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NONLINEAR SINGULAR HAHN–STURM–LIOUVILLE PROBLEMS ON $[\omega_0, \infty)$.

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Abstract. In this paper, we investigate a nonlinear Hahn–Sturm–Liouville problem on $[\omega_0, \infty)$ under the assumption that the Hahn–Sturm–Liouville equation is in the limit-circle case at ∞. The existence and uniqueness of the solutions for these problems are obtained.

1. Introduction

The Hahn difference operator is known as (see [11])

$$D_{\omega,q}g\left(x\right)=\frac{g\left(\omega+qx\right)-g\left(x\right)}{\omega+\left(q-1\right)x},$$

where $q \in (0,1)$ and $\omega > 0$.

In [5], Aldwoah studied a proper inverse of the Hahn difference operator. In [12, 13], Hamza and Ahmet established the theory of linear Hahn difference equations. Sitthiwirattham [16] investigated a nonlinear Hahn difference equation. Recently, in [7], Annaby et al. established the Hahn–Sturm–Liouville theory. In [3], the authors established the Titchmarsh–Weyl theory of the singular Hahn–Sturm–Liouville equation.

At present, there have been some studies on the Hahn difference operator because it unifies the forward difference operator and the quantum difference operator. Although the existence problem of solutions of equations is one of the important research areas in mathematics (see [1, 2, 4, 9, 14, 16, 8, 15, 12, 13]), to the authors' knowledge, there is only one study on the existence of solutions for singular nonlinear Hahn–Sturm–Liouville problems (see [4]). Our problem is very different from the papers in literature since the Hahn–Sturm–Liouville equation on $[\omega_0, \infty)$ is in the limit-circle case at ∞. In the analysis that follows, we will largely follow the development of the theory in [9, 1, 2, 4].

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2. Preliminaries

In this part, we introduce the essentials of the theory of Hahn calculus [6, 10, 11, 7]. Throughout the paper, let $q \in (0,1)$, $\omega_0 := \omega/(1-q)$, $\omega > 0$ and let I be a real interval containing ω_0.

Definition 2.1 ([10], [11])**.** Let $g : I \to \mathbb{R} := (-\infty, \infty)$ be a function. $D_{\omega,q} g$ is defined by

$$D_{\omega,q} g(x) = \begin{cases} \frac{g(\omega+qx)-g(x)}{\omega+(q-1)x}, & x \neq \omega_0, \\ g'(\omega_0), & x = \omega_0, \end{cases}$$

provided that g is differentiable at ω_0.

Definition 2.2 ([6])**.** Let $g : I \to \mathbb{R}$ be a function and $a, b, \omega_0 \in I$. We define the ω, q-integral of g by

$$\int_a^b f(x)\, d_{\omega,q}x := \int_{\omega_0}^b f(x)\, d_{\omega,q}x - \int_{\omega_0}^a f(x)\, d_{\omega,q}x,$$

where

$$\int_{\omega_0}^x f(t)\, d_{\omega,q}t := ((1-q)x - \omega) \sum_{n=0}^{\infty} q^n f\left(\omega \frac{1-q^n}{1-q} + xq^n\right), x \in I$$

provided that the series converges at $x = a$ and $x = b$. Similarly, one can define the ω, q-integral of g over (ω_0, ∞) by

$$\int_{\omega_0}^{\infty} g(x)\, d_{\omega,q}x := \lim_{b\to\infty} \int_{\omega_0}^b g(x)\, d_{\omega,q}x.$$

Let $L^2_{\omega,q}(\omega_0, \infty)$ be the space of all real-valued functions defined on $[\omega_0, \infty)$ such that

$$\|g\| := \left(\int_{\omega_0}^{\infty} |g(x)|^2\, d_{\omega,q}x\right)^{1/2} < \infty.$$

The space $L^2_{\omega,q}(\omega_0, \infty)$ is a separable Hilbert space with the inner product

$$(g_1, g_2) := \int_{\omega_0}^{\infty} g_1(x)\, g_2(x)\, d_{\omega,q}x,\ \ g_1, g_2 \in L^2_{\omega,q}(\omega_0, \infty)$$

(see [6]).

The ω, q-Wronskian of $g_1(.)$, $g_2(.)$ is defined by

$$W_{\omega,q}(g_1, g_2)(x) := g_1(x)\, D_{\omega,q} g_2(x) - g_2(x)\, D_{\omega,q} g_1(x), \tag{2.1}$$

where $x \in [\omega_0, \infty)$.

A nonlinear Hahn–Sturm–Liouville equation can be given as

$$l(y) := -q^{-1} D_{-\omega q^{-1}, q^{-1}} (p(x) D_{\omega,q} y(x)) + r(x)\, y(x) = f(x, y(x)), \tag{2.2}$$

where p and r are real-valued functions such that continuous at ω_0, and $\frac{1}{p}, r \in L^1_{\omega,q,loc}(\omega_0, \infty)$.

Let

$$\mathcal{D} = \left\{ y \in L^2_{\omega,q}(\omega_0, \infty) : \begin{array}{c} y \text{ and } pD_{\omega,q} y \text{ are continuous at } \omega_0 \\ \text{and } l(y) \in L^2_{\omega,q}(\omega_0, \infty). \end{array} \right\}$$

The *maximal* operator L is defined by $Ly = l(y)$ on $\mathcal{D}$. Then for arbitrary two functions g_1 and g_2 in $\mathcal{D}$ we have ω, q-*Green's formula*

$$\int_{\omega_0}^{t} (Lg_1)g_2 d_{\omega,q}x - \int_{\omega_0}^{t} g_1(Lg_2)d_{\omega,q}x = [g_1, g_2]_t - [g_1, g_2]_{\omega_0}, \tag{2.3}$$

where $\omega_0 < t < \infty$ and

$$[g_1, g_2]_x := p(x)\{g_1(x)D_{-\omega q^{-1},q^{-1}}g_2(x) - D_{-\omega q^{-1},q^{-1}}g_1(x)g_2(x)\}$$

(see [7]).

Suppose the following conditions are satisfied.

(A1**)** limit-circle case conditions are satisfied for the Hahn–Sturm–Liouville equation

$$l\,(y) = 0 \tag{2.4}$$

([3]).

(A2**)** $f\,(x, \zeta)$ is a real-valued and continuous in $(x, \zeta) \in [\omega_0, \infty) \times \mathbb{R}$, and, for all (x, ζ) in $[\omega_0, \infty) \times \mathbb{R}$, $f\,(x, \zeta)$ satisfies the following condition:

$$|f\,(x, \zeta)| \le g\,(x) + \vartheta\,|\zeta|\,, \tag{2.5}$$

where $g\,(x) \ge 0$, $g \in L^2_{\omega,q}(\omega_0, \infty)$, and $\vartheta > 0$.

Let $u\,(x)$ and $v\,(x)$ be the solutions of Eq. (2.4) satisfying the conditions

$$u\,(\omega_0) = 0,\ \left(pD_{-\omega q^{-1},q^{-1}}u\right)(\omega_0) = 1, \tag{2.6}$$

$$v\,(\omega_0) = -1,\ \left(pD_{-\omega q^{-1},q^{-1}}v)\right)(\omega_0) = 0.$$

Since $W_{\omega,q}\,(u, v) = 1$, u and v are linearly independent and form a fundamental system of solutions of Eq. (2.4). It follows from (A1) that $u, v \in L^2_{\omega,q}(\omega_0, \infty)$ and $u,\ v \in \mathcal{D}$. Then $[y, u]_\infty$ and $[y, v]_\infty$ exist and are finite for every $y \in \mathcal{D}$. By virtue of (2.3) and (2.6), we conclude that

$$\begin{aligned}[y, u]_\infty &= y\,(\omega_0) + \int_{\omega_0}^{\infty} u\,(x)\,l(y\,(x))d_{\omega,q}x,\\ [y, v]_\infty &= \left(pD_{-\omega q^{-1},q^{-1}}y\right)(\omega_0) + \int_{\omega_0}^{\infty} v\,(x)\,l(y\,(x))d_{\omega,q}x.\end{aligned} \tag{2.7}$$

Now, we will impose the following conditions

$$\begin{aligned}y\,(\omega_0)\cos\alpha + \left(pD_{-\omega q^{-1},q^{-1}}y\right)(\omega_0)\sin\alpha &= d_1,\\ [y, u]_\infty \cos\beta + [y, v]_\infty \sin\beta &= d_2,\end{aligned} \tag{2.8}$$

where $\alpha, \beta \in \mathbb{R}$,

$$\text{(A3)} \qquad \rho := \cos\alpha\sin\beta - \cos\beta\sin\alpha\ \ne 0,$$

and d_1, d_2 are arbitrary given real numbers.

Since the function y in (2.8) satisfies Eq. (2.2), we have

$$[y,u]_{\infty}=y(\omega_0)+\int_{\omega_0}^{\infty}u(x)\,f(x,y(x))\,d_{\omega,q}x,$$

$$[y,v]_{\infty}=(pD_{-\omega q^{-1},q^{-1}}y)(\omega_0)+\int_{\omega_0}^{\infty}v(x)\,f(x,y(x))\,d_{\omega,q}x.$$

3. Green's function

Let us consider the following problem

$$l(y)=h(x)\,,\ x\in[\omega_0,\infty), \tag{3.1}$$

$$\begin{aligned} &y(\omega_0)\cos\alpha+(pD_{-\omega q^{-1},q^{-1}}y)(\omega_0)\sin\alpha=0,\\ &[y,u]_{\infty}\cos\beta+[y,v]_{\infty}\sin\beta=0, \end{aligned} \tag{3.2}$$

where $h\in L^2_{\omega,q}(\omega_0,\infty)$ and $\alpha,\beta\in\mathbb{R}$.

Define

$$\varphi(x)=\cos\alpha u(x)+\sin\alpha v(x)\,,\ \psi(x)=\cos\beta u(x)+\sin\beta v(x)\,, \tag{3.3}$$

where $W_{\omega,q}(\varphi,\psi)=\cos\alpha\sin\beta-\cos\beta\sin\alpha$. It is evident that $\varphi(x)$ and $\psi(x)$ are solutions of Eq. (2.4) and $\varphi,\psi\in L^2_{\omega,q}(\omega_0,\infty)$. Then, we infer that

$$[\varphi,u]_x=\varphi(\omega_0)=-\sin\alpha,\ [\varphi,v]_x=(pD_{-\omega q^{-1},q^{-1}}\varphi)(\omega_0)=\cos\alpha, \tag{3.4}$$

$$[\psi,u]_x=\psi(a)=-\sin\beta,\ [\psi,v]_x=(pD_{-\omega q^{-1},q^{-1}}\psi)(\omega_0)=\cos\beta, \tag{3.5}$$

$$[\psi,u]_{\infty}=-\sin\beta,\ [\psi,v]_{\infty}=\cos\beta. \tag{3.6}$$

Theorem 3.1. *The solution of the boundary-value problem (3.1)-(3.2) as*

$$y(x)=\int_{\omega_0}^{\infty}G(x,t)\,h(t)\,d_{\omega,q}t, \tag{3.7}$$

where $x\in[\omega_0,\infty)$ *and*

$$G(x,t)=\begin{cases} -\frac{\varphi(x)\psi(t)}{W_{\omega,q}(\varphi,\psi)}, & \text{if } t\le x,\\ -\frac{\varphi(t)\psi(x)}{W_{\omega,q}(\varphi,\psi)}, & \text{if } t> x. \end{cases} \tag{3.8}$$

Proof. The variation of constants formula yields

$$\begin{aligned} y(x)=k_1\varphi(x)+k_2\psi(x)+\frac{q}{W_{\omega,q}(\varphi,\psi)}\psi(x)\int_{\omega_0}^{x}\varphi(qt+\omega)\,h(t)\,d_{\omega,q}t\\ -\frac{q}{W_{\omega,q}(\varphi,\psi)}\varphi(x)\int_{\omega_0}^{x}\psi(qt+\omega)\,h(t)\,d_{\omega,q}t, \end{aligned} \tag{3.9}$$

where k_1 and k_2 are arbitrary constants.

By (3.9), we get

$$
\begin{aligned}
&\left(pD_{-\omega q^{-1},q^{-1}}y\right)(x)\\
&=k_1\left(pD_{-\omega q^{-1},q^{-1}}\varphi\right)(x)+k_2\left(pD_{-\omega q^{-1},q^{-1}}\psi\right)(x)\\
&+\frac{q}{W_{\omega,q}(\varphi,\psi)}\left(pD_{-\omega q^{-1},q^{-1}}\psi\right)(x)\int_{\omega_0}^{x}\varphi(qt+\omega)h(t)\,d_{\omega,q}t\\
&-\frac{q}{W_{\omega,q}(\varphi,\psi)}\left(pD_{-\omega q^{-1},q^{-1}}\varphi\right)(x)\int_{\omega_0}^{x}\psi(qt+\omega)h(t)\,d_{\omega,q}t.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
y(\omega_0)&=k_1\varphi(\omega_0)+k_2\psi(\omega_0)=-k_1\sin\alpha-k_2\sin\beta,\\
\left(pD_{-\omega q^{-1},q^{-1}}y\right)(\omega_0)&=k_1\left(pD_{-\omega q^{-1},q^{-1}}\varphi\right)(\omega_0)\\
&+k_2\left(pD_{-\omega q^{-1},q^{-1}}\psi\right)(\omega_0)\\
&=k_1\cos\alpha+k_2\cos\beta.
\end{aligned}
\tag{3.10}
$$

Substituting (3.10) into (3.2), we get

$$
k_2(\cos\alpha\sin\beta-\sin\alpha\cos\beta)=0,\ \ k_2=0,
$$

Moreover, we see that

$$[y,u]_x = p(x)\left\{y(x)D_{-\omega q^{-1},q^{-1}}u(x) - D_{-\omega q^{-1},q^{-1}}y(x)u(x)\right\}$$

$$= k_1[\varphi,u]_x + k_2[\psi,u]_x$$

$$+\frac{q}{W_{\omega,q}(\varphi,\psi)}[\psi,u]_x\int_{\omega_0}^{x}\varphi(qt+\omega)h(t)\,d_{\omega,q}t$$

$$-\frac{q}{W_{\omega,q}(\varphi,\psi)}[\varphi,u]_x\int_{\omega_0}^{x}\psi(qt+\omega)h(t)\,d_{\omega,q}t$$

$$= -k_1\sin\alpha - \frac{q}{W_{\omega,q}(\varphi,\psi)}\sin\beta\int_{\omega_0}^{x}\varphi(qt+\omega)h(t)\,d_{\omega,q}t$$

$$+\frac{q}{W_{\omega,q}(\varphi,\psi)}\sin\alpha\int_{\omega_0}^{x}\psi(qt+\omega)h(t)\,d_{\omega,q}t = -k_1\sin\alpha$$

$$+\frac{q}{W_{\omega,q}(\varphi,\psi)}\int_{\omega_0}^{x}\left(-\sin\beta\varphi(qt+\omega)+\sin\alpha\psi(qt+\omega)\right)h(t)\,d_{\omega,q}t$$

$$= -k_1\sin\alpha + \frac{q}{W_{\omega,q}(\varphi,\psi)}\int_{\omega_0}^{x}u(qt+\omega)h(t)\,d_{\omega,q}t.$$

Thus

$$[y,u]_\infty = -k_1\sin\alpha + q\int_{\omega_0}^{\infty}u(qt+\omega)h(t)\,d_{\omega,q}t.$$

Similarly, we get

$$[y,v]_x = p(x)\left\{y(x)D_{-\omega q^{-1},q^{-1}}v(x) - D_{-\omega q^{-1},q^{-1}}y(x)v(x)\right\}$$

$$= k_1[\varphi,v]_x + \frac{q}{W_{\omega,q}(\varphi,\psi)}[\psi,v]_x\int_{\omega_0}^{x}\varphi(qt+\omega)h(t)\,d_{\omega,q}t$$

$$-\frac{q}{W_{\omega,q}(\varphi,\psi)}[\varphi,v]_x\int_{\omega_0}^{x}\psi(qt+\omega)h(t)\,d_{\omega,q}t$$

and

$$[y,v]_\infty = k_1\cos\alpha + q\int_{\omega_0}^{\infty}v(qt+\omega)h(t)\,d_{\omega,q}t.$$

From the conditions (3.2), we obtain

$$k_1\left(-\sin\alpha\cos\beta+\cos\alpha\sin\beta\right)$$

$$+q\int_{\omega_0}^{\infty}\left[\cos\beta u\left(qt+\omega\right)+\sin\beta v\left(qt+\omega\right)\right]h\left(t\right)d_{\omega,q}t=0.$$

Hence,

$$k_1=-\frac{q}{W_{\omega,q}\left(\varphi,\psi\right)}\int_{\omega_0}^{\infty}\psi\left(qt+\omega\right)h\left(t\right)d_{\omega,q}t.$$

By (3.9), we get

$$y\left(x\right)=-\frac{q}{W_{\omega,q}\left(\varphi,\psi\right)}\int_{\omega_0}^{x}\varphi\left(qt+\omega\right)\psi\left(x\right)h\left(t\right)d_{\omega,q}t$$

$$-\frac{q}{W_{\omega,q}\left(\varphi,\psi\right)}\int_{x}^{\infty}\varphi\left(x\right)\psi\left(qt+\omega\right)h\left(t\right)d_{\omega,q}t.$$

□

Theorem 3.2. *The unique solution of the boundary-value problem (3.1) under the conditions (2.8) is given as*

$$y\left(x\right)=\varsigma\left(x\right)+\int_{\omega_0}^{\infty}G\left(x,t\right)h\left(t\right)d_{\omega,q}t,$$

where

$$\varsigma\left(x\right)=\frac{d_1}{W_{\omega,q}\left(\varphi,\psi\right)}\varphi\left(x\right)-\frac{d_2}{W_{\omega,q}\left(\varphi,\psi\right)}\psi\left(x\right).$$

Proof. It follows from (3.4)-(3.6) that $\varsigma\left(x\right)$ is a unique solution of the problem (3.1), (2.8). □

From Theorem 3.1, the problem (2.2), (2.8) is equivalent to the following equation

$$y\left(x\right)=\varsigma\left(x\right)+\int_{\omega_0}^{\infty}G\left(x,t\right)f\left(t,y\left(t\right)\right)d_{\omega,q}t,\tag{3.11}$$

where $\varsigma\left(x\right)$ and $G\left(x,t\right)$ are defined above. Since $\varphi,\psi\in L^2_{\omega,q}(\omega_0,\infty)$, we have

$$\int_{\omega_0}^{\infty}\int_{\omega_0}^{\infty}\left|G\left(x,t\right)\right|^2d_{\omega,q}xd_{\omega,q}t<\infty.\tag{3.12}$$

Using (2.5) and (3.12), we define an operator

$$T:L^2_{\omega,q}(\omega_0,\infty)\to L^2_{\omega,q}(\omega_0,\infty)$$

as follows:

$$\left(Ty\right)\left(x\right)=\varsigma\left(x\right)+\int_{\omega_0}^{\infty}G\left(x,t\right)f\left(t,y\left(t\right)\right)d_{\omega,q}t,\ x\in\left(\omega_0,\infty\right),\tag{3.13}$$

where y and $\varsigma\in L^2_{\omega,q}(\omega_0,\infty)$. Hence (3.11) can be written as $y=Ty$.

Theorem 3.3. *Suppose that the conditions (A1), (A2), and (A3) are satisfied. Further, let the function $f(x,y)$ satisfy the following Lipschitz condition: there exist a constant $K>0$ such that*

$$\int_{\omega_0}^{\infty} |f(x,y(x)) - f(x,z(x))|^2 \, d_{\omega,q}x \leq K^2 \int_{\omega_0}^{\infty} |y(x)-z(x)|^2 \, d_{\omega,q}x \tag{3.14}$$

for all $y,z \in L^2_{\omega,q}(\omega_0,\infty)$. If

$$K\left(\int_{\omega_0}^{\infty}\int_{\omega_0}^{\infty} |G(x,t)|^2 \, d_{\omega,q}x d_{\omega,q}t\right)^{1/2} < 1, \tag{3.15}$$

then the boundary-value problem (2.2), (2.8) has a unique solution in $L^2_{\omega,q}(\omega_0,\infty)$.

Proof. For $y,z \in L^2_{\omega,q}(\omega_0,\infty)$, we see that

$$\begin{aligned}
&|(Ty)(x) - (Tz)(x)|^2 \\
&= \left|\int_{\omega_0}^{\infty} G(x,t)\left[f(t,y(t)) - f(t,z(t))\right] d_{\omega,q}t\right|^2 \\
&\leq \int_{\omega_0}^{\infty} |G(x,t)|^2 \, d_{\omega,q}t \int_{\omega_0}^{\infty} |f(t,y(t)) - f(t,z(t))|^2 \, d_{\omega,q}t \\
&\leq K^2 \|y-z\|^2 \int_{\omega_0}^{\infty} |G(x,t)|^2 \, d_{\omega,q}t, \ x \in (\omega_0,\infty).
\end{aligned}$$

Thus, we get

$$\|Ty - Tz\| \leq \alpha \|y-z\|,$$

where

$$\alpha = K\left(\int_{\omega_0}^{\infty}\int_{\omega_0}^{\infty} |G(x,t)|^2 \, d_{\omega,q}x d_{\omega,q}t\right)^{1/2} < 1.$$

□

Theorem 3.4. *Suppose that the conditions (A1), (A2) and (A3) are satisfied. In addition, let the function $f(x,y)$ satisfy the following Lipschitz condition: there exist constants $M, \ K>0$ such that*

$$\begin{aligned}
&\int_{\omega_0}^{\infty} |f(x,y(x)) - f(x,z(x))|^2 \, d_{\omega,q}x \\
&\leq K^2 \int_{\omega_0}^{\infty} |y(x)-z(x)|^2 \, d_{\omega,q}x
\end{aligned} \tag{3.16}$$

for all y and z in

$$S_M = \left\{y \in L^2_{\omega,q}(\omega_0,\infty) : \|y\| \leq M\right\},$$

where K may depend on M. If

$$\left\{\int_{\omega_0}^{\infty}|\omega(x)|^2 d_{\omega,q}x\right\}^{1/2}+\left(\int_{\omega_0}^{\infty}\int_{\omega_0}^{\infty}|G(x,t)|^2 d_{\omega,q}x d_{\omega,q}t\right)^{1/2}$$

$$\times\sup_{y\in S_M}\left\{\int_{\omega_0}^{\infty}|f(t,y(t))|^2 d_{\omega,q}t\right\}^{1/2}\leq M \tag{3.17}$$

and

$$K\left(\int_{\omega_0}^{\infty}\int_{\omega_0}^{\infty}|G(x,t)|^2 d_{\omega,q}x d_{\omega,q}t\right)^{1/2}<1, \tag{3.18}$$

then the boundary-value problem (2.2)-(2.8) has a unique solution with

$$\int_{\omega_0}^{\infty}|y(x)|^2 d_{\omega,q}x\leq M^2.$$

Proof. Since S_M is a closed set of $L^2_{\omega,q}(\omega_0,\infty)$, we will show that T maps S_M into itself. For $y\in S_M$ we get

$$\|Ty\|=\left\|\varsigma(.)+\int_{\omega_0}^{\infty}G(.,t)f(t,y(t))d_{\omega,q}t\right\|$$

$$\leq\|\varsigma\|+\left\|\int_{\omega_0}^{\infty}G(.,t)f(t,y(t))d_{\omega,q}t\right\|$$

$$\leq\|\varsigma\|+\left(\int_{\omega_0}^{\infty}\int_{\omega_0}^{\infty}|G(x,t)|^2 d_{\omega,q}x d_{\omega,q}t\right)^{1/2}$$

$$\times\sup_{y\in S_M}\left\{\int_{\omega_0}^{\infty}|f(t,y(t))|^2 d_{\omega,q}t\right\}^{1/2}\leq M.$$

Analysis similar to that in the proof of Theorem 3.3 shows that

$$\|Ty-Tz\|\leq\alpha\|y-z\|,\ y,z\in S_M.$$

From the Banach fixed point theorem, we get the desired result. □

Now, we obtain an existence theorem without the uniqueness of the solution.

Theorem 3.5. *T is the completely continuous operator under the conditions (A1), (A2), and (A3).*

Proof. Let $y_0 \in L^2_{\omega,q}(\omega_0,\infty)$. Then, we obtain

$$\left|(Ty)(x)-(Ty_0)(x)\right|^2$$

$$=\left|\int_{\omega_0}^{\infty} G(x,t)\left[f(t,y(t))-f(t,y_0(t))\right]d_{\omega,q}t\right|^2$$

$$\leq \int_{\omega_0}^{\infty} |G(x,t)|^2 d_{\omega,q}t \int_{\omega_0}^{\infty} |f(t,y(t))-f(t,y_0(t))|^2 d_{\omega,q}t.$$

Thus

$$\|Ty-Ty_0\|^2 \leq K\int_{\omega_0}^{\infty} |f(t,y(t))-f(t,y_0(t))|^2 d_{\omega,q}t, \tag{3.19}$$

where

$$K=\left(\int_{\omega_0}^{\infty}\int_{\omega_0}^{\infty} |G(x,t)|^2 d_{\omega,q}x d_{\omega,q}t\right).$$

It is evident that an operator T defined by $Ty(x)=f(x,y(x))$ is continuous in $L^2_{\omega,q}(\omega_0,\infty)$ under the condition (A2) (see [14]). Hence, for the given $\epsilon>0$, we can find a $\delta>0$ such that $\|y-y_0\|<\delta$ implies

$$\int_{\omega_0}^{\infty} |f(t,y(t))-f(t,y_0(t))|^2 d_{\omega,q}t<\frac{\epsilon^2}{K}.$$

From (3.19), we get

$$\|Ty-Ty_0\|<\epsilon.$$

Let

$$Y=\left\{y\in L^2_{\omega,q}(\omega_0,\infty):\|y\|\leq C\right\}.$$

By (3.13), we have

$$\|Ty\|\leq\|\varsigma\|+\left\{K\int_{\omega_0}^{\infty}|f(t,y(t))|^2 d_{\omega,q}t\right\}^{1/2},$$

for all $y\in Y$. Furthermore, using (2.5), we get

$$\int_{\omega_0}^{\infty}|f(t,y(t))|^2 d_{\omega,q}t\leq\int_{\omega_0}^{\infty}\left[g(t)+\vartheta|y(t)|\right]^2 d_{\omega,q}t$$

$$\leq 2\int_{\omega_0}^{\infty}\left[g^2(t)+\vartheta^2|y(t)|^2\right]d_{\omega,q}t=2\left(\|g\|^2+\vartheta^2\|y\|^2\right)$$

$$\leq 2\left(\|g\|^2+\vartheta^2C^2\right).$$

Therefore, for all $y\in Y$, we see that

$$\|Ty\|\leq\|\varsigma\|+\left[2K\left(\|g\|^2+\vartheta^2C^2\right)\right]^{1/2}.$$

Further, for all $y \in Y$, we have

$$\int_N^\infty |Ty(x)|^2 \, d_{\omega,q}x \le 2\left(\|g\|^2 + \vartheta^2 C^2\right) \int_N^\infty \int_{\omega_0}^\infty |G(x,t)|^2 \, d_{\omega,q}x d_{\omega,q}t.$$

So, from (3.12), we conclude that for given $\epsilon > 0$ there exists a positive number N, depending only on ϵ such that

$$\int_N^\infty |Ty(x)|^2 \, d_{\omega,q}x < \epsilon^2,$$

for all $y \in Y$.

Hence $T(Y)$ is relatively compact in the space $L^2_{\omega,q}(\omega_0, \infty)$. □

Theorem 3.6. *Suppose that the conditions (A1), (A2), and (A3) are satisfied. In addition, let there exist constants $M > 0$ such that*

$$\left\{\int_{\omega_0}^\infty |\varsigma(x)|^2 \, d_{\omega,q}x\right\}^{1/2} + \left(\int_{\omega_0}^\infty \int_{\omega_0}^\infty |G(x,t)|^2 \, d_{\omega,q}x d_{\omega,q}t\right)^{1/2}$$

$$\times \sup_{y \in S_M} \left\{\int_{\omega_0}^\infty |f(t, y(t))|^2 \, d_{\omega,q}t\right\}^{1/2} \le M, \tag{3.20}$$

where $S_M = \left\{y \in L^2_{\omega,q}(\omega_0,\infty) : \|y\| \le M\right\}$. Then the boundary-value problem (2.2), (2.8) has at least one solution with

$$\int_{\omega_0}^\infty |y(x)|^2 \, d_{\omega,q}x \le M^2.$$

Proof. Let $T : L^2_{\omega,q}(\omega_0,\infty) \to L^2_{\omega,q}(\omega_0,\infty)$ be the operator defined as (3.13). It follows from Theorems 3.4, 3.5, and (3.20) that T maps the set S_M into itself. Moreover, the set S_M is bounded, convex and closed. From the Schauder fixed point theorem, we get the desired result. □

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