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AN ESTIMATION OF THE V-CONJUGATION OPERATOR

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ABSTRACT. In this work we prove a new estimation of the V-conjugation operator, which connects it with the difference of the mean values of function $f \in L^1(G)$.

1. Introduction and preliminaries

The V-conjugate function was studied in the works of P. Simon [2], [4] and [3]. In [[2], Theorem 4] it has been proved that the V-conjugation operator is of type (H^1, L^1). The question on whether $\|\tilde{f}\|_{L^1}$ provides a new norm on H^1 was also mentioned in [2]. In [5] it was proved that on the 3-series field H^1 cannot be defined by means of the V-conjugation, i.e. the norms $\|f\|_{H^1}$ and $\|\tilde{f}\|_{L^1}$ are not equivalent in case of the 3-series field and also that the mean value of function $f \in L^1(G)$ on the coset $I_{N-1}(x)$ is dominated by either $\sigma_{M_{N-1}}$ or σ_{M_N} on some translated element. Here we prove a new estimation of the V-conjugation operator, which connects it with the difference of the mean values of function $f \in L^1(G)$.

We use standard notations and many formulae contained in [6] and [1].

Let $(m_0, m_1, \dots, m_n, \dots)$ be a bounded sequence of integers not less than 2. Put $m := \max_n m_n$.

Let $G := \prod_{n=0}^{\infty} \mathbb{Z}_{m_n}$, where $\mathbb{Z}_{m_n}$ denotes the discrete group of order m_n, with addition *mod* m_n. Each element from G can be represented as a sequence $(x_n)_n$, where $x_n \in \{0, 1, \dots, m_n - 1\}$. Addition in G is obtained coordinatewise.

The group of integers of the p-series field is similarly defined by $\prod_{n=0}^{\infty} \mathbb{Z}_p$, where $p \geq 2$ is an integer.

The topology on G is generated by the subgroups
$I_n := \{x = (x_i)_i \in G, x_i = 0 \text{ for } i < n\}$, and their translations
$I_n(y) := \{x = (x_i)_i \in G, x_i = y_i \text{ for } i < n\}$.

The basis $(e_n)_n$ is formed by elements $e_n = (\delta_{in})_i$.

Define the sequence $(M_n)_n$ as follows: $M_0 = 1$ and $M_{n+1} = m_n M_n$.

If $|I_n|$ denotes the normalized product measure of I_n then it can be easily seen that $|I_n| = M_n^{-1}$.

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For every nonnegative integer n, there exists a unique sequence $(n_i)_i$ so that $n = \sum\limits_{i=0}^{\infty} n_i M_i$.

The generalized Rademacher functions are defined by

$$r_n(x) := e^{\frac{2\pi i x_n}{m_n}}, n \in \mathbb{N} \cup \{0\}, x \in G,$$

and the system of Vilenkin functions by

$$\psi_n(x) := \prod_{i=0}^{\infty} r^{n_i}(x), n \in \mathbb{N} \cup \{0\}, x \in G.$$

The Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels and the Fejér means with respect to the Vilenkin system are respectively defined as follows

$$\hat{f}(n) = \int f(x)\bar{\psi}_n(x)dx, \; S_n f = \sum_{k=0}^{n-1} \hat{f}(k)\psi_k,$$

$$D_n = \sum_{k=0}^{n-1} \psi_k, \; \sigma_n f = \frac{1}{n}\sum_{k=1}^{n} S_k f,$$

for every $f \in L^1(G)$. It can be easily seen that

$$S_n f(y) = \int D_n(y-x)f(x)dx,$$

and

$$D_{M_n}(x) = M_n 1_{I_n}(x).$$

We introduce the maximal function $f^*(x) = \sup\limits_n |I_n|^{-1}|\int_{I_n(x)} f(t)dt|$. Generally speaking, the Hardy space $H^p(G)$, $p > 0$ consists of integrable functions f for which $f^* \in L^p(G)$. $H^p(G)$ is a Banach space with the norm

$$\|f\|_{H^p} := \|f^*\|_p.$$

The V-conjugate function as defined in [2] has the form

$$\tilde{f} := \sum_{k=0}^{\infty} f * L_k D_{M_k},$$

where f is an integrable function and

$$L_k := -\sum_{j=1}^{\triangle_k} r_k^j + \sum_{j=\triangle_k+1}^{m_k-1} r_k^j,$$

$\triangle_k = [\frac{m_k-1}{2}]$, if $m_k > 2$, and $\triangle_k = 1$, if $m_k = 2$.
Using the expression of L_k by Simon [4], we have
$a)$

$$L_k(x) = 1 - \frac{1}{i}\frac{(-1)^{x_k}}{sin\dfrac{\pi x_k}{m_k}} + \frac{exp\left(-\dfrac{\pi x_k i}{m_k}\right)}{i}\left(sin\frac{x_k\pi}{m_k}\right)^{-1}$$

$$(x_k \neq 0, m_k \equiv 1(2));$$

$b)$ for $m_k \equiv 0\ (2)(m_k > 2)\ (x_k \neq 0)$ we have

$$L_k(x) = 1 + \frac{1}{i}\frac{exp\left(-\dfrac{\pi x_k i}{m_k}\right)[1-(-1)^{x_k}]}{sin\dfrac{\pi x_k}{m_k}}; \tag{1.1}$$

$c)$ in the case $m_k = 2$ we have $L_k(x) = -(-1)^{x_k}$ $(x \in G)$.
For $m_k \equiv 1(2), s \in \{0, 1, ..., m_k - 1\}$ we write

$$L_k(se_k) = 1 + \frac{i(-1)^s}{sin\dfrac{s\pi}{m_k}} - \frac{icos\dfrac{s\pi}{m_k} + sin\dfrac{s\pi}{m_k}}{sin\dfrac{s\pi}{m_k}}.$$

2. Main results

Theorem 2.1. *Let $G = \Pi_{n=0}^{\infty}\mathbb{Z}_{m_k}$ be a bounded Vilenkin group. Set $m = \max_k m_k$. Then there exists a constant $C > 0$, only depending on m such that*

$$M_k \int_{x+I_k} |f * L_k D_{M_k}| dx \geqslant C \sup_{i=1,\dots,m_k} |S_{M_{k+1}}(f) - S_{M_k}(f)|.$$

Proof. We first establish the proof for the case when m_k is odd and when the function is real.
For fixed k, let $m_k = 2N - 1$.
If $s \in \{2, ..., m_k - 1\}$ is even, we have

$$L_k(se_k) = i\frac{1 - cos\dfrac{s\pi}{2N-1}}{sin\dfrac{s\pi}{2N-1}} = 2i\frac{sin^2\dfrac{s\pi}{4N-2}}{2sin\dfrac{s\pi}{4N-2}cos\dfrac{s\pi}{4N-2}} = i\tan\frac{s\pi}{4N-2},$$

and clearly, $L_k(0) = 0$. If s is odd, we obtain

$$L_k(se_k) = -i\frac{1 + cos\dfrac{s\pi}{2N-1}}{sin\dfrac{s\pi}{2N-1}} = -2i\frac{cos^2\dfrac{s\pi}{4N-2}}{2sin\dfrac{s\pi}{4N-2}cos\dfrac{s\pi}{4N-2}} = -i\cot\frac{s\pi}{4N-2}.$$

Let $\beta_s = \tan\dfrac{\pi s}{4N-2}$ when s is even, and $\beta_s = -\cot\dfrac{\pi s}{4N-2}$, when s is odd. This yields

$$\begin{aligned}
\left(f * L_k D_{M_k}\right)(y) &= \int f(x)(L_k D_{M_k})(y-x)dx = M_k \int\limits_{I_k(y)} f(x)L_k(y-x)dx = \\
&= M_k \sum_{j=0}^{m_{k-1}} \int\limits_{(y-x)\in(s\cdot e_k + I_{k+1})} f(x)L_k(y-x)dx = \\
&= M_k \sum_{j=0}^{m_{k-1}} i\beta_s \int\limits_{t\in(s\cdot e_k + I_{k+1})} f(y-t)dt = i\sum_{s=0}^{m_{k-1}} \beta_s a_s(y),
\end{aligned}$$

when $a_s(y) = M_k \int\limits_{t\in(s\cdot e_k+I_{k+1})} f(y-t)dt$.

We fix $y \in G$ such that $a_{M+1}(y) - a_{M-1}(y) = \max\limits_{i,j} |a_i(y) - a_j(y)|$, for some $M \in \{0, \dots, N-1\}$.

Suppose that $M \geq 1$, and put $a_s := a_s(y)$. Notice that if s is odd then

$$-\beta_s = \cot\frac{\pi s}{4N-2} = \tan\left(\frac{\pi}{2} - \frac{\pi s}{4N-2}\right) = \tan\left(\frac{2N-1-s}{4N-2}\pi\right) = \beta_{2N-1-s}.$$

This is clearly also valid when s is even. Therefore,

$$\sum_{s=0}^{2N-1} \beta_s a_s = \sum_{s=1}^{2N-1} \beta_s a_s = \sum_{s=1}^{2N-1} \beta_s(a_s - a_{2N-1-s}) = \sum_{s=1}^{2N-1} \beta_s(a_s - a_{-s}).$$

Then, it is easily seen that

$$\begin{aligned}
&\sum_{t\in\{\pm M\}} (f * L_k D_{M_k})(y + te_k) - \sum_{t\in\{\pm(M+2)\}} (f * L_k D_{M_k})(y + te_k) = \\
= \quad & \sum_{i=1}^{N-1} \beta_i(a_{M+i} - a_{M-i}) + \sum_{i=1}^{N-1} \beta_i(a_{-M+i} - a_{-M-i}) - \\
- \quad & \sum_{i=1}^{N-1} \beta_i(a_{M+2+i} - a_{M+2-i}) - \sum_{i=1}^{N-1} \beta_i(a_{-M-2+i} - a_{-M-2-i}) = \\
= \quad & \sum_{i=1}^{N-1} \beta_i(a_{M+i} - a_{-M-i}) + \sum_{i=1}^{N-1} \beta_i(a_{-M+i} - a_{M-i}) - \\
- \quad & \sum_{i=1}^{N-1} \beta_i(a_{M+2+i} - a_{-M-2-i}) - \sum_{i=1}^{N-1} \beta_i(a_{-M-2+i} - a_{M+2-i}) = \\
= \quad & I + II + III + IV. \qquad (2.1)
\end{aligned}$$

Furthermore,

$$\begin{aligned} I &= \sum_{i=1}^{N-M-1} \beta_i(a_{M+i} - a_{-M-i}) + \sum_{i=N-M}^{N-1} \beta_i(a_{M+i} - a_{-M-i}) = \\ &= \sum_{i=1}^{N-M-1} \beta_i(a_{M+i} - a_{-M-i}) + \sum_{i=N-M}^{N-1} \beta_i(a_{M-2N+1+i} - a_{2N-M-1-i}) = \\ &= \sum_{i=1}^{N-M-1} \beta_i(a_{M+i} - a_{-M-i}) + \sum_{j=N-2M}^{N-M-1} \beta_{2(N-M)-1-j}(a_{-M-j} - a_{M+j}), \end{aligned}$$

where $j = 2N - 1 - 2M - i$, with $2M \leq N$.
We put

$$L(a_M) = (f * L_k D_{M_k})(y + Me_k),$$

$$L(a_{-M}) = (f * L_k D_{M_k})(y - Me_k),$$

and we get

$$\begin{aligned} L(a_M) + L(a_{-M}) &= \sum_{i=1}^{N-1-M} \beta_i(a_{M+i} - a_{-M-i}) + \sum_{N-M}^{N-1} \beta_i(a_{M+i} - a_{-M-i}) + \\ &+ \sum_{i=1}^{M-1} \beta_i(a_{-M+i} - a_{M-i}) + \sum_{i=M+1}^{N-1} \beta_i(a_{-M+i} - a_{M-i}). \end{aligned}$$

Now,

$$\sum_{i=M+1}^{N-1} \beta_i(a_{i-M} - a_{M-i}) = \sum_{i=M+1}^{2M} \beta_i(a_{i-M} - a_{M-i}) + \sum_{i=2M+1}^{N-1} \beta_i(a_{i-M} - a_{M-i}). \quad (2.2)$$

For the first term on the right side of the equation (2.2) we put $2M - i = j$, and for the second $i - 2M = j$, so we have

$$\begin{aligned} \sum_{i=M+1}^{N-1} \beta_i(a_{-M+i} - a_{M-i}) &= \sum_{j=0}^{M-1} \beta_{2M-j}(a_{M-j} - a_{-M+j}) + \\ &+ \sum_{j=1}^{N-1-2M} \beta_{j+2M}(a_{-M+j} - a_{-M-i}). \end{aligned}$$

Taking substitute $M + i - 2N + 1 = -M - j$ we have $j = 2N - 1 - 2M - i$, so

$$\sum_{i=N-M}^{N-1} \beta_i(a_{M+i} - a_{M-i}) = \sum_{j=N-2M}^{N-2M-1} \beta_{2(N-M)-1-j}(a_{-M-j} - a_{M+j}).$$

Now, we have

$$\begin{aligned} L(a_M)+L(a_{-M}) &= \sum_{i=1}^{N-1-2M}(\beta_i+\beta_{i+2M})(a_{M+i}-a_{-M-i})+ \\ &+ \sum_{i=N-2M}^{N-M-1}(\beta_i+\beta_{2(N-M)-1-i})(a_{M+i}-a_{-M-i})+ \\ &+ \sum_{i=1}^{M-1}(\beta_i+\beta_{2M-i})(a_{-M+i}-a_{M-i}), \end{aligned}$$

and

$$\begin{aligned} L(a_{M+2})+L(a_{-M-2}) &= \sum_{i=1}^{N-1-2M-4}(\beta_i+\beta_{i+2M+4})(a_{M+i+1}-a_{-M-i-1})+ \\ &+ \sum_{i=1-2M-4}^{N-M-3}(\beta_i-\beta_{2(N-M-2)-1-i})(a_{M+2+i}-a_{-M-2-i})+ \\ &+ \sum_{i=1}^{M+1}(\beta_i+\beta_{2M+4-i})(a_{-M-2+i}-a_{M+2-i}). \end{aligned}$$

If we put $i+2=j$, we get

$$\begin{aligned} L(a_{M+2})+L(a_{-M-2}) &= \sum_{j=0}^{N-2M-3}(\beta_{j-2}+\beta_{j+2+2M})(a_{M+j}-a_{-M-j})+ \\ &+ \sum_{j=N-2M-2}^{N-M-1}(\beta_{j-2}+\beta_{2(N-M)-3-j})(a_{M+j}-a_{-M-j})+ \\ &+ \sum_{j=-1}^{M-1}(\beta_{j+2}+\beta_{2M+2-j})(a_{-M+j}-a_{M-j}). \end{aligned}$$

Formula (2.1) now becomes

$$\begin{aligned}
(\quad & \beta_1+\beta_{1+2M})(a_{M+1}-a_{-M-1})+(\beta_2+\beta_{2+2M})(a_{M+2}-a_{-M-2})+\\
+ \quad & (\beta_{N-2-2M}+\beta_{N-2})(a_{N-2-M}-a_{-N+2+M})+\\
+ \quad & (\beta_{N-1-2M}+\beta_{N-1})(a_{N-1-M}-a_{-N+1+M})+\\
+ \quad & \sum_{i=3}^{N-2M-3}\underbrace{[\beta_i+\beta_{i+2M}-\beta_{i-2}-\beta_{i+2M+2}]}_{I'}(a_{M+i}-a_{-M-i})+\\
+ \quad & \sum_{j=N-2M}^{N-M-1}\underbrace{[\beta_i-\beta_{2(N-M)-1-i}-\beta_{i-2}+\beta_{2(N-M)-3-i}]}_{II'}(a_{M+i}-a_{-M-i})-\\
- \quad & (\beta_{N-2M-4}-\beta_{2(N-M)-3-(N-2M-2)})(a_{-M+N-2}-a_{M-N+2})-\\
- \quad & (\beta_{N-2M-3}-\beta_{2(N-M)-3-(N-2M-1)})(a_{N-M-3}-a_{-N+M+3})+\\
+ \quad & \sum_{i=3}^{M-1}\underbrace{[\beta_i+\beta_{2M-1}-\beta_{i+2}-\beta_{2M+2-i}]}_{III'}(a_{-M+i}-a_{M-i})-\\
- \quad & (\beta_1-\beta_{2M+3})(a_{-M-1}-a_{M+1})-(\beta_2-\beta_{2M+2)})(a_{-M}-a_M).
\end{aligned}$$

We estimate I'_{odd} (when i is odd)

$$\begin{aligned}
I'_{odd} &= \cot\frac{i\pi}{4N-2}-\cot\frac{(i-2)\pi}{2N-2}+\cot\left(\frac{i+2M}{2N-2}\pi\right)-\cot\left(\frac{i+2M+2}{4N-2}\pi\right)=\\
&= \frac{\sin\dfrac{-2\pi}{4N-2}}{\sin\left(\dfrac{i\pi}{4N-2}\right)\sin\left(\dfrac{(i-2)\pi}{4N-2}\right)}+\frac{\sin\dfrac{2\pi}{4N-2}}{\sin\left(\dfrac{i+2M}{4N-2}\right)\sin\left(\dfrac{i+2M+2}{4N-2}\pi\right)}<0.
\end{aligned}$$

For i even, we get

$$\begin{aligned}
I'_{even} &= -\tan\frac{i\pi}{4N-2}+\tan\frac{(i-2)\pi}{4N-2}-\\
&- \tan\left(\frac{i+2M}{4N-2}\pi\right)+\tan\left(\frac{i+2M+2}{4N-2}\pi\right)=\\
&= \frac{\sin\dfrac{-2\pi}{4N-2}}{\cos\left(\dfrac{i\pi}{4N-2}\right)\cos\left(\dfrac{(i-2)\pi}{4N-2}\right)}+\\
&+ \frac{\sin\dfrac{2\pi}{4N-2}}{\cos\left(\dfrac{i+2M}{4N-2}\right)\cos\left(\dfrac{i+2M+2}{4N-2}\pi\right)}>0.
\end{aligned}$$

We use the following notations:

$$<k>=\begin{cases} k, & if\ \ k\ \ is\ \ odd \\ k-1, & if\ \ k\ \ is\ \ even \end{cases} \qquad >k<=\begin{cases} k, & if\ \ k\ \ is\ \ odd \\ k+1, & if\ \ k\ \ is\ \ even \end{cases}$$

$$<<k>>=\begin{cases} k, & if\ \ k\ \ is\ \ even \\ k-1, & if\ \ k\ \ is\ \ odd \end{cases} \qquad >>k<<=\begin{cases} k & if\ \ k\ \ is\ \ even \\ k+1 & if\ \ k\ \ is\ \ odd, \end{cases}$$

so

$$\begin{aligned} I' &> [\sum_{\substack{i=3 \\ (i\ odd)}}^{N-2M-3} (\beta_i - \beta_{i-2}) + (\beta_{i+2M} - \beta_{i+2M+2}) \\ &+ \sum_{\substack{i=4 \\ (i\ even)}}^{N-2M-3} (\beta_{i-2} - \beta_i + \beta_{i+2M+2} - \beta_{i+2M})] \\ &\cdot\ (a_{M+1} - a_{-M-1}) \\ &= [(-\beta_1 + \beta_{<N-2M-3>} + \beta_{3+2M} - \beta_{<N-1>}) \\ &+ (\beta_2 + \beta_{<<N-2M-3>>} + \beta_{4+2M} - \beta_{<<N-1>>})] \\ &\cdot\ (a_{M+1} - a_{-M-1}). \end{aligned}$$

As $\beta_{<N-2M-3>} > \beta_{>N-2M-2<}$ we have

$$\beta_{<N-2M-3>} - \beta_{>N-2M-2<} > 0.$$

Now we estimate II' when i is odd.

$$\begin{aligned} II'_{odd} &= \cot\frac{i\pi}{4N-2} - \cot\frac{(i-2)\pi}{4N-2} + \tan\left(\frac{2(N-M)-1-i}{4N-2}\pi\right) - \\ &- \tan\left(\frac{2(N-M)-3-i}{4N-2}\pi\right) = \\ &= \frac{\sin\dfrac{-2\pi}{4N-2}}{\sin\dfrac{i\pi}{4N-2}\sin\dfrac{i-2}{4N-2}\pi} + \\ &+ \frac{\sin\dfrac{2\pi}{4N-2}}{\cos\dfrac{2(N-M)-i-1}{4N-2}\pi\cdot\cos\dfrac{2(N-M-3-i)}{4N-2}\pi} < 0. \end{aligned}$$

For i even, we have

$$\begin{aligned}
II'_{even} &= -\tan\frac{i\pi}{4N-2}+\tan\frac{(i-2)\pi}{4N-2}-\cot\left(\frac{2(N-M)-1-i}{4N-2}\pi\right)+\\
&+\cot\left(\frac{2(N-M)-3-i}{4N-2}\pi\right)=\\
&=\frac{\sin\dfrac{-2\pi}{4N-2}}{\cos\dfrac{i\pi}{4N-2}\cos\dfrac{(i-2)\pi}{4N-2}}+\\
&+\frac{\sin\dfrac{2\pi}{4N-2}}{\sin\dfrac{2(N-M)-i-1}{4N-2}\pi\cos\dfrac{2(N-M-3-i)}{4N-2}\pi}>0.
\end{aligned}$$

So,

$$\begin{aligned}
II' &> \Bigg[\sum_{\substack{i=N-2M\\(i\ odd)}}^{N-M-1}((-\beta_{i-2}+\beta_i)+(\beta_{2(N-M)-3-i}-\beta_{2(N-M)-1-i}))+\\
&+\sum_{\substack{i=N-2M\\(i\ even)}}^{N-M-1}(\beta_{i-2}-\beta_i-\beta_{2(N-M)-3-i}+\beta_{2(N-M)-1-i})\Bigg](a_{M+1}-a_{-M-1})=\\
&= -\beta_{>N-2M-2<}+\beta_{<N-M-1>}+\beta_{>>N-M-2<<}-\beta_{<<N-1>>}\\
&+\beta_{>>N-2M-2<<}-\beta_{<<N-M-1>>}-\beta_{>N-M-2<}+\beta_{<N-1>}\\
&=\beta_{<N-2M-1>}.
\end{aligned}$$

We estimate III' for i odd,

$$\cot\frac{i\pi}{4N-2}-\cot\frac{i+2}{4N-2}\pi+\cot\frac{2M+2-i}{4N-2}\pi-\cot\frac{2M-i}{4N-2}\pi=$$

$$=\frac{\sin\dfrac{2\pi}{4N-2}}{\sin\dfrac{i\pi}{4N-2}\sin\left(\dfrac{i+2}{4N-2}\pi\right)}+\frac{\sin\dfrac{-2\pi}{4N-2}}{\sin\left(\dfrac{2M+2-i}{4N-2}\pi\right)\sin\left(\dfrac{2M-i}{4N-2}\pi\right)}>0.$$

For i even, we have

$$\tan\frac{i+2}{4N-2}\pi-\tan\frac{i\pi}{4N-2}\pi+\tan\frac{2M-i}{4N-2}\pi-\tan\frac{2M+2-i}{4N-2}\pi=$$

$$\frac{\sin\dfrac{2\pi}{4N-2}}{\cos\dfrac{i\pi}{4N-2}\cos\left(\dfrac{i+2}{4N-2}\pi\right)}+\frac{\sin\dfrac{-2\pi}{4N-2}}{\cos\left(\dfrac{2M-i}{4N-2}\pi\right)\cos\left(\dfrac{2M+2-i}{4N-2}\pi\right)}<0.$$

$$III' > \left[\sum_{\substack{i=1\\(i\ odd)}}^{M-1}(-\beta_i+\beta_{i+2}-\beta_{2M+2-i})+\sum_{\substack{i=2\\(i\ even)}}^{M-1}(\beta_i-\beta_{i+2}-\beta_{2M-i}+\beta_{2M+2-i})\right]\cdot$$

$$\cdot(a_{M+1}-a_{-M-1})=$$

$$[-\beta_1+\beta_{<M+1>}+\beta_{>M+1<}-\beta_{2M+1}+\beta_2-\beta_{<<M+1>>}-\ \beta_{>>M+1<<}+\beta_{2M}]\cdot$$

$$\cdot(a_{M+1}-a_{-M-1})>-\beta_1-\beta_{>>N+1<<}.$$

So, we get

$$\begin{aligned}
I'+II'+III' &> 2\beta_1+\beta_{1+2M}+\beta_{3+2M}+2\beta_2+2\beta_{2+2M}-\beta_{<N-1>}+\beta_{>>N-2<<}\\
&- \beta_1+\beta_{<N-2M-3>}+\beta_{3+2M}+2\beta_2-\beta_{<<N-2M-3>>}-\beta_{4+2M}\\
&- \beta_1-\beta_{>>M+1<<}=\\
&= \beta_{2+2M}-\beta_{4+2M}+\beta_{3+2M}-\beta_{<N-1>}+\beta_{3+2M}+\beta_{>>N-2<<}+\\
&+ \beta_2-\beta_{<<N-2M-3>>}+\beta_2-\beta_{>>N+1<<}+\beta_{<N-2M-3>}+\\
&+ \beta_{2+2M}++\beta_{1+2M}+2\beta_2>\beta_{1+2M}+2\beta_2=\\
&= \cot\frac{1+2M}{4N-2}\pi-2\tan\frac{2\pi}{4N-2}\\
&= \frac{\cos\left(\dfrac{1+2M}{4N-2}\pi\right)\cos\left(\dfrac{2\pi}{4N-2}\right)-2\sin\dfrac{1+2M}{4N-2}\sin\dfrac{2\pi}{4n-2}}{\sin\dfrac{1+2M}{4N-2}\pi\cdot\cos\dfrac{2\pi}{4N-2}}\\
&= \frac{\cos\left(\dfrac{3+2M}{4N-2}\right)\pi-\sin\dfrac{1+2M}{4N-2}\sin\dfrac{2\pi}{4N-2}}{\sin\dfrac{1+2M}{4N-2}\pi\cdot\cos\dfrac{2\pi}{4N-2}}\\
&> \frac{\cos\left(\dfrac{3+2M}{4N-2}\right)\pi-\sin\dfrac{2\pi}{4N-2}}{\sin\dfrac{1+2M}{4N-2}\pi\cdot\cos\dfrac{2\pi}{4N-2}}\\
&= \frac{\sin\left(\dfrac{2N-1-3-2M}{4N-2}\right)\pi-\sin\dfrac{2\pi}{4N-2}}{\sin\dfrac{1+2M}{4N-2}\pi\cdot\cos\dfrac{2\pi}{4N-2}}\\
&> C,
\end{aligned}$$

if $2(N-M)-4 \geq 3 \Leftrightarrow 2(N-M) \geq 7 \Rightarrow N-M \geq 4 \Rightarrow N-1-M \geq 3$, valid as $2M+3 \leq N-1$, because $2(M+2) \leq N-1$.
It's not hard to show that the sum $I' + II' + III'$ is bounded from below in the cases when $2(M+2) > N-1 \geq 2M$, or when $2M > N-1$.
The second part of the proof concerns $m_k = 2N$, $N \neq 1$. From the formula b) (1.1), we get $\gamma_k = L_k(se_k) = 1$, if s is even, and $\gamma_k = L_k(se_k) = -1 - 2i \cot \frac{s}{m_k}\pi$, if s is odd.
The proof can be done by the same method by taking only the complex part of $\int\limits_{x+I_k} (L_k * f)$. Two possible situations occur. The first one is that

$$a_{M+1}(y) - a_{-M-1}(y) = \max_{i,j} |a_i(y) - a_j(y)|,$$

and the second one is

$$a_{M+1}(y) - a_{-M}(y) = \max_{i,j} |a_i(y) - a_j(y)|.$$

We first study the first case. The same method can be used as in the case when $m_k = 2N-1$, except the part of the sum where $N-M \geq i+2M \geq 2(N-M)-N = N-2M$.
For such odd indices i, the factor in the sum that is multiplied by $(a_{M+i} - a_{-M-i})$ becomes

$$\gamma_i - \gamma_{2(N-M)-i} - \gamma_{i-2} + \gamma_{2(N-M)-2-i} =$$

$$\cot \frac{i\pi}{2N} - \cot \frac{(i-2)\pi}{2N} + \cot \left(\frac{2(N-M)-2-i}{2N}\pi \right) - \cot \left(\frac{2(N-M)-i}{2N}\pi \right) =$$

$$\frac{\sin \frac{-2\pi}{2N}}{\sin \frac{i\pi}{2N} \sin \frac{(i-2)\pi}{2N}} + \frac{\sin \frac{2\pi}{2N}}{\sin \frac{2(N-M)-i-2}{2N}\pi \sin \frac{2(N-M)-i}{2N}\pi} < 0.$$

If f is a real function, it is easily seen that the real part of

$$\sum_{t \in \{M,-M\}} (f * L_k D_{M_k})(y + te_k) - \sum_{t \in \{M+2,-M-2\}} (f * L_k D_{M_k})(y + te_k)$$

vanishes. Using the same arguments as before, we get

$$\frac{1}{2} Im \left(\sum_{t \in \{M,-M\}} (f * L_k D_{M_k})(y + te_k) - \sum_{t \in \{M+2,-M-2\}} (f * L_k D_{M_k})(y + te_k) \right) =$$

$$
\begin{aligned}
= & \Big[\sum_{\substack{i=3\\(i\ odd)}}^{N-2M-3}(\underbrace{\gamma_i-\gamma_{i-2}+\gamma_{i+2M}-\gamma_{i+2M+2}}_{<0})+\\
+ & \sum_{\substack{i=N-2M\\(i\ odd)}}^{N-M-1}(\underbrace{\gamma_i-\gamma_{i-2}+\gamma_{2(N-M)-2-i}-\gamma_{2(N-M)-i}}_{<0})+\\
+ & \sum_{\substack{i=1\\(i\ odd)}}^{M-1}(\underbrace{\gamma_{i+2}-\gamma_i+\gamma_{2M-i}-\gamma_{2M-i+2}}_{<0})\Big]\cdot(a_{M-i}-a_{-M+i})+\\
+ & (2\gamma_1+\gamma_{2M+1}-\gamma_{2M+3})\cdot(a_{M+1}-a_{-M-1})+\\
+ & (\underbrace{\gamma_i-\gamma_{i-2}+\gamma_{i+2M}+\gamma_{2(N-M)-2-i}}_{<0,\ (i=>N-2M-2<)})\cdot(a_{>N-M-2<}-a_{->N-M-2<})>\\
> & \Big[\sum_{\substack{i=3\\(i\ odd)}}^{N-2M-3}(\underbrace{\gamma_i-\gamma_{i-2}+\gamma_{i+2M}-\gamma_{i+2M+2}}_{<0})+\\
+ & \sum_{\substack{i=N-2M\\(i\ odd)}}^{N-M-1}(\underbrace{\gamma_i-\gamma_{i-2}+\gamma_{2(N-M)-2-i}-\gamma_{2(N-M)-i}}_{<0})+\\
+ & \sum_{\substack{i=1\\(i\ odd)}}^{M-1}(\underbrace{\gamma_{i+2}-\gamma_i+\gamma_{2M-i}-\gamma_{2M-i+2}}_{<0})\Big]\cdot(a_{M+1}-a_{-M-1})+\\
+ & [(2\gamma_1+\gamma_{2M+1}-\gamma_{2M+3})+(\gamma_{>N-2M-2<}-\gamma_{>N-2M-4<}+\gamma_{>N-2<}+\gamma_{>N-1<})]\cdot\\
\cdot & (a_{M+1}-a_{-M-1})=-\gamma_1+\gamma_{<N-2M-3>}+\gamma_{2M+3}-\gamma_{<N-1>}-\gamma_{>N-2M-2<}+\\
+ & \gamma_{<N-M-1>}+\gamma_{>N-M-1<}-\gamma_{<N>}-\gamma_1+\gamma_{<M+1>}+\gamma_{<M+1>}-\gamma_{2M+1}+\\
+ & 2\gamma_1+\gamma_{2M+1}-\gamma_{2M+3}+\gamma_{>N-2M-2<}-\gamma_{>N-2M-4<}+\\
+ & \gamma_{>N-2<}+\gamma_{>N-1<}>C,
\end{aligned}
$$

because $\gamma_{>N-1<}=\gamma_{<N>}$, $\gamma_{>N-2<}=\gamma_{<N-1>}$ and $\gamma_{<N-2M-3>}=\gamma_{>N-2M-4<}$.
In the second situation, namely when

$$\max_{i,j}|a_i(y)-a_j(y)|=a_{M+1}(y)-a_{-M}(y),$$

for some $M\geq 0$, we prove the result following the steps done before. □

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