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ON α-SEMIDERIVATIONS AND COMMUTATIVITY OF Γ-RINGS

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ABSTRACT. In this paper, we introduce the notion of a α-semiderivation on Γ-rings, and we try to extend some results for derivations of Γ-rings or near-rings to a more general case for α-semiderivations of prime Γ-rings.

1. Introduction

In 1994, Nobusawa [9] introduced the notion of a Γ-ring, an object more general than a ring. Barnes [3] slightly weakened the conditions in the definition of a Γ-rings and in the sense of Nobusawa. Since then, many researchers have done a lot of results on Γ-rings and have obtained some generalizations of the corresponding results in ring theory. Aydin [2] studied generalized derivations of prime rings. Ceven and Ozturk [8]have dealt with Jordan generalized derivations in Γ-rings and they proved that every Jordan generalized derivation on some Γ-rings is a generalized derivation. In this paper, we introduce the notion of a α-semiderivation on Γ-ring, and we try to extend some results for derivations of Γ-rings or near-rings to a more general case for α-semiderivations of prime Γ-rings.

2. Preliminaries

. Let M and Γ be additive abelian groups. Then M is called a Γ-*ring* if the following conditions are satisfied:

(M1) $x\gamma y \in M$,
(M2) $(x+y)\gamma z = x\gamma z + y\gamma z, x(\gamma+\beta)y = x\gamma y + x\beta y, x\gamma(y+z) = x\gamma y + x\gamma z$,
(M3) $(x\gamma y)\beta z = x\gamma(y\beta z)$, for all $x, y, z \in M$ and $\gamma, \beta \in \Gamma$.

Let M be a Γ-ring with center $Z(M)$. For any $x, y \in M$, the notation $[x, y]_\gamma$ denotes the commutator $x\gamma y - y\gamma x$ while the symbol $(x \circ y)_\gamma$ denotes $x\gamma y + y\gamma x$, respectively. We know that $[x\beta y, z]_\gamma = x\beta[y, z]_\gamma + [x, z]_\gamma\beta y + x[\beta, \gamma]_z y$ and $[x, y\beta z]_\gamma = y\beta[x, z]_\gamma + [x, y]_\gamma\beta z + y[\beta, \gamma]_x z$, for all $x, y, z \in M$ and $\gamma, \beta \in \Gamma$. We take an assumption (*) $x\gamma y\beta z = x\beta y\gamma z$ for all $x, y, z \in M$ and $\gamma, \beta \in \Gamma$. Using the assumption (*), identities $[x\beta y, z]_\gamma = x\beta[y, z]_\gamma + [x, z]_\gamma\beta y$ and $[x, y\beta z]_\gamma = y\beta[x, z]_\gamma + [x, y]_\gamma\beta z$ for all $x, y, z \in M$ and $\gamma, \beta \in \Gamma$ are used extensively in our

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results. Let M be a Γ-ring. M is said to be *prime* if $x\Gamma M\Gamma y = 0$ implies $x = 0$, or $y = 0$, for all $x, y \in M$. M is said to be 2-*torsion free* if $2x = 0$ implies $x = 0$, for all $x \in M$. Let M be a Γ-ring. An additive mapping $d : M \to M$ is called a *derivation* if

$$d(x\gamma y) = d(x)\gamma y + x\gamma d(y)$$

for all $x, y \in M$ and $\gamma \in \Gamma$.

3. α-SEMIDERIVATION OF Γ-RINGS

Throughout this paper, we assume that M is a Γ-ring satisfying an assumption $(*)$.

Definition 3.1. Let M be a Γ-ring and let α be an automorphism of M. An additive mapping $d : M \to M$ is called a α-*semiderivation* associated with a surjective function $g : M \to M$ if
(a) $d(x\gamma y) = d(x)\gamma g(y) + \alpha(x)\gamma d(y) = d(x)\gamma\alpha(y) + g(x)\gamma d(y)$,
(b) $d(g(x)) = g(d(x))$, for all $x, y \in M$ and $\gamma \in \Gamma$.

Lemma 3.2. *Let d be a α-semiderivation on Γ-ring M and $a \in M$ be such that $a\gamma d(x) = 0$ (or $d(x)\gamma a = 0$) for all $x \in M$ and $\gamma \in \Gamma$. Then $a = 0$ or $d = 0$.*

Proof. By hypothesis, we have $a\gamma d(x\beta y) = 0$ for all $x, y \in M$ and $\gamma, \beta \in \Gamma$. Then we have

$$a\gamma(d(x)\beta g(y) + \alpha(x)\beta d(y)) = 0.$$

By the hypothesis, we have $a\gamma d(x)\beta g(y) + a\gamma\alpha(x)\beta d(y) = a\gamma\alpha(x)\beta d(y) = 0$, which implies $a\Gamma M\Gamma d(y) = \{0\}$ Since M is prime, we have $a = 0$ or $d = 0$. Similarly, if $d(x)\gamma a = 0$ for all $x \in M$ and $\gamma \in \Gamma$, then we can get $a = 0$ or $d = 0$. □

Lemma 3.3. *Let M be a 2-torsion free prime Γ-ring and d be a α-semiderivation on M such that $d \circ \alpha = \alpha \circ d$. If $d^2 = 0$, then $d = 0$.*

Proof. By hypothesis, we have, for all $x, y \in M$ and $\beta, \gamma \in \Gamma$,

$$\begin{aligned}
0 &= d^2(x\gamma y) = d(d(x\gamma y)) = d(d(x)\gamma g(y) + \alpha(x)\gamma d(y)) \\
&= d^2(x)\gamma g^2(y) + \alpha(d(x))\gamma d(g(y)) + d(\alpha(x))\gamma g(d(y)) + \alpha^2(x)\gamma d^2(y) \\
&= \alpha(d(x))\gamma d(g(y)) + d(\alpha(x))\gamma g(d(y)) \\
&= d(\alpha(x))\gamma d(g(y)) + d(\alpha(x))\gamma d(g(y)) \\
&= 2d(\alpha(x))\gamma d(g(y))
\end{aligned}$$

By the fact that M is 2-torsion free, we get $d(\alpha(x))\gamma d(g(y)) = 0$ for all $x, y \in M$ and $\gamma \in \Gamma$. Again, replacing x by $\alpha^{-1}(x)$ in the last relation, we have $d(x)\gamma d(g(y)) = 0$ for all $x, y \in R$. Since g is onto, we obtain $d(x)\gamma d(y) = 0$ for all $x, y \in M$. Hence by Lemma 3.2, we have $d = 0$. □

Lemma 3.4. *Let d be a α_1-semiderivation associated with g on M and let α_2 be an automorphism of M, which commute with d and $g \circ \alpha_2 = \alpha_2 \circ g$. Then $(\alpha_1 \circ \alpha_2)(x)\gamma(d \circ \alpha_2)(y) = (\alpha_2 \circ \alpha_1)(x)\gamma(\alpha_2 \circ d)(y)$ for all $x, y \in M$ and $\gamma \in \Gamma$.*

Proof. Let $x, y \in M$ and $\gamma \in \Gamma$. Then we have

$$\begin{aligned}\alpha_2 \circ d(x\gamma y) &= \alpha_2(d(x)\gamma g(y) + \alpha_1(x)\gamma d(y)) \\ &= \alpha_2(d(x)\gamma g(y)) + \alpha_2(\alpha_1(x)\gamma d(y)) \\ &= \alpha_2(d(x))\gamma\alpha_2(g(y)) + \alpha_2(\alpha_1(x))\gamma\alpha_2(d(y))\end{aligned}$$

Also, we have

$$\begin{aligned}d \circ \alpha_2(x\gamma y) &= d(\alpha_2(x)\gamma\alpha_2(y)) \\ &= d(\alpha_2(x))\gamma g(\alpha_2(y)) + \alpha_1(\alpha_2(x))\gamma d(\alpha_2(y)) \\ &= d(\alpha_2(x))\gamma\alpha_2(g(y)) + \alpha_1(\alpha_2(x))\gamma d(\alpha_2(y))\end{aligned}$$

Since d commutes α_2, we have $(\alpha_2 \circ \alpha_1)(x)\gamma(\alpha_2 \circ d)(y) = (\alpha_1 \circ \alpha_2)(x)\gamma(d \circ \alpha_2)(y)$. □

Theorem 3.5. *Let d be a nonzero α-semiderivation associated with g on a prime Γ-ring M. If $d([x, y]_\gamma) = 0$ for all $x, y \in M$ and $\gamma \in \Gamma$, then M is commutative.*

Proof. By hypothesis, we have

$$d([x, y]_\gamma) = 0, \ \forall\ x, y \in M, \gamma \in \Gamma. \tag{3.1}$$

Replacing y by $y\beta x$ in (1), we have $d([x, y\beta x]_\gamma) = d([x, y]_\gamma\beta x) = 0$ for all $x, y \in M$ and $\gamma, \beta \in \Gamma$, which implies that $d([x, y]_\gamma)\beta g(x) + \alpha([x, y]_\gamma)\beta d(x) = 0$ for all $x, y \in M$ and $\gamma, \beta \in \Gamma$. By (1), we have $\alpha([x, y]_\gamma)\beta d(x) = 0$ for all $x, y \in M$ and $\gamma, \beta \in \Gamma$. Substituting $y\delta x$ for y in the last relation, we have $\alpha([x, y\delta x]_\gamma)\beta d(x) = \alpha([x, y]_\gamma)\delta\alpha(x)\beta d(x) = 0$ or all $x, y \in M$ and $\gamma, \beta, \delta \in \Gamma$. Since α is an automorphism of M, we have $\alpha([x, y]_\gamma)\Gamma M\Gamma d(x) = 0$ or all $x, y \in M$ and $\gamma \in \Gamma$. Since M is prime, we have $d(x) = 0$ for all $x \in M$ or $\alpha([x, z]_\gamma) = 0$ for all $x, z \in M$ and $\gamma \in \Gamma$. But $d \neq 0$, and so $\alpha([x, z]_\gamma) = 0$ for all $x, z \in M$ and $\gamma \in \Gamma$. Replacing $[x, z]_\gamma$ by $\alpha^{-1}([x, z]_\gamma)$ in the last relation, we get $[x, z]_\gamma = 0$ for all $x, z \in M$ and $\gamma \in \Gamma$, which implies that M is commutative. □

Theorem 3.6. *Let d be a nonzero α-semiderivation associated with g on a prime Γ-ring M. If $d(x \circ y)_\gamma) = 0$ for all $x, y \in M$ and $\gamma \in \Gamma$, then M is commutative.*

Proof. By hypothesis, we have

$$d(x \circ y)_\gamma) = 0, \ \forall\ x, y \in M, \gamma \in \Gamma. \tag{3.2}$$

Replacing y by $y\beta x$ in (2), we have $d(x \circ y\beta x)_\gamma = d((x \circ y)_\gamma\beta x) = 0$ for all $x, y \in M$ and $\gamma, \beta \in \Gamma$, which implies that $d(x \circ y)_\gamma\beta g(x) + \alpha(x \circ y)_\gamma\beta d(x) = 0$ for all $x, y \in M$ and $\gamma, \beta \in \Gamma$. By (2), we have $\alpha(x \circ y)_\gamma\beta d(x) = 0$ for all $x, y \in M$ and $\gamma, \beta \in \Gamma$. Taking $y\delta x$ instead of y in the last relation, we have $\alpha(x \circ y)_\gamma\delta\alpha(x)\beta d(x) = 0$ for all $x, y \in M$ and $\gamma, \beta, \delta \in \Gamma$. Since α is an automorphism of M, we have $\alpha(x \circ y)_\gamma\Gamma M\Gamma d(x) = 0$ for all $x, y \in M$ and $\gamma \in \Gamma$. Since M is prime, we have $d(x) = 0$ for all $x \in M$ or $\alpha(x \circ y)_\gamma = 0$ for all $x, y \in M$ and $\gamma \in \Gamma$. But $d \neq 0$, and so $\alpha(x \circ y)_\gamma = 0$ for all $x, y \in M$ and $\gamma \in \Gamma$. Replacing $(x \circ y)_\gamma$ by $\alpha^{-1}((x \circ y)_\gamma)$ in the last relation, we have $(x \circ y)_\gamma = 0$ for all $x, y \in M$ and $\gamma \in \Gamma$, which implies that $x\gamma y = -y\gamma x$ for all $x, y \in M$ and $\gamma \in \Gamma$. Again, replacing x by $x\beta z$ in the last relation, we have $x\beta z\gamma y = -y\gamma x\beta z = x\gamma y\beta z$ for all $x, y, z \in M$ and $\beta, \gamma \in \Gamma$. Hence we obtain $x\beta[z, y]_\gamma = 0$ for all $x, y, z \in M$

and $\beta, \gamma \in \Gamma$. This implies $M\Gamma[z, y]_\gamma = \{0\}$ for all $y, z \in M$ and $\gamma \in \Gamma$. Hence we have $[z, y]_\gamma \Gamma M \Gamma [z, y]_\gamma = \{0\}$ for all $y, z \in M$ and $\gamma \in \Gamma$. Since M is prime, we have $[z, y]_\gamma = 0$, for all $y, z \in M$, which implies that M is commutative.

□

Theorem 3.7. *Let d be a nonzero α-semiderivation associated with g on a prime Γ-ring M. If $[d(x), y]_\gamma = 0$ for all $x, y \in M$ and $\gamma \in \Gamma$, then M is commutative.*

Proof. By hypothesis, we have

$$[d(x), y]_\gamma = 0, \ \forall\, x, y \in M, \gamma \in \Gamma. \tag{3.3}$$

Replacing x by $x\beta z$ in (3) and using (3), we have

$$\begin{aligned} 0 &= [d(x\beta z), y]_\gamma \\ &= [d(x)\beta g(z) + \alpha(x)\beta d(z), y] \\ &= [d(x)\beta g(z), y]_\gamma + [\alpha(x)\beta d(z), y]_\gamma \\ &= d(x)\beta[g(z), y]_\gamma + [\alpha(x), y]_\gamma \beta d(z) \end{aligned} \tag{3.4}$$

Taking $g(z)$ instead of y in (4), we have $[\alpha(x), g(z)]_\gamma \beta d(z) = 0$ for all $x, z \in M$ and $\gamma, \beta \in \Gamma$. Substituting $t\delta x$ for x with $t \in M$ in this relation, we obtain $[\alpha(t), g(z)]_\gamma \delta \alpha(x) \beta d(y) = 0$ for all $x, y, z, t \in M$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. That is, $[\alpha(t), g(z)]_\gamma \Gamma M \Gamma d(y) = \{0\}$ for all $t, y, z \in M$ and $\gamma \in \Gamma$. Since M is prime, we have $d(y) = 0$ or $[\alpha(x), g(z)]_\gamma = 0$ for all $x, z \in M$ and $\alpha, \gamma \in \Gamma$. But $d \neq 0$, and so $[\alpha(t), g(z)]_\gamma = 0$ for all $t, z \in M$ and $\gamma \in \Gamma$. Replacing t by $\alpha^{-1}(t)$ in the last relation, we have $[t, g(z)]_\gamma = 0$ for all $t, z \in M$ and $\gamma \in \Gamma$. Since g is onto, we get $[t, z]_\gamma = 0$ for all $t, z \in M$ and $\gamma \in \Gamma$, which implies that M is commutative.

□

Theorem 3.8. *Let d be a nonzero α-semiderivation associated with g on a prime Γ-ring M. If $(d(x) \circ y)_\gamma = 0$ for all $x, y \in M$ and $\gamma \in \Gamma$, then M is commutative.*

Proof. By hypothesis, we have

$$(d(x) \circ y)_\gamma = 0, \ \forall\, x, y \in M, \gamma \in \Gamma. \tag{3.5}$$

Replacing x by $x\beta z$ in (5) and using (5), we have

$$\begin{aligned} 0 &= (d(x\beta z) \circ y)_\gamma \\ &= (d(x)\beta g(z) + \alpha(x)\beta d(z) \circ y)_\gamma \\ &= (d(x)\beta g(z) \circ y)_\gamma + (\alpha(x)\beta d(z) \circ y)_\gamma \\ &= d(x)\beta[g(z), y]_\gamma - [\alpha(x), y]_\gamma \beta d(z) \end{aligned} \tag{3.6}$$

Taking $g(z)$ instead of y in (6), we have $[\alpha(x), g(z)]_\gamma \beta d(z) = 0$ for all $x, z \in M$ and $\gamma, \beta \in \Gamma$. Substituting $t\delta x$ for x with $t \in M$ in this relation, we obtain $[\alpha(t), g(z)]_\gamma \delta \alpha(x) \beta d(y) = 0$ for all $x, y, z, t \in M$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. That is, $[\alpha(t), g(z)]_\gamma \Gamma M \Gamma d(y) = \{0\}$ for all $t, y, z \in M$ and $\gamma \in \Gamma$. Since M is prime, we have $d(y) = 0$ or $[\alpha(t), g(z)]_\gamma = 0$ for all $x, z \in M$ and $\alpha, \gamma \in \Gamma$. But $d \neq 0$, and so $[\alpha(t), g(z)]_\gamma = 0$ for all $t, z \in M$ and $\gamma \in \Gamma$. Replacing t by $\alpha^{-1}(t)$ in the last relation, we have we have $[t, g(z)]_\gamma = 0$ for all $t, z \in M$ and $\gamma \in \Gamma$. Since g

is onto, we get $[t, z]_\gamma = 0$ for all $t, z \in M$ and $\gamma \in \Gamma$, which implies that M is commutative.

□

Theorem 3.9. *Let d be an α-semiderivation associated with g on a Γ-ring M, If $d(x\beta y) = d(x)\beta d(y)$ for all $x, y \in M$, then $d = 0$.*

Proof. By hypothesis, we have

$$d(x\beta y) = d(x)\beta d(y),\ \forall\ x, y \in M, \tag{3.7}$$

which implies that $d(x)\beta g(y)+\alpha(x)\beta d(y) = d(x)\beta d(y)$ for all $x, y \in M$. Replacing x by $x\delta y$ in this relation, we have

$$\begin{aligned} d(x)\delta d(y)\beta g(y) + \alpha(x)\delta\alpha(y)\beta d(y) &= d(x\delta y)\beta d(y) \\ &= d(x)\delta d(y)\beta d(y) \\ &= d(x)\delta d(y\beta y) \\ &= d(x)\delta(d(y)\beta g(y) + \alpha(y)\beta d(y)) \\ &= d(x)\delta d(y)\beta g(y) + d(x)\delta\alpha(y)\beta d(y) \end{aligned}$$

for every $x, y \in M$ and $\beta, \delta \in \Gamma$. Hence we have $(\alpha(x) - d(x))\delta\alpha(y)\beta d(y) = 0$ for all $x, y \in M$ and $\beta, \delta \in \Gamma$. Since α is an automorphism of M, we have $(\alpha(x) - d(x))\Gamma M\Gamma d(y) = \{0\}$ for all $x, y \in M$. Using the fact that M is prime, we get $\alpha(x) - d(x) = 0$ or $d(y) = 0$ for all $x, y \in M$. In the former case, we have $\alpha(x) = d(x)$ for all $x \in M$. Again, replacing $x\gamma y$ in the last relation, we get $d(x\gamma y) = \alpha(x\gamma y)$ for all $x, y \in M$ and $\gamma \in \Gamma$. Hence we obtain $d(x)\gamma g(y) + \alpha(x)\gamma d(y) = \alpha(x)\gamma\alpha(y)$ for all $x, y \in M$ and $\gamma \in \Gamma$. By the fact that $d(x) = \alpha(x)$ for all $x \in M$, we get $d(x)\gamma g(y) = 0$ for all $x, y \in M$ and $\gamma \in \Gamma$. Using the Lemma 3.2, we have $d = 0$. In the second case, $d(y) = 0$ for all $y \in M$, that is, $d = 0$.

□

Theorem 3.10. *Let d be an α-semiderivation associated with g on a Γ-ring M, If $d(x\beta y) = d(y)\beta d(x)$ for all $x, y \in M$ and $\beta \in \Gamma$, then $d = 0$.*

Proof. By hypothesis, we have

$$d(x\beta y) = d(y)\beta d(x),\ \forall\ x, y \in M, \beta \in \Gamma, \tag{3.8}$$

which implies that $d(x)\beta g(y) + \alpha(x)\beta d(y) = d(y)\beta d(x)$ for all $x, y \in M$ and $\beta \in \Gamma$. Replacing x by $x\delta y$ in this relation, we have

$$\begin{aligned} d(y)\delta d(x)\beta g(y) + \alpha(x)\delta\alpha(y)\beta d(y) &= d(y)\beta d(x\delta y) \\ &= d(y)\beta(d(x)\delta g(y) + \alpha(x)\delta d(y)) \\ &= d(y)\beta d(x)\delta g(y) + d(y)\beta\alpha(x)\delta d(y) \end{aligned} \tag{3.9}$$

for every $x, y \in M$ and $\beta, \delta \in \Gamma$. Hence we get

$$\alpha(x)\beta\alpha(y)\delta d(y) = d(y)\beta\alpha(x)\delta d(y), x, y \in M, \beta, \delta \in \Gamma. \tag{3.10}$$

Taking $x\sigma z$ instead of x in (10) and using (10), we have $[\alpha(x), d(y)]_\beta\sigma\alpha(z)\delta d(y) = 0$ for all $x, y, z \in M$ and $\beta, \delta, \sigma \in \Gamma$. Hence $[\alpha(x), d(y)]_\beta\Gamma M\Gamma d(y) = \{0\}$ for all $x, y \in M$ and $\beta \in \Gamma$. Since M is prime, we have $[\alpha(x), d(y)]_\beta = 0$ or $d(y) = 0$

for all $x, y \in M$ and $\beta \in \Gamma$. Let $K = \{y \in M | d(y) = 0\}$ and $L = \{y \in M | [\alpha(x), d(y)] = 0, \forall x \in M\}$. Then K and L are both additive subgroups and $K \cup L = M$, but $(M, +)$ is not union of two its proper subgroups, which implies that either $K = M$ or $L = M$. In the former case, we have $d = 0$. If $L = M$, then $[\alpha(x), d(y)] = 0$ for all $x, y \in M$. Replacing x by $\alpha^{-1}(x)$ in the last relation, we have $[x, d(y)] = 0$ for all $x, y \in M$. Thus we have $d(y) \in Z(M)$. Since d acts as an endomorphism of M, it follows that $d = 0$ by Theorem 3.9.

□

Proposition 3.11. *Let M be a prime Γ-ring and let d be a α-semiderivation on M. If $d(M) \subseteq Z(M)$, then $d = 0$ or M is commutative.*

Proof. By hypothesis, we have $d(x\gamma y) \in Z(M)$ for all $x, y \in M$ and $\gamma \in \Gamma$. That is, $d(x)\gamma g(y) + \alpha(x)\gamma d(y) \in Z(M)$ for all $x, y \in M$ and $\gamma \in \Gamma$. Commuting this term with x, we obtain

$$\begin{aligned} 0 &= [d(x)\gamma g(y) + \alpha(x)\gamma d(y), x]_\beta \\ &= [d(x)\gamma g(y), x]_\beta + [\alpha(x)\gamma d(y), x]_\beta \\ &= d(x)\gamma[g(y), x]_\beta + [d(x), x]_\beta\gamma g(y) + \alpha(x)\gamma[d(y), x]_\beta + [\alpha(x), x]_\beta\gamma d(y). \end{aligned}$$

Since $d(x) \in Z(M)$, we have $d(x)\gamma[g(y), x]_\beta + [\alpha(x), x]_\beta\gamma d(y) = 0$ for all $x, y \in M$ and $\beta, \gamma \in \Gamma$. Replacing x by $\alpha(x)$ in this relation, we obtain

$$d(\alpha(x))\gamma[g(y), \alpha(x)]_\beta = 0$$

for all $x, y \in M$ and $\gamma, \beta \in \Gamma$. Since g is onto, we have $d(\alpha(x))\gamma[y, \alpha(x)]_\beta = 0$ for all $x, y \in M$ and $\gamma, \beta \in \Gamma$. Taking $y\delta z$ instead of y with $z \in M$ in the last relation, we obtain $d(\alpha(x))\gamma y\delta[z, \alpha(x)]_\beta = 0$ for all $x, y, z \in M$ and $\gamma, \beta, \delta \in \Gamma$. Replacing x by $\alpha^{-1}(x)$ in the last relation, we have $d(x)\Gamma M\Gamma[z, x]_\beta = \{0\}$ for all $x, z \in M$ and $\beta \in \Gamma$. Since M is prime, we have $d(x) = 0$ or $[z, x]_\beta = 0$ for all $x, y \in M$ and $\beta \in \Gamma$. Let $K = \{x \in M | d(x) = 0\}$ and $L = \{x \in M | [z, x] = 0, \forall z \in M\}$. Then K and L are both additive subgroups and $K \cup L = M$, but $(M, +)$ is not union of two its proper subgroups, which implies that either $K = M$ or $L = M$. In the former case, we have $d(x) = 0$ for all $x \in M$, that is, $d = 0$. If $L = M$, then we get $[x, z] = 0$ for all $x, z \in M$, which implies that M is commutative.

□

Theorem 3.12. *Let d be an α-semiderivation associated with g on a Γ-ring M such that g is an automorphism of M. If $d([x, y]_\gamma) = \alpha([x, y]_\gamma)$ for all $x, y \in M$ and $\gamma \in \Gamma$, then $d = 0$ or M is commutative.*

Proof. By hypothesis, we have

$$d([x, y]_\gamma) = \alpha([x, y]_\gamma), \ \forall\ x, y \in M, \gamma \in \Gamma. \tag{3.11}$$

Replacing y by $x\beta y$ in (11), we have $d([x, x\beta y]_\gamma) = \alpha([x, x\beta y]_\gamma)$ for all $x, y \in M$ and $\gamma, \beta \in \Gamma$, which implies that $d(x)\beta g([x, y]_\gamma) + \alpha(x)\beta d([x, y]_\gamma) = \alpha(x)\beta\alpha([x, y]_\gamma)$ for all $x, y \in M$ and $\gamma, \beta \in \Gamma$. Using (11), we get $d(x)\beta g([x, y]_\gamma) = 0$ for all $x, y \in M$ and $\gamma, \beta \in \Gamma$. Taking $y\delta x$ instead of x in this relation, we have $d(x)\beta g(y)\delta g([x, y]_\gamma) = 0$ for all $x, y \in M$ and $\gamma, \beta, \delta \in \Gamma$. Since g is an automorphism of M, we get $d(x)\Gamma M\Gamma g([x, y]_\gamma) = \{0\}$ for all $x, y \in M$ and $\gamma \in \Gamma$. Since

M is prime, we have $d(x) = 0$ or $g([x, y]_\gamma) = 0$ for all $x, y \in M$ and $\gamma \in \Gamma$. Let $K = \{x \in M | d(x) = 0\}$ and $L = \{x \in M | g([x, y]_\gamma) = 0, \forall y \in M\}$. Then K and L are both additive subgroups and $K \cup L = M$, but $(M, +)$ is not union of two its proper subgroups, which implies that either $K = M$ or $L = M$. In the former case, we have $d(x) = 0$ for all $x \in M$, that is, $d = 0$. If $L = M$, then we get $g([x, y]_\gamma) = 0$ for all $x, y \in M$. Since g is onto, we have $[x, y]_\gamma = 0$ for all $x, y \in M$, which implies that M is commutative.

□

Theorem 3.13. *Let d be an α-semiderivation associated with g on a Γ-ring M such that g is an automorphism of M. If $d(x \circ y_\gamma) = \alpha(x \circ y)_\gamma$ for all $x, y \in M$ and $\gamma \in \Gamma$, then $d = 0$ or M is commutative.*

Proof. By hypothesis, we have

$$d(x \circ y_\gamma) = \alpha(x \circ y)_\gamma,\ \forall\ x, y \in M, \gamma \in \Gamma. \tag{3.12}$$

Replacing y by $x\beta y$ in (12), we have $d((x \circ x\beta y)_\gamma) = \alpha((x \circ x\beta y)_\gamma)$ for all $x, y \in M$ and $\gamma, \beta \in \Gamma$, which implies that $d(x)\beta g(x \circ y)_\gamma + \alpha(x)\beta d(x \circ y)_\gamma = \alpha(x)\beta\alpha(x \circ y)_\gamma$ for all $x, y \in M$ and $\gamma, \beta \in \Gamma$. Using (12), we get $d(x)\beta g(x \circ y)_\gamma = 0$ for all $x, y \in M$ and $\gamma, \beta \in \Gamma$. Taking $y\delta x$ instead of x in this relation, we have $d(x)\beta g(y)\delta g(x \circ y)_\gamma = 0$ for all $x, y \in M$ and $\gamma, \beta, \delta \in \Gamma$. Since g is an automorphism of M, we get $d(x)\Gamma M\Gamma g(x \circ y)_\gamma = \{0\}$ for all $x, y \in M$ and $\gamma \in \Gamma$. Since M is prime, we have $d(x) = 0$ or $g(x \circ y)_\gamma = 0$ for all $x, y \in M$ and $\gamma \in \Gamma$. Let $K = \{x \in M | d(x) = 0\}$ and $L = \{x \in M | g(x \circ y)_\gamma = 0, \forall y \in M\}$. Then K and L are both additive subgroups and $K \cup L = M$, but $(M, +)$ is not union of two its proper subgroups, which implies that either $K = M$ or $L = M$. In the former case, we have $d(x) = 0$ for all $x \in M$, that is, $d = 0$. If $L = M$, then we get $g(x \circ y)_\gamma = 0$ for all $x, y \in M$. Since g is onto, we obtain $(x \circ y)_\gamma = 0$ for all $x, y \in M$ and $\gamma \in \Gamma$, which implies that $x\gamma y = -y\gamma x$ for all $x, y \in M$ and $\gamma \in \Gamma$. Again, replacing x by $x\beta z$ in the last relation, we have $x\beta z\gamma y = -y\gamma x\beta z = x\gamma y\beta z$ for all $x, y, z \in M$ and $\beta, \gamma \in \Gamma$. Hence we obtain $x\beta[z, y]_\gamma = 0$ for all $x, y, z \in M$ and $\beta, \gamma \in \Gamma$. This implies $M\Gamma[z, y]_\gamma = \{0\}$ for all $y, z \in M$ and $\gamma \in \Gamma$. Hence we have $[z, y]_\gamma\Gamma M\Gamma[z, y]_\gamma = \{0\}$ for all $y, z \in M$ and $\gamma \in \Gamma$. Since M is prime, we have $[z, y]_\gamma = 0$, for all $y, z \in M$, which implies that M is commutative.

□

Theorem 3.14. *Let d be a nonzero α-semiderivation associated with g on a Γ-ring M such that g is an automorphism of M. If $d([x, y]_\gamma) = \alpha(-x\beta y + y\beta x)$ for all $x, y \in M$ and $\gamma, \beta \in \Gamma$, then M is commutative.*

Proof. By hypothesis, we have

$$d([x, y]_\gamma) = \alpha(-x\beta y + y\beta x),\ \forall\ x, y \in M, \gamma \in \Gamma. \tag{3.13}$$

Replacing y by $x\delta y$ in (13), we have $d([x, x\delta y]_\gamma) = \alpha(-x\beta x\delta y + x\delta y\beta x)$ for all $x, y \in M$ and $\gamma, \beta, \delta \in \Gamma$. Hence $d(x\delta[x, y]_\gamma) = \alpha(x)\alpha(-x\beta y + y\beta x)$ for all $x, y \in M$ and $\gamma, \beta, \delta \in \Gamma$, which means that $d(x)\delta g([x, y]_\gamma) + \alpha(x)\delta d([x, y]_\gamma) = \alpha(x)\alpha(-x\beta y + y\beta x)$ for all $x, y \in M$ and $\gamma, \beta, \delta \in \Gamma$. By using (13), we get $d(x)\delta g([x, y]_\gamma) = 0$ for all $x, y \in M$ and $\gamma, \delta \in \Gamma$. Taking $y\beta x$ instead of x in the last relation, we have $d(x)\delta g(y)\beta g([x,y]_\gamma) = 0$ for all $x, y \in M$ and $\gamma, \beta, \delta \in \Gamma$. Since g is an automorphism of M, we get $d(x)\Gamma M\Gamma g([x,y]_\gamma) = \{0\}$ for all $x, y \in M$ and $\gamma \in \Gamma$. By the fact that M is prime, we get $d(x) = 0$ or $g([x,y]_\gamma) = 0$ for all $x, y \in M$ and $\gamma \in \Gamma$. But $d \neq 0$, and so $g([x,y]_\gamma) = 0$ for all $x, y \in M$ and $\gamma \in \Gamma$. Since g is an automorphism of M, we have $[x,y]_\gamma = 0$, for all $x, y \in M$ and $\gamma \in \Gamma$, which implies that M is commutative.

□

Theorem 3.15. *Let d be a nonzero α-semiderivation associated with g on a Γ-ring M. If $[d(x), \alpha(y)]_\gamma = 0$ for all $x, y \in M$ and $\gamma \in \Gamma$, then M is commutative.*

Proof. By hypothesis, we have

$$[d(x), \alpha(y)]_\gamma = 0, \ \forall\ x, y \in M, \gamma \in \Gamma. \tag{3.14}$$

Replacing x by $y\beta x$ in (14) and using (14), we get

$$\begin{aligned}
0 &= [d(y\beta x), \alpha(y)]_\gamma \\
&= [d(y)\beta g(x) + \alpha(y)\beta d(x), \alpha(y)]_\gamma \\
&= [d(y)\beta g(x), \alpha(y)]_\gamma + [\alpha(y)\beta d(x), \alpha(y)]_\gamma \\
&= d(y)\beta[g(x), \alpha(y)]_\gamma + [d(x), \alpha(y)]_\gamma\beta g(x) + \alpha(y)\beta[d(x), \alpha(y)] + [\alpha(y), \alpha(y)]_\gamma\beta d(x) \\
&= d(y)\beta[g(x), \alpha(y)]_\gamma \qquad (3.15)
\end{aligned}$$

for all $x, y \in M$ and $\gamma, \beta \in \Gamma$. Taking $y\delta z$ instead of y in (15), we have

$$d(y)\beta\alpha(y)\delta[g(x), \alpha(z)]_\gamma = 0$$

for all $x, y, z \in M$ and $\beta, \delta, \gamma \in \Gamma$. This implies that $d(y)\Gamma M\Gamma[g(x), \alpha(z)]_\gamma = \{0\}$ for all $x, z \in M$ and $\gamma \in \Gamma$. Since M is prime, we get $d(y) = 0$ or $[g(x), \alpha(z)]_\gamma = 0$ for all $x, y, z \in M$. But $d \neq 0$, and so $[g(x), \alpha(z)]_\gamma = 0$ for all $x, z \in M$ and $\gamma \in \Gamma$. Replacing z by $\alpha^{-1}(z)$ in the last relation, we have $[g(x), z]_\gamma = 0$ for all $x, z \in M$ and $\gamma \in \Gamma$. Since g is onto, we have $[x, z]_\gamma = 0$ for all $x, z \in M$ and $\gamma \in \Gamma$, which implies that M is commutative.

□

Theorem 3.16. *Let d be a nonzero α-semiderivation associated with g on a Γ-ring M. If $(d(x) \circ \alpha(y))_\gamma = 0$ for all $x, y \in M$ and $\gamma \in \Gamma$, then M is commutative.*

Proof. By hypothesis, we have

$$(d(x) \circ \alpha(y))_\gamma = 0, \ \forall\ x, y \in M, \gamma \in \Gamma. \tag{3.16}$$

Replacing x by $y\beta x$ in (16) and using (16), we get

$$\begin{aligned}
0 &= (d(y\beta x), \alpha(y))_\gamma \\
&= (d(y)\beta g(x) + \alpha(y)\beta d(x), \alpha(y))_\gamma \\
&= (d(y)\beta g(x) \circ \alpha(y))_\gamma + (\alpha(y)\beta d(y) \circ \alpha(y))_\gamma \\
&= (d(y) \circ \alpha(y))_\gamma\beta g(x) + d(y)\beta[g(x), \alpha(y)]_\gamma + \alpha(y)\beta(d(y) \circ \alpha(y)) - [\alpha(y), \alpha(y)]_\gamma\beta d(y) \\
&= d(y)\beta[g(x), \alpha(y)]_\gamma \qquad (3.17)
\end{aligned}$$

for all $x, y \in M$ and $\gamma, \beta \in \Gamma$. Replacing y by $\alpha^{-1}(y)$ in (17), we have $d(y)\beta[g(x), y]_\gamma = 0$ for all $x, y \in M$ and $\beta, \gamma \in \Gamma$. Substituting $y\delta z$ for y in the last relation, we have

$d(y)\beta y\delta[g(x), z]_\gamma = 0$ for all $x, y, z \in M$ and $\beta, \delta, \gamma \in \Gamma$. Hence $d(y)\Gamma M\Gamma[g(x), z]_\gamma = \{0\}$ for all $x, y, z \in M$ and $\gamma \in \Gamma$. Since M is prime, we have $d(y) = 0$ or $[g(x), z]_\gamma = 0$ for all $x, y, z \in M$ and $\gamma \in \Gamma$. But $d \neq 0$, and so $[g(x), z]_\gamma = 0$ for all $x, z \in M$ and $\gamma \in \Gamma$. Since g is onto, we have $[x, z]_\gamma = 0$ for all $x, z \in M$ and $\gamma \in \Gamma$, which implies that M is commutative.

□

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