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TWO TOPICS IN NUMBER THEORY. THE GREATEST k-FREE NUMBER THAT DIVIDES n AND FORMULAS FOR COMPOSITE AND PRIME NUMBERS

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Abstract. In the first part we study the greatest k-free number that divides n. Consequently, we generalize the kernel of the positive integer n. That is, the greatest square-free number that divides n. In the second part we obtain formulas for composite and prime numbers and prove some inequalities.

1. Introduction and Preliminary Notes

In the first part (sections 2 and 3) we study the greatest k-free number that divides n. Consequently, we generalize the kernel of the positive integer n. That is, the greatest square-free number that divides n. In the second part (section 4) we obtain formulas for composite and prime numbers and prove some inequalities. Let p_k be the k-th prime. We show that if we know either the prime factorization of $1, 2, \ldots, n-1$ or the primes p_k not exceeding $n-1$ then we can obtain the prime factorization of n. We also show that if we know the primes $p_1, p_2, \ldots, p_{n-1}$ then we can obtain the prime p_n. Finally The inequalities $p_{n+1} < p_n + n + 1$ $(n \geq 1)$, $p_n < \frac{n^2}{2} + \frac{n}{2} + 1$ $(n \geq 2)$ and $p_n < n^2$ $(n \geq 2)$ are proved. The composite numbers also can be studied in short intervals (see, for example, [6])

Let us consider the prime factorization of a positive integer $n = q_1^{s_1} \cdots q_r^{s_r}$ where the q_i $(i = 1, \ldots, r)$ $(r \geq 1)$ are the different primes in the prime factorization and the s_i $(i = 1, \ldots, r)$ are the multiplicities or exponents. Let $k \geq 2$ an arbitrary but fixed positive integer. A k-free number is a number such that $1 \leq s_i \leq k-1$, we consider 1 a k-free number. If $k = 2$ we obtain the square-free numbers, if $k = 3$ we obtain the cube-free numbers, etc. We shall denote a k-free number in the form q_{k-1}. Let $Q_k(x)$ be the number of k-free numbers not exceeding x. It is well-known the following asymptotic formula (see, for example, [4] for a simple proof)

$$Q_k(x) = \frac{1}{\zeta(k)} x + o(x). \tag{1.1}$$

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where $\zeta(s)$ denotes the Riemann zeta function.

We shall need the following well-known theorems.

Theorem 1.1. *(Inclusion-exclusion principle)Let S be a set of N distinct elements, and let $S_1, \ldots, S_r$ be arbitrary subsets of S containing $N_1, \ldots, N_r$ elements, respectively. For $1 \leq i < j < \ldots < l \leq r$, let $S_{ij\ldots l}$ be the intersection of $S_i, S_j, \ldots, S_l$ and let $N_{ij\ldots l}$ be the number of elements of $S_{ij\ldots l}$. Then the number K of elements of S not in any of $S_1, \ldots, S_r$ is*

$$K = N - \sum_{1\leq i\leq r} N_i + \sum_{1\leq i<j\leq r} N_{ij} - \sum_{1\leq i<j<k\leq r} N_{ijk} + \ldots + (-1)^r N_{12\ldots r}.$$

Proof. See, for example, [7, page 84] or [2, page 233]. □

Theorem 1.2. *(The second Möbius inversion formula) Let $f(x)$ and $g(x)$ be functions defined for $x \geq 1$. If*

$$g(x) = \sum_{n\leq x} f\left(\frac{x}{n}\right) \qquad (x \geq 1)$$

then

$$f(x) = \sum_{n\leq x} \mu(n) g\left(\frac{x}{n}\right) \qquad (x \geq 1)$$

where $\mu(n)$ is the Möbius function.

Proof. See, for example, [2, Chapter XVI, Theorem 268]. □

Theorem 1.3. *The following formula holds*

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^k} = \frac{1}{\zeta(k)}.$$

Proof. See, for example, [2, Chapter XVII, Theorem 287 and page 245]. □

Theorem 1.4. *Let us consider a strictly increasing sequence of positive integers, we denote b a positive integer in this sequence. Let $A(x)$ be the number of positive integers in this sequence not exceeding x. That is $A(x) = \sum_{b\leq x} 1$. Suppose that $A(x) = \rho x + o(x)$, where ρ is a positive real number, that is, ρ is the positive density of these integers. Then*

$$\sum_{b\leq x} b^k = \frac{\rho}{k+1} x^{k+1} + o\left(x^{k+1}\right),$$

where k is an arbitrary but fixed positive integer.

Proof. See [3]. □

2. k-Free Numbers multiple of a Set of Primes

Let $k \geq 2$ an arbitrary but fixed positive integer. Let $q_1, \ldots, q_s$ be $s \geq 1$ distinct primes. Let $M^k_{q_1\cdots q_s}(x)$ be the number of positive integers n not exceeding x such that in their prime factorization appear the primes $q_1, \ldots, q_s$ with multiplicity not multiple of k. We have the following theorem.

Theorem 2.1. *The following asymptotic formulas holds.*

$$M^k_{q_1\cdots q_s}(x) = \left(\prod_{i=1}^{s} \frac{q_i^{k-1}-1}{q_i^k-1}\right) x + o(x). \tag{2.1}$$

Proof. The number of positive integers n not exceeding x relatively prime with $q_1\cdots q_s$ will be (inclusion exclusion principle)

$$\lfloor x \rfloor - \sum_{1\le i\le s}\left\lfloor \frac{x}{q_i}\right\rfloor + \sum_{1\le i<j\le s}\left\lfloor \frac{x}{q_iq_j}\right\rfloor - \cdots = \prod_{i=1}^{s}\left(1-\frac{1}{q_i}\right)x + o(x). \tag{2.2}$$

Let us consider the numbers whose prime factorization is of the form $q_1^{r_1}\cdots q_s^{r_s}$ where r_i $(i=1,\ldots,s)$ is not multiple of k. We have

$$\begin{aligned}
&\prod_{i=1}^{s}\left(1-\frac{1}{q_i}\right)\sum_{q_1^{r_1}\cdots q_s^{r_s}}\frac{1}{q_1^{r_1}\cdots q_s^{r_s}}\\
&= \prod_{i=1}^{s}\left(1-\frac{1}{q_i}\right)\left(\sum_{i=1}^{\infty}\frac{1}{q_1^i}-\sum_{i=1}^{\infty}\frac{1}{q_1^{ki}}\right)\cdots\left(\sum_{i=1}^{\infty}\frac{1}{q_s^i}-\sum_{i=1}^{\infty}\frac{1}{q_s^{ki}}\right)\\
&= \prod_{i=1}^{s}\frac{q_i^{k-1}-1}{q_i^k-1}.
\end{aligned} \tag{2.3}$$

Let $\epsilon > 0$. We shall choose the number A such that

$$\sum_{q_1^{r_1}\cdots q_s^{r_s}>A}\frac{1}{q_1^{r_1}\cdots q_s^{r_s}} \le \epsilon. \tag{2.4}$$

Therefore we have (see (2.2) and (2.4))

$$\begin{aligned}
M^k_{q_1\cdots q_s}(x) &= \sum_{q_1^{r_1}\cdots q_s^{r_s}\le A}\left(\prod_{i=1}^{s}\left(1-\frac{1}{q_i}\right)\frac{x}{q_1^{r_1}\cdots q_s^{r_s}}+o(x)\right)+F(x)\\
&= x\prod_{i=1}^{s}\left(1-\frac{1}{q_i}\right)\sum_{q_1^{r_1}\cdots q_s^{r_s}\le A}\frac{1}{q_1^{r_1}\cdots q_s^{r_s}}+o(x)+F(x)\\
&= \prod_{i=1}^{s}\frac{q_i^{k-1}-1}{q_i^k-1}x - x\prod_{i=1}^{s}\left(1-\frac{1}{q_i}\right)\sum_{q_1^{r_1}\cdots q_s^{r_s}>A}\frac{1}{q_1^{r_1}\cdots q_s^{r_s}}+o(x)+F(x),
\end{aligned} \tag{2.5}$$

where (see (2.4))

$$0 \le F(x) \le \sum_{q_1^{r_1}\cdots q_s^{r_s}>A}\frac{x}{q_1^{r_1}\cdots q_s^{r_s}} \le \epsilon x. \tag{2.6}$$

Equations (2.5), (2.4) and (2.6) give

$$\left|\frac{M^k_{q_1\cdots q_s}(x)}{x} - \prod_{i=1}^{s}\frac{q_i^{k-1}-1}{q_i^k-1}\right| \le \prod_{i=1}^{s}\left(1-\frac{1}{q_i}\right)\epsilon+\epsilon+\epsilon \le 3\epsilon.$$

That is (2.1), since ϵ can be arbitrarily small. The theorem is proved. □

Let $Q^k_{q_1\cdots q_s}(x)$ be the number of k-free numbers not exceeding x multiple of $q_1\cdots q_s$. We have the following theorem.

Theorem 2.2. *The following asymptotic formula holds.*

$$Q^k_{q_1\cdots q_s}(x)=\frac{1}{\zeta(k)}\prod_{i=1}^{s}\frac{q_i^{k-1}-1}{q_i^k-1}x+o(x). \tag{2.7}$$

Proof. We have (see Theorem 2.1 and [2, Chapter XVIII, Theorem 333]).

$$M^k_{q_1\cdots q_s}(y^k)=cy^k+o(y^k)=cy^k+f(y^k)y^k=\sum_{d\le y}Q^k_{q_1\cdots q_s}\left(\left(\frac{y}{d}\right)^k\right).$$

where for sake of simplicity we put $c=\prod_{i=1}^s\frac{q_i^{k-1}-1}{q_i^k-1}$. Besides $\lim_{x\to\infty}f(x)=0$ and $|f(x)|<M$. By Theorem 1.2 and Theorem 1.3 we have

$$\begin{aligned}M^k_{q_1\cdots q_s}(y^k)&=\sum_{d\le y}\mu(d)\left(c\left(\frac{y}{d}\right)^k+f\left(\left(\frac{y}{d}\right)^k\right)\frac{y^k}{d^k}\right)=y^kc\sum_{d\le y}\frac{\mu(d)}{d^k}\\ &+\ y^k\sum_{d\le y}f\left(\left(\frac{y}{d}\right)^k\right)\frac{\mu(d)}{d^k}=\frac{1}{\zeta(k)}cy^k+O(y)+o(y^k)\\ &=\ \frac{1}{\zeta(k)}\prod_{i=1}^{s}\frac{q_i^{k-1}-1}{q_i^k-1}y^k+o(y^k).\end{aligned}$$

If we put $y^k=x$ then we obtain equation (2.7). Note that

$$y^kc\sum_{d>y}\frac{\mu(d)}{d^k}=O(y)$$

and

$$\begin{aligned}y^k\sum_{d\le y}f\left(\left(\frac{y}{d}\right)^k\right)\frac{\mu(d)}{d^k}&=y^k\sum_{d\le\sqrt[k]{y}}f\left(\left(\frac{y}{d}\right)^k\right)\frac{\mu(d)}{d^k}\\ &+\ y^k\sum_{\sqrt[k]{y}<d\le y}f\left(\left(\frac{y}{d}\right)^k\right)\frac{\mu(d)}{d^k}=o(y^k).\end{aligned}$$

Since for all $\epsilon>0$ we have

$$\begin{aligned}\left|\sum_{d\le y}f\left(\left(\frac{y}{d}\right)^k\right)\frac{\mu(d)}{d^k}\right|&\le\sum_{d\le\sqrt[k]{y}}\left|f\left(\left(\frac{y}{d}\right)^k\right)\right|\frac{1}{d^k}\\ &+\ \sum_{\sqrt[k]{y}<d\le y}\left|f\left(\left(\frac{y}{d}\right)^k\right)\right|\frac{1}{d^k}\le\epsilon\zeta(k)+Mo(1)\le 3\epsilon.\end{aligned}$$

The theorem is proved. □

Let $R^k_{q_1\cdots q_s}(x)$ be the number of k-free numbers not exceeding x, relatively prime to $q_1\cdots q_s$. The following theorem holds.

Theorem 2.3. *The following asymptotic formula holds.*

$$R^k_{q_1\cdots q_s}(x) = \frac{1}{\zeta(k)} \prod_{i=1}^{s} \frac{q_i^{k-1}(q_i-1)}{q_i^k - 1} x + o(x). \tag{2.8}$$

Proof. By the inclusion-exclusion principle, Theorem 2.2 and (1.1) we have

$$\begin{aligned} R^k_{q_1\cdots q_s}(x) &= \frac{1}{\zeta(k)}x + o(x) - \sum_{1\le i\le s}\left(\frac{1}{\zeta(k)}\frac{q_i^{k-1}-1}{q_i^k-1}x + o(x)\right) \\ &+ \sum_{1\le i<j\le s}\left(\frac{1}{\zeta(k)}\frac{q_i^{k-1}-1}{q_i^k-1}\frac{q_j^{k-1}-1}{q_j^k-1}x + o(x)\right) - \cdots \\ &= \frac{1}{\zeta(k)}x\prod_{i=1}^{s}\left(1 - \frac{q_i^{k-1}-1}{q_i^k-1}\right) + o(x). \end{aligned}$$

That is, equation (2.8). The theorem is proved. □

3. The Greatest k-Free Number that Divides n

Let $k \ge 2$ a positive integer. A number is k-full if all the distinct primes in its prime factorization have multiplicity (or exponent) greater than or equal to k. That is, the number $q_1^{r_1}\cdots q_s^{r_s}$ $(s \ge 1)$ is k-full if $r_i \ge k$ $(i = 1, \ldots, s)$. If $k = 2$ then the number is called square-full, etc.

The greatest k-free number that divides $n = q_1^{r_1}\cdots q_s^{r_s}$ $(s \ge 1)$ will be denoted $u_k(n)$. Note that $u_k(n) = q_1^{t_1}\cdots q_s^{t_s}$, where $t_i = min\{r_i, k-1\}$ $(i = 1, \ldots, s)$. If $k = 2$ then the greatest square-free that divides n $(u_2(n))$ is called kernel or radical of n. In [3] the following theorem on the kernel function is proved (in [3] the notation $u(n) = u_2(n)$ is used).

Theorem 3.1. *Let h be an arbitrary but fixed positive integer. The following asymptotic formula holds*

$$\sum_{n\le x} u_2(n)^h = \frac{C_{2,h}}{h+1}x^{h+1} + o\left(x^{h+1}\right), \tag{3.1}$$

where

$$C_{2,h} = \prod_p \left(1 - \frac{p^h - 1}{p\left(p^{h+1}-1\right)}\right). \tag{3.2}$$

In the following theorem we generalize the former theorem to the greatest k-free number that divides n.

Theorem 3.2. *Let $k \ge 2$ an arbitrary but fixed positive integer and let h be an arbitrary but fixed positive integer. The following asymptotic formula holds*

$$\sum_{n\le x} u_k(n)^h = \frac{C_{k,h}}{h+1}x^{h+1} + o\left(x^{h+1}\right), \tag{3.3}$$

where

$$C_{k,h}=\prod_{p}\left(1-\frac{p^h-1}{p^{k-1}\left(p^{h+1}-1\right)}\right). \tag{3.4}$$

Proof. In this proof we shall denote $q_1^{r_1}\cdots q_s^{r_s}$ the prime factorization of a k-full number. The q_i $(i=1,\ldots,s)$ are the different primes in the prime factorization and the r_i $(i=1,\ldots,s)$ are the multiplicities or exponents.

Let us consider the series

$$\sum_{q_1^{r_1}\cdots q_s^{r_s}}\frac{\left(q_1^{k-1}\cdots q_s^{k-1}\right)^h}{\left(q_1^{r_1}\cdots q_s^{r_s}\right)^{h+1}}, \tag{3.5}$$

where the sum run on all k-full numbers $q_1^{r_1}\cdots q_s^{r_s}$. The series (3.5) converges since we have

$$\sum_{q_1^{r_1}\cdots q_s^{r_s}}\frac{\left(q_1^{k-1}\cdots q_s^{k-1}\right)^h}{\left(q_1^{r_1}\cdots q_s^{r_s}\right)^{h+1}}=\sum_{q_1^{r_1}\cdots q_s^{r_s}}\frac{1}{q_1^{k-1}\cdots q_s^{k-1}}\frac{1}{\left(q_1^{r_1-(k-1)}\cdots q_s^{r_s-(k-1)}\right)^{h+1}},$$

where the numbers $q_1^{r_1-(k-1)}\cdots q_s^{r_s-(k-1)}$ run on all positive integers n greater than or equal to 2 and the series $\sum_{n=2}^{\infty}\frac{1}{n^{h+1}}$ converges. In this proof (see the introduction) q_{k-1} denotes a k-free number.

We have (see equation (1.1) and Theorem 1.4)

$$\sum_{q_{k-1}\leq x}\left(q_{k-1}\right)^h=\frac{1}{\zeta(k)}\frac{x^{h+1}}{h+1}+o\left(x^{h+1}\right). \tag{3.6}$$

Let us consider the number

$$q_{k-1}.q_1^{r_1}\cdots q_s^{r_s},$$

where q_{k-1} and $q_1\cdots q_s$ are relatively primes. The greatest k-free number that divides this number is

$$q_{k-1}.q_1^{k-1}\cdots q_s^{k-1}.$$

Therefore if the k-full number $q_1^{r_1}\cdots q_s^{r_s}$ is fixed and the k-free number q_{k-1} is variable we have (see Theorem 2.3 and Theorem 1.4)

$$\begin{aligned}\sum_{q_{k-1}q_1^{r_1}\cdots q_s^{r_s}\leq x}\left(q_{k-1}q_1^{k-1}\cdots q_s^{k-1}\right)^h &= \left(q_1^{k-1}\cdots q_s^{k-1}\right)^h\sum_{q_{k-1}\leq\frac{x}{q_1^{r_1}\cdots q_s^{r_s}}}\left(q_{k-1}\right)^h\\ &= \left(q_1^{k-1}\cdots q_s^{k-1}\right)^h\frac{1}{(h+1)\zeta(k)}\prod_{i=1}^{s}\frac{q_i^{k-1}(q_i-1)}{q_i^k-1}\frac{x^{h+1}}{\left(q_1^{r_1}\cdots q_s^{r_s}\right)^{h+1}}\\ &+ o\left(x^{h+1}\right).\end{aligned} \tag{3.7}$$

We denote $B_{k,h}$ the sum of the series

$$B_{k,h}=1+\sum_{q_1^{r_1}\cdots q_s^{r_s}}\prod_{i=1}^{s}\frac{q_i^{k-1}(q_i-1)}{q_i^k-1}\frac{\left(q_1^{k-1}\cdots q_s^{k-1}\right)^h}{\left(q_1^{r_1}\cdots q_s^{r_s}\right)^{h+1}}. \tag{3.8}$$

Note that this series converges since the series (3.5) converges (Comparison Criterion).

Let $\epsilon > 0$. Since the series (3.5) converges we shall choose a k-full number A such that

$$\sum_{q_1^{r_1}\cdots q_s^{r_s}>A} \frac{\left(q_1^{k-1}\cdots q_s^{k-1}\right)^h}{\left(q_1^{r_1}\cdots q_s^{r_s}\right)^{h+1}} < \epsilon. \tag{3.9}$$

Now, we have (see (3.6), (3.7), (3.8) and (3.9))

$$\begin{aligned}\sum_{n\leq x} u_k(n)^h &= \frac{x^{h+1}}{(h+1)\zeta(k)}B_{k,h} + o\left(x^{h+1}\right)\\ &- \frac{x^{h+1}}{(h+1)\zeta(k)} \sum_{q_1^{r_1}\cdots q_s^{r_s}>A} \prod_{i=1}^{s} \frac{q_i^{k-1}(q_i-1)}{q_i^k-1} \frac{\left(q_1^{k-1}\cdots q_s^{k-1}\right)^h}{\left(q_1^{r_1}\cdots q_s^{r_s}\right)^{h+1}} + F(x). \end{aligned}\tag{3.10}$$

Note that

$$1^h + 2^h + \cdots + n^h \leq \int_0^n x^h\, dx + n^h \leq 2n^{h+1} \qquad (n\geq 1) \qquad (h\geq 1) \tag{3.11}$$

Hence if A_x is the greatest k-full number not exceeding x then we have (see (3.11) and (3.9))

$$\begin{aligned} 0 \leq F(x) &\leq \sum_{A<q_1^{r_1}\cdots q_s^{r_s}\leq A_x} \sum_{q_{k-1}q_1^{r_1}\cdots q_s^{r_s}\leq x} \left(q_{k-1}q_1^{k-1}\cdots q_s^{k-1}\right)^h\\ &\leq \sum_{A<q_1^{r_1}\cdots q_s^{r_s}\leq A_x} \sum_{nq_1^{r_1}\cdots q_s^{r_s}\leq x} \left(nq_1^{k-1}\cdots q_s^{k-1}\right)^h\\ &= \sum_{A<q_1^{r_1}\cdots q_s^{r_s}\leq A_x} \left(q_1^{k-1}\cdots q_s^{k-1}\right)^h \sum_{1\leq n\leq \frac{x}{q_1^{r_1}\cdots q_s^{r_s}}} n^h\\ &\leq 2x^{h+1} \sum_{A<q_1^{r_1}\cdots q_s^{r_s}} \frac{\left(q_1^{k-1}\cdots q_s^{k-1}\right)^h}{\left(q_1^{r_1}\cdots q_s^{r_s}\right)^{h+1}} \leq 2\epsilon x^{h+1}. \end{aligned}\tag{3.12}$$

Equations (3.10), (3.9) and (3.12) give

$$\left| \frac{\sum_{n\leq x} u_k(n)^h}{\frac{x^{h+1}}{(h+1)\zeta(k)}} - B_{k,h}\right| \leq \epsilon + \epsilon + 2(h+1)\zeta(k)\epsilon \leq \epsilon' \qquad (x\geq x_{\epsilon'}), \tag{3.13}$$

where ϵ' can be arbitrarily small since ϵ can be arbitrarily small. Therefore equation (3.13) gives

$$\sum_{n\leq x} u_k(n)^h = \frac{1}{\zeta(k)} \frac{B_{k,h}}{h+1} x^{h+1} + o\left(x^{h+1}\right), \tag{3.14}$$

where (see (3.8))

$$\begin{aligned} C_{k,h} &= \frac{1}{\zeta(k)} B_{k,h} \\ &= \frac{1}{\zeta(k)} \left(1 + \sum_{q_1^{r_1} \cdots q_s^{r_s}} \prod_{i=1}^{s} \frac{(q_i - 1)}{q_i^k - 1} \frac{1}{\left(q_1^{r_1-(k-1)} \cdots q_s^{r_s-(k-1)} \right)^{h+1}} \right) \\ &= \prod_p \left(1 - \frac{1}{p^k} \right) \prod_p \left(1 + \frac{p-1}{p^k - 1} \left(\frac{1}{p^{h+1}} + \frac{1}{(p^2)^{h+1}} + \cdots \right) \right) \\ &= \prod_p \left(1 - \frac{p^h - 1}{p^{k-1} \left(p^{h+1} - 1 \right)} \right). \end{aligned} \tag{3.15}$$

Thus, equations (3.14) and (3.15) are equations (3.3) and (3.4). The theorem is proved. □

4. Formulas for Composite Numbers and Prime Numbers

We denote p_n the n-th prime number, $\gcd(a_1, a_2)$ the greatest common divisor of a_1 and a_2 and $lcm(a_1, a_2, \ldots, a_s)$ the least common multiple of $a_1, a_2, \ldots, a_s$. In this section we show that if we know the prime factorization of $1, 2, \ldots, n-1$ then we can obtain the prime factorization of n. In this way, by mathematical induction, if we begin with 1 and $p_1 = 2$ then we can build the sequence of all prime factorizations of the positive integers. We need only three simple rules.

Rule 1) If $lcm\left(\gcd(1, n-1), \gcd(2, n-2), \ldots, \gcd(n-1, 1)\right) = 1$, that is

$$\gcd(k, n-k) = 1 \qquad (k = 1, 2, \ldots, n-1),$$

then n is prime.

Rule 2) If $lcm\left(\gcd(1, n-1), \gcd(2, n-2), \ldots, \gcd(n-1, 1)\right) = p^k$, where p denotes a positive prime and k denotes a positive integer, then

$$n = p\ lcm\left(\gcd(1, n-1), \gcd(2, n-2), \ldots, \gcd(n-1, 1)\right) = p^{k+1}.$$

Rule 3) If $lcm\left(\gcd(1, n-1), \gcd(2, n-2), \ldots, \gcd(n-1, 1)\right) \neq 1$ and

$$lcm\left(\gcd(1, n-1), \gcd(2, n-2), \ldots, \gcd(n-1, 1)\right) \neq p^k$$

, then

$$n = lcm\left(\gcd(1, n-1), \gcd(2, n-2), \ldots, \gcd(n-1, 1)\right).$$

The proof of these rules is very simple since $k + (n-k) = n$ $(k = 1, 2, \ldots, n-1)$.

If we begin with $1, p_1$ then we obtain $lcm(\gcd(1, p_1)) = lcm(1) = 1$ and by rule 1 $n = p_2$ is prime. Therefore we have $1, p_1, p_2$ and consequently

$lcm(\gcd(1, p_2), \gcd(p_1, p_1)) = lcm(1, p_1) = p_1$ and by rule 2 $n = p_1^2$. Therefore we have $1, p_1, p_2, p_1^2$ and consequently $lcm(\gcd(1, p_1^2), \gcd(p_1, p_2)) = lcm(1, 1) = 1$ and by rule 1 $n = p_3$ is prime. Therefore we have $1, p_1, p_2, p_1^2, p_3$ and consequently

$lcm(\gcd(1, p_3), \gcd(p_1, p_1^2), \gcd(p_2, p_2)) = lcm(1, p_1, p_2) = p_1 p_2$ and by rule 3 $n = p_1 p_2$. Therefore we have $1, p_1, p_2, p_1^2, p_3, p_1 p_2$, etc.

We can traduce these rules in theoretical formulas. Let us consider the primes p such that $2 \leq p \leq n-1$. Now, consider the formula

$$f(p) = \sum_{k=1}^{n-1} \left\lfloor \left| cos\left(\pi\left(\frac{k}{p}\right)\right)\right| \left| cos\left(\pi\left(\frac{n-k}{p}\right)\right)\right| \right\rfloor,$$

where $\lfloor\rfloor$ is the integer part function and $||$ is the absolute value function. Let A be the set of primes p such that $2 \leq p \leq n-1$ and $f(p) \neq 0$. If A is empty then n is prime. If A is an unitary set, that is, $A = \{p\}$, then

$$n = p^{\left(1+\sum_{i=1}^{\left\lfloor \frac{\log(n-1)}{\log p}\right\rfloor} \frac{1}{\left\lfloor \frac{n-1}{p^i}\right\rfloor} \sum_{k=1}^{n-1} \left\lfloor \left| cos\left(\pi\left(\frac{k}{p^i}\right)\right)\right| \left| cos\left(\pi\left(\frac{n-k}{p^i}\right)\right)\right| \right\rfloor\right)}.$$

If A has at least two distinct primes, then

$$n = \prod_{p\in A} p^{\left(\sum_{i=1}^{\left\lfloor \frac{\log(n-1)}{\log p}\right\rfloor} \frac{1}{\left\lfloor \frac{n-1}{p^i}\right\rfloor} \sum_{k=1}^{n-1} \left\lfloor \left| cos\left(\pi\left(\frac{k}{p^i}\right)\right)\right| \left| cos\left(\pi\left(\frac{n-k}{p^i}\right)\right)\right| \right\rfloor\right)}.$$

Note that the inequality $p^i \leq n-1$ hols for $1 \leq i \leq \frac{\log(n-1)}{\log p}$.

Another approach is by use of the well-known Legendre's rule (see [2]). Namely, if the prime $p \leq n$ then the exponent of p in the prime factorization of $n!$ is

$$\sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor,$$

where clearly the sum has only a finite number of nonzero terms. Consequently we have the following criterion. Let us consider the primes $p \leq n-1$ and consider the function

$$f(n) = \prod_{p\leq n-1} \left(1 + \sum_{k=1}^{\infty} \left(\left\lfloor \frac{n}{p^k}\right\rfloor - \left\lfloor \frac{n-1}{p^k}\right\rfloor\right)\right).$$

Clearly if $f(n) = 1$ then n is prime. On the other hand, if $f(n) > 1$ then n is composite and its prime factorization is

$$n = \prod_{p\leq n-1} p^{\sum_{k=1}^{\infty}\left(\left\lfloor \frac{n}{p^k}\right\rfloor - \left\lfloor \frac{n-1}{p^k}\right\rfloor\right)}.$$

Also, suppose that we know the first $n-1$ primes. Namely, $p_1, p_2, \ldots, p_{n-1}$. Then we can determine the prime p_n by use of the following criterion. The prime p_n is the first positive integer $s \geq p_{n-1}+1$ such that

$$\prod_{p\leq p_{n-1}} \left(1 + \sum_{k=1}^{\infty} \left(\left\lfloor \frac{s}{p^k}\right\rfloor - \left\lfloor \frac{s-1}{p^k}\right\rfloor\right)\right) = 1.$$

In this form we can determine p_n if we know $p_1, p_2, \ldots, p_{n-1}$. A formula of this type was first proved by Gandhi (see [8]) and another formula of this type is given in [5].

Let p_n be the n-th prime. Bertrand conjectured in 1845 the inequality $p_{n+1} < 2p_n$ for $n \geq 1$. This statement has been known as Bertrand's postulate and it was proved by Tschebycheff in 1852. In this article we prove the stronger statement $p_{n+1} < p_n + n + 1$ also for $n \geq 1$. For this purpose we need strong inequalities proved by Rosser and Schoenfeld in 1962 and by Dusart in 1999. That is, we need the following fundamental lemma.

Lemma 4.1. *We have the following inequalities.*

$$p_n < n \log n + n \log \log n - \frac{1}{2} n \qquad (n \geq 20) \tag{4.1}$$

$$p_n > n \log n + n \log \log n - n \qquad (n \geq 2) \tag{4.2}$$

Proof. Inequality (4.1) is proved in [9]. Inequality (4.2) is proved in [1]. The lemma is proved. □

Theorem 4.2. *The following inequality holds*

$$p_{n+1} < p_n + n + 1 \qquad (n \geq 1) \tag{4.3}$$

Proof. By use of a small table of primes p_n $(n \leq 20)$ we can prove that inequality (4.3) holds for $n \leq 19$. Therefore we have to prove that inequality (4.3) holds for $n \geq 20$.

Let us consider the function

$$f(x) = 2\log(x+1) + 2\log\log(x+1) + \frac{2}{\log x} - x - 1 \qquad (x \geq 20)$$

We have

$$f'(x) = \frac{2}{x+1} + \frac{2}{(x+1)\log(x+1)} - \frac{2}{x \log^2 x} - 1 < \frac{2}{x+1} + \frac{2}{x+1} - 1 < 0$$

for $x \geq 20$. On the other hand, $f(20) < 0$. Therefore $f(n) < 0$ for $n \geq 20$. The inequality $f(n) < 0$ can be written in the equivalent form

$$2\left(\log(n+1) + 1\right) + 2\left(\log\log(n+1) + \frac{1}{\log n}\right) < n + 3 \qquad (n \geq 20) \tag{4.4}$$

On the other hand, $D(x \log x) = \log x + 1$ and $D(x \log\log x) = \log\log x + \frac{1}{\log x}$. Therefore the mean value theorem and inequality (4.4) give

$$\begin{aligned} & 2\left((n+1)\log(n+1) - n\log n\right) + 2\left((n+1)\log\log(n+1) - n\log\log n\right) \\ < \ & 2\left(\log(n+1) + 1\right) + 2\left(\log\log(n+1) + \frac{1}{\log n}\right) < n + 3 \qquad (n \geq 20) \end{aligned} \tag{4.5}$$

Inequality (4.5) can be written in the form

$$\begin{aligned} & (n+1)\log(n+1) + (n+1)\log\log(n+1) - \frac{1}{2}(n+1) \\ < \ & n\log n + n\log\log n - n + n + 1 \qquad (n \geq 20) \end{aligned} \tag{4.6}$$

Finally, Lemma 4.1 and inequality (4.6) give

$$\begin{aligned} p_{n+1} &< (n+1)\log(n+1)+(n+1)\log\log(n+1)-\frac{1}{2}(n+1) \\ &< n\log n+n\log\log n-n+n+1<p_n+n+1 \qquad (n\geq 20) \end{aligned}$$

The theorem is proved. □

Corollary 4.3. *Let $d_n = p_{n+1} - p_n$ be. Let k be an arbitrary but fixed positive integer. The following inequality holds for $n \geq 2$.*

$$\sum_{i=1}^{n-1} d_i^k < \frac{(n+1)^{k+1}}{k+1}$$

Proof. Since x^k is a function strictly increasing in the interval $[0,\infty)$ we have $i^k < \int_i^{i+1} x^k\,dx$ and consequently if $n \geq 2$ then $\sum_{i=1}^n i^k < \int_0^{n+1} x^k\,dx$. Therefore if $n \geq 2$ then inequality (4.3) gives

$$\sum_{i=1}^{n-1} d_i^k < \sum_{i=1}^{n-1}(i+1)^k = \left(\sum_{i=1}^{n} i^k\right) - 1 < \sum_{i=1}^{n} i^k < \int_0^{n+1} x^k\,dx = \frac{(n+1)^{k+1}}{k+1}.$$

The corollary is proved. □

The prime number theorem in the form $p_n \sim n\log n$ implies that from a certain value of n the inequality $p_n < n^2$ holds. Now, we give a very simple proof of the following more precise results.

Corollary 4.4. *If $n \geq 2$ then $p_n < \frac{n^2}{2} + \frac{n}{2} + 1$ and $p_n < n^2$.*

Proof. Let us consider the following $n-1$ inequalities (4.3) $p_{i+1} < p_i + i + 1$ $(1 \leq i \leq n-1)$ $(n \geq 2)$. If we add these inequalities we obtain the inequality $p_n < \frac{n^2}{2} + \frac{n}{2} + 1$. We have $p_2 = 3 < 2^2$. Suppose that if $n \geq 2$ then $p_n < n^2$. Inequality (4.3) gives $p_{n+1} < p_n + n + 1 < n^2 + n + 1 < (n+1)^2$. Now, inequality $p_n < n^2$ follows by mathematical induction. The corollary is proved. □

Corollary 4.5. *If $n \geq 2$ then $\prod_{i=1}^{n-1} d_i < n!$.*

Proof. By inequality (4.3) we have $\prod_{i=1}^{n-1} d_i = \prod_{i=1}^{n-1}(p_{i+1}-p_i) < \prod_{i=1}^{n-1}(i+1) = n!$. The corollary is proved. □

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