}^{\infty}\{\frac{t^{(p+1)\alpha}}{\Gamma((p+1)\alpha+1)}\left(\frac{1}{2^{(p+1)}}\sigma_1^{2(p+1)}\max(\bar{B}_1e^x,0)+\frac{1}{2^{(p+1)}}\sigma_2^{2(p+1)}\max(\bar{B}_2e^y,0)\right)\\
&+e^{x+y}\frac{t^{(p+2)\alpha}\Gamma(\alpha+1)}{\Gamma((p+2)\alpha+1)}\left(\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\rho\sigma_1\sigma_2\right)^{(p+1)}\\
&-e^{(x+y)}\frac{t^{(p+1)\alpha}\Gamma(\alpha+1)}{\Gamma((p+1)\alpha+1)}\left(\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\rho\sigma_1\sigma_2\right)^{p}\}.
\end{aligned} \tag{5.11}$$

In this way the closed form solution of equation (3.5) is given as:

$$\begin{aligned}
v(x,y,t) =& \max(\bar{B}_1e^x+\bar{B}_2e^y-K,0)+e^{x+y}t^\alpha+\max(\bar{B}_1e^x,0)\frac{t^\alpha\sigma_1^2}{2}E_{\alpha,\alpha+1}\left(\frac{t^\alpha\sigma_1^2}{2}\right)\\
&+\max(\bar{B}_2e^y,0)\frac{t^\alpha\sigma_2^2}{2}E_{\alpha,\alpha+1}\left(\frac{t^\alpha\sigma_2^2}{2}\right)\\
&+e^{x+y}\left(\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\rho\sigma_1\sigma_2\right)\Gamma(\alpha+1)t^{2\alpha}E_{\alpha,2\alpha+1}\left(t^\alpha\left(\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\rho\sigma_1\sigma_2\right)\right)\\
&-e^{x+y}\Gamma(\alpha+1)t^\alpha E_{\alpha,\alpha+1}\left(t^\alpha\left(\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\rho\sigma_1\sigma_2\right)\right),
\end{aligned} \tag{5.12}$$

where $E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}$ is the two parameter Mittag-Leffler function [34] with α and β as constants.

6. Results & Discussion

This section gives the MATLAB computation of series solution obtained in equation (5.12). The parameters used to carry out the numerical simulations are given as follows. The graphical representation of closed form solution v and original solution f are plotted in Figures $1-6$.

Table 1. Parameters for Numerical simulation of European call options with two assets through NSTIM

Parameters	Value
B_1	3
B_2	2
Risk free rate of interest, r	4%
Strike price, K	50
Maturity Time, T	1 $Year$
Volatility of 1st asset, σ_1	15%
Volatility of 2nd asset, σ_2	10%
Correlation, ρ	0.4

Figure $1(a, b)$ represents the surface plot of European call option with strike price $K = 50$ and the correlation parameter $\rho = 0.4$. The solutions v and f are plotted over a range of $0 \leq S_1 \leq 150$ and $0 \leq S_2 \leq 150$ at a day before an expiration date with order $\alpha = 0.9$. The results show an increase in the call option value with increasing stock prices S_1 and S_2. Figure $2(a, b)$ shows the solutions v and f with order of $\alpha = 0.9$. With $S_2 = 10$, the option price becomes 0 as the stock price increases from $0 - 20$. Afterwards the option price increases significantly with increasing stock price from $20 - 150$. The solution v becomes 0 when x reaches from $0 - 1.5$. After that the value of v increases with increasing x and y.

Figure $3(a, b)$ represents the solution of call option with order $\alpha = 0.9$. With $S_1 = 10$ and time $0 \leq t \leq 1$, the option price becomes 0 for S_2 ranging from $0 - 20$. When the stock price is larger than 20, the option price grows linearly. The solution v becomes zero for $0 \leq y \leq 3$. The value of v increases as y is greater than 3. In Figure $4(a, b)$, the solution v is investigated for different values of $\alpha = 0.3, 0.5, 0.7$ with respect to x and y. Effects of fractional order α on v are shown.

In Figure 5, the solution v is investigated with correlation coefficient $0 \leq \rho \leq 3$ for different values of x and y. Effects of x and y on v with fixed y and different values of x are shown in figure $5(a)$. Figure $5(b)$ represents the value of v for fixed x and different values of y are taken. In Figure $6(a)$, the solution v is plotted against σ_1 for different values of x and fixed value of y. In Figure $6(b)$, v is plotted for fixed x and different values of y.

In Table 2, the numerical analysis of the NSTIM solutions for different values of fractional value α are given.

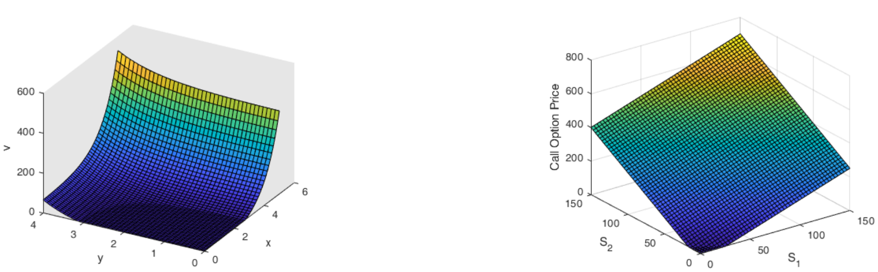

FIGURE 1. (a) Closed form solution v, and (b) call option price f for $\alpha = 0.9$.

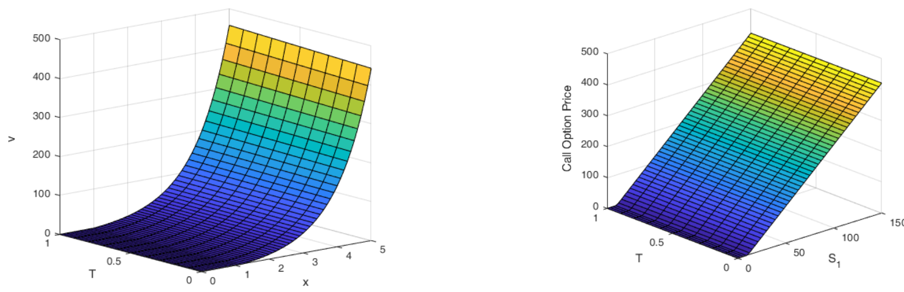

FIGURE 2. (a) Closed form solution v, and (b) call option price f for $\alpha = 0.9$ and $S_2 = 10$

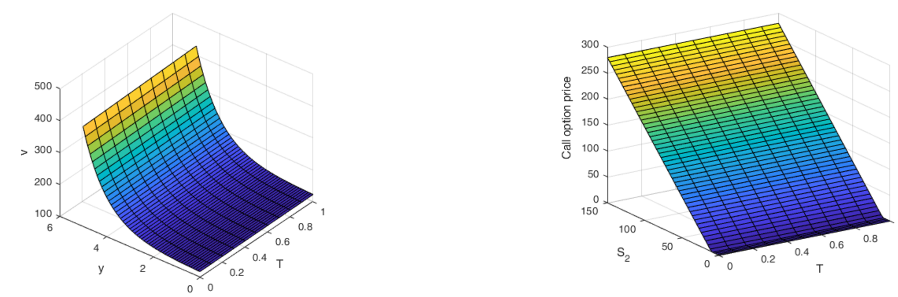

FIGURE 3. (a) Closed form solution v, and (b) call option price f for $\alpha = 0.9$ and $S_1 = 10$.

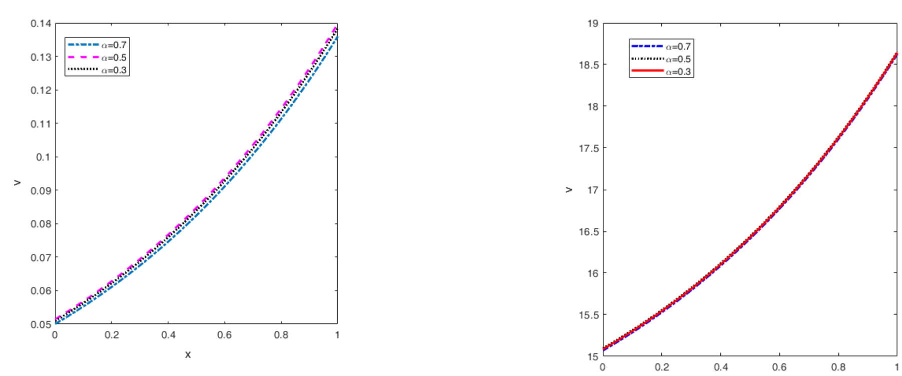

FIGURE 4. Solution plots of v for different values of $\alpha = 0.3, 0.5, 0.7$ (a) with respect to x, and (b) with respect to y.

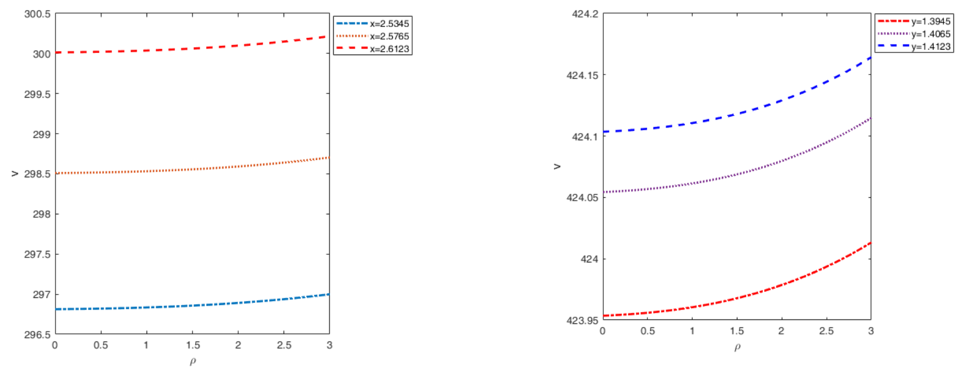

FIGURE 5. Solution plots of v with respect to correlation coefficient ρ, (a) for different values of x, and (b) for different values of y.

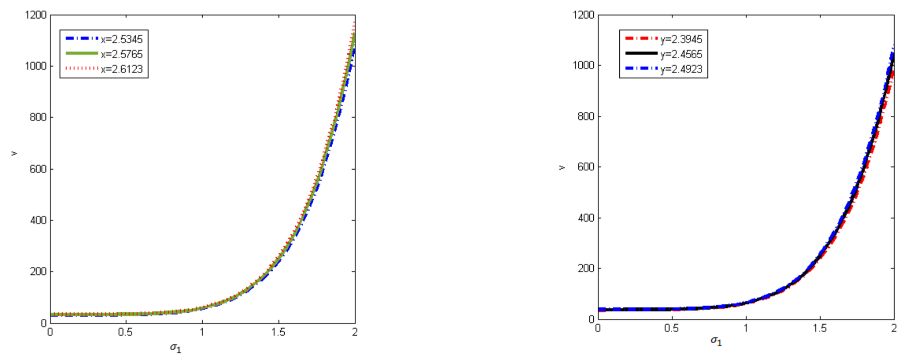

FIGURE 6. Relationship between v and volatility σ_1: (a) for $x = 2.5345,\ 2.5765,\ 2.6127$ and (b) for$y = 2.3945,\ 2.4565,\ 2.4923.$

TABLE 2. Numerical simulation of European call options with two assets through NSTIM.

t	x	y	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
0.25	1	2	0.9071	0.0695	0.0512	0.0367
	2	3	14.6445	14.5866	14.5371	14.4974
	3	4	125.7050	125.5466	125.4118	125.3040
0.50	1	2	0.1119	0.0984	0.0834	0.0685
	2	3	14.7023	14.6654	14.6245	14.5840
	3	4	125.8635	125.7614	125.6946	125.5393
0.75	1	2	0.1265	0.1207	0.1109	0.0988
	2	3	14.7423	14.7262	14.6994	14.6663
	3	4	125.9740	125.9277	125.8537	125.7633
1.0	1	2	0.1381	0.1396	0.1359	0.1282
	2	3	14.7740	14.7777	14.7674	14.7462
	3	4	126.0619	126.0691	126.0392	125.9808

7. Conclusion

It is well known that financial models can be outstandingly explained using fractional partial differential equations. Solution of fractional partial differential equations involving financial models is a recent and prominent area of investigation. Improvisation in financial models for better output necessitated the advancement in existing techniques to solve fractional partial differential equations. This article proposes the new Sumudu transform iterative method to find the explicit solution of time fractional Black-Scholes model governing European options with two assets. The advantage of NSTIM is to combine the Sumudu transform and new iterative method to get the explicit solution of time fractional Black-Scholes equation. The solution is obtained in the form of generalized Mittag-Leffler function. By using NSTIM for this problem, the transformed solution can be easily applied to simulate the European call option based on two assets. Additionally, numerical results have been presented to show the efficiency and performance of the proposed method. Thus, it can be concluded that NSTIM is an easy and efficient method to intensify the capability of reducing computational work. This method can be used to solve fractional order partial differential equations in different fields of applied mathematics.

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