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ON MULTIPLICITY-FREE INDUCED REPRESENTATIONS OF A HOMOGENEOUS TREE

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Abstract. Let $T_{q,n}$ be a homogeneous rooted tree, let K be a subgroup of $Aut(T_{q,n})$, the group of automorphisms of $T_{q,n}$, and let $\tau_{q,\ell}$ be a unitary irreducible representation of K. In this paper, we construct some multiplicity-free irreducible representations of $Aut(T_{q,n})$. We also give a necessary condition for $(Aut(T_{q,n}), K)$ to be a strong Gelfand pair.

1. Introduction and preliminaries

1.1. **Introduction.** Multiplicity-free representations appear in various contexts of mathematics. A distinguishing feature of multiplicity-free representation is that the representation space decomposes canonically into irreducibles. Also, the multiplicity-free involves the commutativity of some group algebras. Indeed, if G is a finite group and K a subgroup of G, the convolution algebra $L(K\backslash G/K)$ of bi-K-invariant functions is commutative if only if the quasi-regular representation $Ind_K^G(\mathbf{1_K})$ is multiplicity-free. In this case, the pair (G, K) is called a Gelfand pair [8]. The notion of the Gelfand pair has been extended to commutative triples. A triple (G, K, τ) is commutative if the algebra $L(G, K, \tau, \tau)$ of τ-radial functions on G is commutative for the convolution product. It is shown [10], in this condition, that the induced representation $Ind_K^G\tau$ is multiplicity-free. We can notice that the one-dimensional trivial representation of K is replaced by a non-trivial unitary irreducible representation τ. The notion of commutative triple has been studied by many authors for infinite groups [4, 11, 22] and finite groups [1, 10]. In this paper, our goal is to construct some multiplicity-free induced representations or equivalently some commutative triples on a homogeneous tree. From the work of Grigorchuck introduced in 1980 [17, 18], we can notice that the study of trees has been extended to many areas of mathematics [21, 16]. The study of trees has many applications for instance in dynamical systems [3], geometric groups [7], statistics and probability [8]. When the action of the group of automorphisms of a homogeneous tree $Aut(T_{q,n})$ on the space of the leaves Σ^n is 2-point homogeneous, the pair $(Aut(T_{q,n}),\ Stab_{Aut(T_{q,n})}(0^n))$ is a Gelfand pair [8]. The main question is: " can we obtain a subgroup K of $Aut(T_{q,n})$ and a

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unitary irreducible representation τ of K such that $Ind_K^{Aut(T_{q,n})}\tau$ is multiplicity-free? " Our purpose is to investigate this question. The paper is organized as follows. In the next subsection, we give definitions and notations useful for the understanding of the paper. In section 2 we obtain some multiplicity-free induced representations of $Aut(T_{q,n})$.

1.2. **Preliminaries.** We use the notations and setup of this subsection in the rest of the paper without mentioning it. Let G be a finite group, let K be a subgroup of G, and let τ be a unitary irreducible representation of K on the finite-dimensional vector space V. The induced representation of the representation τ on G is the G-representation generally denoted by $Ind_K^G\tau$. Its space of realization is:

$$W = \{f : G \longrightarrow V \ / \ f(gk) = \tau(k^{-1})f(g), \quad \forall g \in G, k \in K\}$$

The action of G on W is given by:

$$((Ind_K^G\tau)(g_1)f)(g_2) = f(g_1^{-1}g_2), \quad \forall g_1, g_2 \in G \text{ and } f \in W.$$

Let $\widehat{G}$ denote the set of equivalence classes of unitary irreducible representations of G. The triple (G, K, τ) is commutative if only if for each $\pi \in \widehat{G}$, the multiplicity of τ in $\pi_{|K}$ is at most one. The pair (G, K) is a strong Gelfand pair if, for any $\tau \in \widehat{K}$, the triple (G, K, τ) is commutative (see [12]).
Let F be a finite group and let X be a finite set. We denote by F^X the set of all maps f: $X \longrightarrow F$. Let us assume that G acts on X. This action induces an action of G on F^X. The set $F^X \times G$ equipped with the following multiplication:

$$(f, g)(f', g') = (f.gf', gg'), \quad \forall \ (f, g) \ ; \ (f', g') \in F^X \times G$$
$$\text{where } (f.gf')(x) = f(x)f'(g^{-1}x), \quad \forall \ x \in X,$$

is a group (see [9]). The identity element is $(1_F,1_G)$ and the inverse of (f, g) is given by $(g^{-1}f^{-1}, g^{-1})$. This group is called the wreath product of F by G with respect to X and is denoted by $F \wr_X G$ or by $F \wr G$ if there is no confusion on X.
The subgroup

$$\overline{F^X} = \{(f, 1_G) : f \in F^X\}$$

is called the base group. It is naturally identified with F^X. The subgroup diag$F^X := \{f \in F^X : f$ is constant on $X\}$ is isomorphic to F and called the diagonal subgroup of the base group (see [9]). When G is replaced by the permutation group S_q and we take $X = \{0, 1, ..., q-1\}$ the wreath product $F \wr S_q$ is defined by:

$$F \wr S_q = \{(h; \sigma)/ \ h :\{0, 1, ...q-1\} \longrightarrow F, \ \sigma \in S_q\}.$$

The base group $\overline{F} := \{(h; \mathbf{1}_{S_q})/ \ h :\{0, 1, ...q-1\} \longrightarrow F\}$ is a normal subgroup of $F \wr S_q$ and is the direct product of q copies of F. It is denoted by F^q. The elements of diag$\overline{F}$ are of the form: $(h; \mathbf{1}_{S_q}) = (h(0), h(1), ..., h(q-1); \mathbf{1}_{S_q}) = (f, f, ..., f; \mathbf{1}_{S_q})$. We deduce that diag$\overline{F}$ is isomorphic to F and we write:

$$diag\overline{F} = \{(f, f, ..., f; \mathbf{1}_{S_q}) \ / \ f \in F\}.$$

The complement of the base group is:

$$S_q' := \{(e; \sigma)/ \ \sigma \in S_q, \quad \forall \ i \in \{0; 1; ...; q-1\}, \quad e(i) \ = \mathbf{1}_{S_q}\}.$$

$S_q^{'}$ is isomorphic to S_q and $(\mathrm{diag}\overline{F}) \times S_q^{'} = \{(f, f, ..., f; \sigma)/f \in F, \sigma \in S_q\}$. So, $F \wr S_q = F^q \rtimes S_q$ (for more details about $F \wr S_q$, the reader can consult [19]).
Let N be a normal subgroup of G and $\rho \in \widehat{G/N}$. An inflation representation $\overline{\rho}$ of ρ is the unitary representation of G defined by: $\overline{\rho}(g) = \rho(gN), \quad$ for all $g \in G$.
An extension of $\tau \in \widehat{K}$ is the representation of G denoted $\widetilde{\tau}$ such that the restriction of $\widetilde{\tau}$ to K is equal to τ.
Let (σ, V) and (ρ, W) be two representations of G. The inner tensor product of σ and ρ is the representation of G on $V \otimes W$, defined by:

$$(\sigma \otimes \rho)(g)(v \otimes w) = \sigma(g)v \otimes \rho(g)w, \quad \text{for all } g \in G,\ v \in V,\ w \in W.$$

We denote it by $\sigma \otimes \rho$.
Let (σ, V) be a representation of G_1 and (ρ, W) be a representation of G_2. The outer tensor product of σ and ρ is the representation of $(G_1 \times G_2)$ on $V \otimes W$, defined by:

$$(\sigma \boxtimes \rho)(g_1, g_2)(v \otimes w) = \sigma(g_1)v \otimes \rho(g_2)w, \quad \text{for all } g_1 \in G_1,\ g_2 \in G_2,\ v \in V,\ w \in W.$$

We denote it by $\sigma \boxtimes \rho$.
If ρ is an irreducible representation of F, then the tensor representation $\rho^{\boxtimes q} = \rho \boxtimes \rho \boxtimes ... \boxtimes \rho$ of F^q is irreducible (see [9],[23]).
Let $q \in \mathbb{N}$. A composition of q of length t is the t-uple $\lambda = (q_1, q_2, ..., q_t)$ such that $\sum_{i=1}^{t} q_i = q$ with $q_i \in \mathbb{N}$. If $q_1 \geq q_2 \geq ... \geq q_t$, then λ is a partition of q. We write $\lambda \vdash q$. It is well known that the irreducible representations of S_q are labelled by integer partitions of q (see [20]). Therefore, we denote by S^{λ} the irreducible representation associated with the partition λ. Let $(q_1, q_2, ..., q_t)$ be a composition of q. If $r_1 \vdash q_1; ...; r_t \vdash q_t$, then $(r_1, ..., r_t)$ is a multipartition of q. We denote it by $(r_1, ..., r_t) \Vdash q$.
If $\rho_1, ..., \rho_t$ is the complete list of pairwise inequivalent irreducible representations of F, $(q_1, q_2, ..., q_t)$ a composition of length t of q and $(r_1, ..., r_t) \Vdash q$ a multipartition of q, then $Ind_{F \wr (S_{q_1} \times ... \times S_{q_t})}^{F \wr S_q} (\widetilde{\rho_1^{\boxtimes q_1}} \boxtimes ... \boxtimes \widetilde{\rho_t^{\boxtimes q_t}}) \otimes (\overline{S^{r_1}} \boxtimes ... \boxtimes \overline{S^{r_t}})$ is an irreducible representation of $F \wr S_q$ (see [5],[13],[23]).
Let $\Sigma = \{0; 1; ...; q-1\}$, where $q \in \mathbb{N}^*$. We call the set Σ the alphabet. A word over Σ of length k is a sequence $\omega = \sigma_1\sigma_2...\sigma_k$ where $\sigma_i \in \Sigma$ for $i = 1,...,k$. The set of all the words of length n is denoted by Σ^n. A graph is a pair (X, E) consisting of a set of vertices X and a family E of two-element subsets of X, called edges [2]. A tree is a connected graph without circuits. A tree is said to be rooted if it has a fixed vertex called the root of the tree. The root is denoted by $\emptyset$. Two vertices x, y are neighbours, and we write $x \sim y$ if they are connected by an edge. The degree of a vertex x is the number of its neighbours. A leaf is a vertex of degree one. On a tree, we define a distance from two vertices x and y as the length of the shortest path joining x and y. The distance between x and y is denoted by $d(x, y)$. A homogeneous tree is a tree where all the vertices which are not leaves have the same degree (For more details about a homogeneous tree see [15, 8, 6]).

In this paper, we are interested in finite the homogeneous rooted trees. We denote by $T_{q,n}$ a finite homogeneous rooted tree whose root is of degree q and the distance from the root to a leaf is n. We designate by $Aut(T_{q,n})$ the group of automorphisms of the homogeneous rooted tree $T_{q,n}$. We have
$Aut(T_{q,n}) \cong \underbrace{Aut(T_{q-1,n-1}) \times ... \times Aut(T_{q-1,n-1})}_{q factors} \rtimes S_q = (Aut(T_{q-1,n-1}))^q \rtimes S_q$
(see [14]). Also from ([9, 13]) we have for $n \geqslant 2$,
$Aut(T_{q-1,n-1}) = \underbrace{S_{q-1} \wr S_{q-1} ... \wr S_{q-1}}_{(n-1) factors}$ and $Aut(T_{q,n}) = Aut(T_{q-1,n-1}) \wr S_q$.

2. MULTIPLICITY-FREE INDUCED REPRESENTATIONS

Using proposition 3.12 ([23], page 15), we have

Proposition 2.1. *Let ℓ be an integer less than q and let $\sigma_1, ..., \sigma_\ell$ be the complete list of pairwise inequivalent irreducible representations of $Aut(T_{q-1,n-1})$. Let $(q_1, ..., q_\ell)$ be a composition of q and let $\alpha = (\alpha_1, ..., \alpha_\ell)$ be a multipartition of q. A representation $\pi_{q,\alpha}$ of $Aut(T_{q,n})$ is defined by:*

$$\pi_{q,\alpha} = Ind_{Aut(T_{q-1,n-1}) \wr (S_{q_1} \times ... \times S_{q_\ell})}^{Aut(T_{q,n})} \, (\widetilde{\sigma_1^{\boxtimes q_1}} \otimes \overline{S^{\alpha_1}}) \boxtimes ... \boxtimes (\widetilde{\sigma_\ell^{\boxtimes q_\ell}} \otimes \overline{S^{\alpha_\ell}}).$$

Throughout the rest of the paper, we set $\zeta_{\alpha_i}^{q_i} = \widetilde{\sigma_i^{\boxtimes q_i}} \otimes \overline{S^{\alpha_i}}$ for $i \in (1, 2,...,\ell)$ and $\zeta_1^{1,j} = \widetilde{\sigma_j^{\boxtimes 1}} \otimes \overline{S^1}$ for $j \in (1, 2,...,\ell)$.

Remark 2.2. The set $\{\pi_{\beta,\phi}$ / β is some integer composition of length ℓ of q, ϕ is a multipartition of q $\}$ is a complete list of the irreducible representations of $Aut(T_{q,n})$ (see [23]).

The following result is a special case of the Littlewood-Richardson rule for a wreath product. Considering a certain partition of q, we obtain a multiplicity-free representation of $G \wr S_q$. In all that follows, we assume that ℓ is less than q.

Theorem 2.3. *Let G be a finite group, let $(\ell, q-\ell)$ be a composition of q, let $(1, ..., 1) \vdash \ell$ be a partition of ℓ, let $\delta \vdash (q-\ell)$ be a partition of $(q-\ell)$ and, let $\Theta_{(1,...,1)}^{\ell} \boxtimes \Theta_{\delta}^{q-\ell}$ be a representation of $G \wr (S_\ell \times S_{q-\ell})$. Then:*

$$Ind_{G \wr (S_\ell \times S_{q-\ell})}^{G \wr S_q} \, (\Theta_{(1,...,1)}^{\ell} \boxtimes \Theta_{\delta}^{q-\ell}) = \bigoplus_{\beta \vdash q} \Theta_{\beta}^{q}.$$

Proof. Let σ be a unitary irreducible representation of G. We have

$$Ind_{G \wr (S_\ell \times S_{q-\ell})}^{G \wr S_q} (\Theta_{(1,...,1)}^{\ell} \boxtimes \Theta_{\delta}^{q-\ell}) = Ind_{G \wr (S_\ell \times S_{q-\ell})}^{G \wr S_q} ((\widetilde{\sigma^{\boxtimes \ell}} \otimes \overline{S^{(1,...,1)}}) \boxtimes (\widetilde{\sigma^{\boxtimes (q-\ell)}} \otimes \overline{S^{\delta}}))$$

$$= Ind_{G\wr(S_\ell\times S_{q-\ell})}^{G\wr S_q}((\widetilde{\sigma^{\boxtimes \ell}} \boxtimes \widetilde{\sigma^{\boxtimes (q-\ell)}}) \otimes (\overline{S^{(1,\dots,1)}} \boxtimes \overline{S^{\delta}}))$$
$$= Ind_{G\wr(S_\ell\times S_{q-\ell})}^{G\wr S_q}((\widetilde{\sigma^{\boxtimes q}}) \otimes (\overline{S^{(1,\dots,1)} \boxtimes S^{\delta}}))$$
$$= Ind_{G\wr S_\ell\times G\wr S_{q-\ell}}^{G\wr S_q}((Res_{G\wr S_\ell\times G\wr S_{q-\ell}}^{G\wr S_q}(\widetilde{\sigma^{\boxtimes q}})) \otimes (\overline{S^{(1,\dots,1)} \boxtimes S^{\delta}}))$$
$$= \widetilde{\sigma^{\boxtimes q}} \otimes \overline{Ind_{S_\ell\times S_{q-\ell}}^{S_q}(S^{(1,\dots,1)} \boxtimes S^{\delta})}.$$

Moreover, thanks to corollary 3.7 ([20], page 105) if ε_ℓ is a permutation representation then:

$$\begin{aligned} Ind_{S_\ell\times S_{q-\ell}}^{S_q}(S^{(1,\dots,1)} \boxtimes S^{\delta}) &= Ind_{S_\ell\times S_{q-\ell}}^{S_q}(\varepsilon_\ell \boxtimes S^{\delta}) \\ &= \sum_{\beta\vdash q} S^{\beta}. \end{aligned}$$

Hence

$$\begin{aligned} Ind_{G\wr(S_\ell\times S_{q-\ell})}^{G\wr S_q}(\Theta^{\ell}_{(1,\dots,1)} \boxtimes \Theta^{q-\ell}_{\delta}) &= \widetilde{\sigma^{\boxtimes q}} \otimes \overline{\sum_{\beta\vdash q} S^{\beta}} \\ &= \sum_{\beta\vdash q}(\widetilde{\sigma^{\boxtimes q}} \otimes \overline{S^{\beta}}) \\ &= \sum_{\beta\vdash q}\Theta^{q}_{\beta}. \end{aligned}$$

□

In the following result, we obtain a multiplicity-free representation of $Aut(T_{q-1,n-1}) \wr S_q$.

Theorem 2.4. *Let $\sigma_1, ..., \sigma_\ell$ be a set of all pairwise inequivalent irreducible representations of $Aut(T_{q-1,n-1})$, let $(1, ..., 1)$; $(r_1, ..., r_\ell)$ and $(1+r_1, ..., 1+r_\ell)$ be some compositions of respectively ℓ, $(q-\ell)$ and q, let $(1, ..., 1) \Vdash \ell$ be a multipartition of ℓ and let $\xi = (\delta_1, ..., \delta_\ell) \Vdash (q-\ell)$ be a multipartition of $q-\ell$. Then:*

$$Ind_{Aut(T_{q-1,n-1})\wr(S_\ell\times S_{q-\ell})}^{Aut(T_{q-1,n-1})\wr S_q}\ (\Theta_{(1,\dots,1)} \boxtimes \Theta_\xi) = \bigoplus_{\nu\Vdash q} \Theta_\nu$$

where $\Theta_{(1,\dots,1)} = Ind_{Aut(T_{q-1,n-1})\wr(S_1\times\dots\times S_1)}^{Aut(T_{q-1,n-1})\wr S_\ell}(\zeta_1^{1,1} \boxtimes \zeta_1^{1,2} \boxtimes \dots \boxtimes \zeta_1^{1,\ell})$ *and* $\Theta_\xi = Ind_{Aut(T_{q-1,n-1})\wr(S_{r_1}\times\dots\times S_{r_\ell})}^{Aut(T_{q-1,n-1})\wr S_{q-\ell}}(\zeta_{\delta_1}^{r_1} \boxtimes \zeta_{\delta_2}^{r_2} \boxtimes \dots \boxtimes \zeta_{\delta_\ell}^{r_\ell})$.

Proof. Let us set:
$K = Aut(T_{q-1,n-1}) \wr (S_\ell \times S_{q-\ell})$; $K_1 = Aut(T_{q-1,n-1}) \wr S_\ell$;
$K_2 = Aut(T_{q-1,n-1}) \wr S_{q-\ell}$; $F_1 = Aut(T_{q-1,n-1}) \wr (S_1 \times ... \times S_1)$;
$F_2 = Aut(T_{q-1,n-1}) \wr (S_{r_1} \times ... \times S_{r_\ell})$;
$H = Aut(T_{q-1,n-1}) \wr (S_1 \times S_{r_1)} \times ... \times Aut(T_{q-1,n-1}) \wr (S_1 \times S_{r_\ell})$;
$H_i = Aut(T_{q-1,n-1}) \wr (S_1 \times S_{r_i})$ for $i \in \{1, ..., \ell\}$ and

$E = Aut(T_{q-1,n-1}) \wr (S_{1+r_1} \times ... \times S_{1+r_\ell})$.
Thanks to proposition 2, we have

$$\begin{aligned} Ind_{Aut(T_{q-1,n-1})\wr(S_\ell\times S_{q-\ell})}^{Aut(T_{q-1,n-1})\wr S_q}(\Theta_{(1,...,1)} \boxtimes \Theta_\xi) &= Ind_K^{Aut(T_{q-1,n-1})\wr S_q}(Ind_{F_1}^{K_1}(\zeta_1^{1,1} \boxtimes \zeta_1^{1,2} \boxtimes ... \boxtimes \zeta_1^{1,\ell}) \boxtimes Ind_{F_2}^{K_2}(\zeta_{\delta_1}^{r_1}\boxtimes \zeta_{\delta_2}^{r_2} \boxtimes ... \boxtimes \zeta_{\delta_\ell}^{r_\ell})) \\ &= Ind_K^{Aut(T_{q-1,n-1})\wr S_q}(Ind_{F_1\times F_2}^{K}(\zeta_1^{1,1} \boxtimes \zeta_1^{1,2} \boxtimes ... \boxtimes \zeta_1^{1,\ell}) \boxtimes (\zeta_{\delta_1}^{r_1} \boxtimes \zeta_{\delta_2}^{r_2} \boxtimes ... \boxtimes \zeta_{\delta_\ell}^{r_\ell})) \\ &= Ind_{F_1\times F_2}^{Aut(T_{q-1,n-1})\wr S_q}((\zeta_1^{1,1} \boxtimes \zeta_1^{1,2} \boxtimes ... \boxtimes \zeta_1^{1,\ell}) \boxtimes (\zeta_{\delta_1}^{r_1} \boxtimes \zeta_{\delta_2}^{r_2} \boxtimes ... \boxtimes \zeta_{\delta_\ell}^{r_\ell})) \\ &= Ind_H^{Aut(T_{q-1,n-1})\wr S_q}((\zeta_1^{1,1}\boxtimes \zeta_{\delta_1}^{r_1}) \boxtimes ... \boxtimes (\zeta_1^{1,\ell} \boxtimes \zeta_{\delta_\ell}^{r_\ell})) \\ &= Ind_H^{Aut(T_{q-1,n-1})\wr S_q} Ind_{H_1\times..\times H_\ell}^{H}((\zeta_1^{1,1} \boxtimes \zeta_{\delta_1}^{r_1}) \boxtimes ... \boxtimes (\zeta_1^{1,\ell} \boxtimes \zeta_{\delta_\ell}^{r_\ell})). \end{aligned}$$

So,

$$Ind_{Aut(T_{q-1,n-1})\wr(S_\ell\times S_{q-\ell})}^{Aut(T_{q-1,n-1})\wr S_q}(\Theta_{(1,...,1)} \boxtimes \Theta_\xi) = Ind_E^{Aut(T_{q-1,n-1})\wr S_q}(Ind_{H_1}^{Aut(T_{q-1,n-1})\wr S_{1+r_1}}((\zeta_1^{1,1} \boxtimes \zeta_{\delta_1}^{r_1}) \boxtimes ... \boxtimes Ind_{H_\ell}^{Aut(T_{q-1,n-1})\wr S_{1+r_\ell}}(\zeta_1^{1,\ell} \boxtimes \zeta_{\delta_\ell}^{r_\ell}))).$$

According to theorem 2.3, we have for any $i \in \{1, ..., \ell\}$

$$Ind_{Aut(T_{q-1,n-1})\wr S_1\times S_{r_i}}^{Aut(T_{q-1,n-1})\wr S_{1+r_i}}(\zeta_1^{1,i} \boxtimes \zeta_{\delta_i}^{r_i}) = \bigoplus_{\nu_1\vdash(1+r_i)} \Theta_{\nu_i}$$

hence

$$\begin{aligned} Ind_E^{Aut(T_{q-1,n-1})\wr S_q}((\bigoplus_{\nu_1\vdash(1+r_1)} \Theta_{\nu_1}) \boxtimes ... \boxtimes (\bigoplus_{\nu_\ell\vdash(1+r_\ell)} \Theta_{\nu_\ell})) &= \bigoplus_{\nu_1\vdash(1+r_1)} \cdots \bigoplus_{\nu_\ell\vdash(1+r_\ell)} Ind_E^{Aut(T_{q-1,n-1})\wr S_q}(\Theta_{\nu_1}\boxtimes ... \boxtimes \Theta_{\nu_\ell}) \\ &= \bigoplus_{\nu \Vdash q} \Theta_\nu. \end{aligned}$$

□

As a consequence of theorem 2.4, we have the following remark.

Remark 2.5. Let $Aut(T_{q,n})$ be the group of automorphisms of the homogeneous rooted tree $T_{q,n}$ and let $Aut(T_{q-1,n-1}) \wr (S_\ell \times S_{q-\ell})$ be a subgroup of $Aut(T_{q,n})$. Considering the representation $\tau_{q,\ell} = (\Theta_{(1,...,1)} \boxtimes \Theta_\xi)$ of $Aut(T_{q-1,n-1}) \wr (S_\ell \times S_{q-\ell})$, the triple $(Aut(T_{q,n}), Aut(T_{q-1,n-1}) \wr (S_\ell \times S_{q-\ell}), \tau_{q,\ell})$ is commutative.

Using Theorem 4.1 in [5], we deduce the following remark.

Remark 2.6. Let $q \geqslant 2$, let $Aut(T_{q,n})$ be the group of automorphisms of the homogeneous rooted tree $T_{q,n}$, and let $Aut(T_{q-1,n-1}) \wr (S_\ell \times S_{q-\ell})$ be a subgroup of $Aut(T_{q,n})$. If $\ell = q-1$ or $\ell = q-2$ then the pair $(Aut(T_{q,n})\ ,\ Aut(T_{q-1,n-1}) \wr (S_\ell \times S_{q-\ell}))$ is a strong Gelfand pair. So, if $\ell = q-1$ or $\ell = q-2$, for each representation τ of $Aut(T_{q-1,n-1}) \wr (S_\ell \times S_{q-\ell})$, the induced representation $Ind_{Aut(T_{q-1,n-1}) \wr (S_\ell \times S_{q-\ell})}^{Aut(T_{q,n})} \tau$ is multiplicity-free.

According to [8], $Aut(T_{q,n})$ acts on each vertex x of $T_{q,n}$ and the action is defined by:

$$g(x) = g(x_1 x_2 \ldots . x_n) = g_\emptyset(x_1) g_{x_1}(x_2) \ldots g_{x_1 x_2 \ldots x_{n-1}}(x_n),$$
$$\text{for all } g \in Aut(T_{q,n}),\ x = x_1 x_2 \ldots . x_n \in T_{q,n}.$$

Let us designate by $Stab_{Aut(T_{q,n})}(x)$ the stabilizer of x. We have the following result.

Theorem 2.7. *Let Σ^1 be the first level of the homogeneous rooted tree $T_{q,n}$ and $x_0 \in \Sigma^1$. Then $Stab_{Aut(T_{q,n})}(x_0) = Aut(T_{q-1,n-1})$.*

Proof. Suppose that x_0 is a vertex of Σ^1 and $g \in Stab_{Aut(T_{q,n})}(x_0)$. By definition of the stabilizer, $g(x_0) = x_0$, hence x_0 is the root of the subtree $T_{q-1,n-1}$ of the homogeneous rooted tree $T_{q,n}$. Hence $g \in Aut(T_{q-1,n-1})$ and consequently $Stab_{Aut(T_{q,n})}(x_0) \subset Aut(T_{q-1,n-1})$. Conversely, assume that $g \in Aut(T_{q-1,n-1})$. Any vertex $x_0 \in \Sigma^1$ is the root of the subtree $T_{q-1,n-1}$ therefore $g(x_0) = x_0$. We deduce that $g \in Stab_{Aut(T_{q-1,n-1})}(x_0)$. Moreover, any automorphism of a subtree $T_{q-1,n-1}$ is an automorphism of the tree $T_{q,n}$, so $g \in Stab_{Aut(T_{q,n})}(x_0)$ and $Aut(T_{q-1,n-1}) \subset Stab_{Aut(T_{q,n})}(x_0)$. Consequently
$Stab_{Aut(T_{q,n})}(x_0) = Aut(T_{q-1,n-1})$. □

In the next result, we identify the stabilizer of a leaf of a homogeneous rooted tree $T_{q,n}$ with some wreath product.

Theorem 2.8. *Let $x_1 x_2 \ldots x_n$ be a leaf of the homogeneous rooted tree $T_{q,n}$. We have $Stab_{Aut(T_{q,n})}(x_1 x_2 \ldots x_n) \cong (Aut(T_{q-1,n-2}) \times Aut(T_{q-1,n-3}) \times \ldots \times Aut(T_{q-1,1}) \times S_1) \wr S_{q-1}$.*

Proof. By definition,

$$\begin{aligned} Stab_{Aut(T_{q,n})}(x_1 x_2 \ldots x_n) &= \{g \in Aut(T_{q,n}) : g(x_1 x_2 \ldots x_n) = x_1 x_2 \ldots x_n\}. \\ &= \{g \in Aut(T_{q,n}) : g_\emptyset(x_1) g_{x_1}(x_2) \ldots g_{x_1 x_2 \ldots x_{n-1}}(x_n) = \\ &\quad x_1 x_2 \ldots x_n\}. \end{aligned}$$

Therefore,

$$\begin{aligned} g_\emptyset(x_1) &= x_1; \\ g_{x_1}(x_2) &= x_2; \\ &\ \ \vdots \\ g_{x_1 x_2 \ldots x_{n-2}}(x_{n-1}) &= x_{n-1}; \\ g_{x_1 x_2 \ldots x_{n-1}}(x_n) &= x_n. \end{aligned}$$

Moreover, $g_\emptyset$ is a permutation induced by g on the first level of the rooted tree $T_{q,n}$, hence $g_\emptyset$ is an automorphism of the subtree $T_{q-1,n-1}$. Similarly, g_{x_1} is a permutation induced by g on the first level of the subtree rooted in $g_\emptyset(x_1)$ that is g_{x_1} is an automorphism of the tree $T_{q-1,n-2}$. So on and using the same reasoning, $g_{x_1x_2...x_{n-1}}$ is an automorphism of $Aut(T_{q-1,1})$. Let us consider the map:

$$\begin{array}{rccl} \phi & : & Stab_{Aut(T_{q,n})}(x_1...x_n) & \rightarrow & Aut(T_{q-1,n-1}) \times ... \times Aut(T_{q-1,1}) \\ & & g & \mapsto & (g_\emptyset, g_{x_1}, ..., g_{x_1x_2...x_{n-2}}) \end{array}$$

ϕ is injective.
Indeed, let $g \in Stab_{Aut(T_{q,n})}(x_1x_2...x_n)$ and $h \in Stab_{Aut(T_{q,n})}(x_1x_2...x_n)$ such that $\phi(g) = \phi(h)$. Then $(g_\emptyset, g_{x_1}, ..., g_{x_1x_2...x_{n-2}}) = (h_\emptyset, h_{x_1}, ..., h_{x_1x_2...x_{n-2}})$ hence

$$\begin{array}{c} g_\emptyset = h_\emptyset; \\ . \\ . \\ . \\ g_{x_1x_2...x_{n-2}} = h_{x_1x_2...x_{n-2}} \end{array}$$

so $h = g$.
Also, ϕ is surjective by construction. Thus ϕ is bijective. Therefore $Stab_{Aut(T_{q,n})}(x_1x_2...x_n) \cong Aut(T_{q-1,n-1}) \times Aut(T_{q-1,n-2}) \times ... \times Aut(T_{q-1,1})$. Since

$$\begin{array}{c} Aut(T_{q-1,n-1}) = \underbrace{S_{q-1} \wr ... \wr S_{q-1}}_{(n-1)}; \\ Aut(T_{q-1,n-2}) = \underbrace{S_{q-1} \wr ... \wr S_{q-1}}_{(n-2)}; \\ . \\ . \\ . \\ Aut(T_{q-1,2}) = S_{q-1} \wr S_{q-1}; \\ Aut(T_{q-1,1}) = S_{q-1} \end{array}$$

then

$$\begin{aligned} Stab_{Aut(T_{q,n})}(x_1x_2...x_n) &\cong (\underbrace{S_{q-1} \wr ... \wr S_{q-1}}_{(n-1)}) \times (\underbrace{S_{q-1} \wr ... \wr S_{q-1}}_{(n-2)}) \times ... \\ &\times (S_{q-1} \wr S_{q-1}) \times S_{q-1} \\ &\cong (\underbrace{S_{q-1} \wr ... \wr S_{q-1}}_{(n-2)} \times \underbrace{S_{q-1} \wr ... \wr S_{q-1}}_{(n-3)} \times ... \\ &\times S_{q-1} \times S_1) \wr S_{q-1} \\ &\cong (Aut(T_{q-1,n-2}) \times Aut(T_{q-1,n-3}) \times ... \times \\ &Aut(T_{q-1,1}) \times S_1) \wr S_{q-1} \end{aligned}$$

$\square$

Corollary 2.9. *Let* $\underbrace{000...0}_{n} = 0^n$ *be the leftmost leaf of the homogeneous rooted tree* $T_{q,n}$. *Then* $Stab_{Aut(T_{q,n})}(0^n) \cong (Aut(T_{q-1,n-2})\times Aut(T_{q-1,n-3})\times ...\times Aut(T_{q-1,1})\times S_1) \wr S_{q-1}$.

The following result proves that there exists a representation ρ of $Stab_{Aut(T_{q,n})}(0^n)$ such that $Ind^{Aut(T_{q,n})}_{Stab_{Aut(T_{q,n})}(0^n)}\ \rho$ is multiplicity free.

Theorem 2.10. *Let* $Aut(T_{q,n})$ *be the group of automorphisms of the homogeneous rooted tree* $T_{q,n}$, *let* $Stab_{Aut(T_{q,n})}(0^n)$ *be a subgroup of* $Aut(T_{q,n})$, C_i ; $i = 1, ..., n-2$ *be a unitary irreducible representation of* $Aut(T_{q-1,n-(i+1)}) \wr S_{q-1}$ *and let* D *be a representation of* S_{q-1}. *Then,* $Ind^{Aut(T_{q,n})}_{Stab_{Aut(T_{q,n})}(0^n)}(C_1 \boxtimes ... \boxtimes C_{n-2} \boxtimes D)$ *is multiplicity-free.*

Proof. According to corollary 2.9, we have
$Stab_{Aut(T_{q,n})}(0^n) \cong (Aut(T_{q-1,n-2})\times Aut(T_{q-1,n-3})\times ...\times Aut(T_{q-1,1})\times S_1) \wr S_{q-1}$. Let us denote by $\sigma^i_1, ..., \sigma^i_{\ell_i}$, a set of pairwise inequivalent irreducible representations of $Aut(T_{q-1,n-(i+1)})$ for $i = 1, ..., n-2$. We set:

$$\begin{aligned} H_1 &= (Aut(T_{q-1,n-2}) \times Aut(T_{q-1,n-3}) \times ... \times Aut(T_{q-1,1}) \times S_1) \wr S_{q-1} \\ &= Aut(T_{q-1,n-2}) \wr S_{q-1} \times Aut(T_{q-1,n-3}) \wr S_{q-1} \times ...\times \\ &\quad Aut(T_{q-1,1}) \wr S_{q-1} \times S_{q-1}. \end{aligned}$$

$$H_2 = Aut(T_{q-1,n-2}) \wr (S_{\ell_1} \times S_{q-\ell_1-1}) \times ... \times Aut(T_{q-1,1}) \wr (S_{\ell_{n-2}} \times S_{q-\ell_{n-2}-1}) \times S_{q-1}.$$

$$H_3 = Aut(T_{q-1,n-2}) \wr (S_{1+\alpha^1_1} \times ... \times S_{1+\alpha^1_{\ell_1}}) \times ... \times Aut(T_{q-1,1}) \wr (S_{1+\alpha^{n-2}_1} \times ... \times S_{1+\alpha^{n-2}_{\ell_{n-2}}}) \times S_{q-1}.$$

$$H_4 = Aut(T_{q-1,n-2}) \wr (S_1 \times S_{\alpha^1_1} \times ... \times S_1 \times S_{\alpha^1_{\ell_1}}) \times ... \times Aut(T_{q-1,1}) \wr (S_1 \times S_{\alpha^{n-2}_1} \times ... \times S_1 \times S_{\alpha^{n-2}_{\ell_{n-2}}}) \times S_{q-1}.$$

We have,
$Ind^{Aut(T_{q,n})}_{Stab_{Aut(T_{q,n})}(0^n)}(C_1 \boxtimes ... \boxtimes C_{n-2} \boxtimes D) = Ind^{Aut(T_{q,n})}_{H_1}(C_1 \boxtimes ... \boxtimes C_{n-2} \boxtimes D)$
where

$$C_1 = Ind^{Aut(T_{q-1,n-2})\wr S_{q-1}}_{Aut(T_{q-1,n-2})\wr(S_{\ell_1}\times S_{q-\ell_1-1})}(Ind^{Aut(T_{q-1,n-2})\wr S_{\ell_1}}_{Aut(T_{q-1,n-2})\wr(S_1\times...\times S_1)}(\widetilde{\sigma^{1\boxtimes 1}_1} \otimes \overline{S^1}) \boxtimes ...\boxtimes (\widetilde{\sigma^{1\boxtimes 1}_{\ell_1}} \otimes \overline{S^1})) \boxtimes (Ind^{Aut(T_{q-1,n-2})\wr S_{q-\ell_1-1}}_{Aut(T_{q-1,n-2})\wr(S_{\alpha^1_1}\times...\times S_{\alpha^1_{\ell_1}})}(\widetilde{\sigma^{1\boxtimes\alpha^1_1}_1} \otimes \overline{S^{r^1_1}}) \boxtimes ... \boxtimes (\widetilde{\sigma^{1\boxtimes\alpha^1_{\ell_1}}_{\ell_1}} \otimes \overline{S^{r^1_{\ell_1}}}));$$

$$\vdots$$

$$C_{n-2} = Ind_{Aut(T_{q-1,1})\wr(S_{\ell_{n-2}}\times S_{q-\ell_{n-2}-1})}^{Aut(T_{q-1,1})\wr S_{q-1}}(Ind_{Aut(T_{q-1,1})\wr(S_1\times\ldots\times S_1)}^{Aut(T_{q-1,1})\wr S_{\ell_{n-2}}}(\widetilde{\sigma_1^{n-2\boxtimes 1}}\otimes\overline{S^1})\boxtimes$$
$$\ldots\boxtimes(\widetilde{\sigma_{\ell_{n-2}}^{n-2\boxtimes 1}}\otimes\overline{S^1}))\boxtimes(Ind_{Aut(T_{q-1,1})\wr(S_{\alpha_1^{n-2}}\times\ldots\times S_{\alpha_{\ell_{n-2}}^{n-2}})}^{Aut(T_{q-1,1})\wr S_{q-\ell_{n-2}-1}}(\widetilde{\sigma_1^{n-2\boxtimes\alpha_1^{n-2}}}\otimes\overline{S^{r_1^{n-2}}})\boxtimes$$
$$\ldots\boxtimes(\widetilde{\sigma_{\ell_{n-2}}^{n-2\boxtimes\alpha_{\ell_{n-2}}^{n-2}}}\otimes\overline{S^{r_{\ell_{n-2}}^{n-2}}})).$$

For $j=1,\ldots,n-2$, setting:

$$A_j^{'} = Ind_{Aut(T_{q-1,n-1-j})\wr(S_1\times\ldots\times S_1)}^{Aut(T_{q-1,n-1-j})\wr S_{\ell_j}}(\widetilde{\sigma_1^{j\boxtimes 1}}\otimes\overline{S^1})\boxtimes\ldots\boxtimes(\widetilde{\sigma_{\ell_j}^{j\boxtimes 1}}\otimes\overline{S^1})$$
$$= Ind_{Aut(T_{q-1,n-1-j})\wr(S_1\times\ldots\times S_1)}^{Aut(T_{q-1,n-1-j})\wr S_{\ell_j}}B_j^{'};$$

$$A_j^{''} = Ind_{Aut(T_{q-1,n-1-j})\wr(S_{\alpha_1^j}\times\ldots\times S_{\alpha_{\ell_j}^j})}^{Aut(T_{q-1,n-1-j})\wr S_{q-\ell_j-1}}(\widetilde{\sigma_1^{j\boxtimes\alpha_1^j}}\otimes\overline{S^{r_1^j}})\boxtimes\ldots\boxtimes(\widetilde{\sigma_{\ell_j}^{j\boxtimes\alpha_{\ell_j}^j}}\otimes\overline{S^{r_{\ell_j}^j}})$$
$$= Ind_{Aut(T_{q-1,n-1-j})\wr(S_{\alpha_1^j}\times\ldots\times S_{\alpha_{\ell_j}^j})}^{Aut(T_{q-1,n-1-j})\wr S_{q-\ell_j-1}}B_j^{''},$$

we obtain:

$$Ind_{H_1}^{Aut(T_{q,n})}(C_1\boxtimes\ldots\boxtimes C_{n-2}\boxtimes D) = Ind_{H_1}^{Aut(T_{q-1,n-1})\wr S_q}(Ind_{Aut(T_{q-1,n-2})\wr(S_{\ell_1}\times S_{q-\ell_1-1})}^{Aut(T_{q-1,n-2})\wr S_{q-1}}$$
$$(A_1^{'}\boxtimes A_1^{''})\boxtimes\ldots\boxtimes Ind_{Aut(T_{q-1,1})\wr(S_{\ell_{n-2}}\times S_{q-\ell_{n-2}-1})}^{Aut(T_{q-1,1})\wr S_{q-1}}($$
$$A_{n-2}^{'}\boxtimes A_{n-2}^{''})\boxtimes D)$$
$$= Ind_{H_1}^{Aut(T_{q-1,n-1})\wr S_q}Ind_{H_2}^{H_1}(A_1^{'}\boxtimes A_1^{''})\boxtimes\ldots\boxtimes($$
$$A_{n-2}^{'}\boxtimes A_{n-2}^{''})\boxtimes D$$
$$= Ind_{H_2}^{Aut(T_{q-1,n-1})\wr S_q}(A_1^{'}\boxtimes A_1^{''})\boxtimes\ldots\boxtimes($$
$$A_{n-2}^{'}\boxtimes A_{n-2}^{''})\boxtimes D$$
$$= Ind_{H_2}^{Aut(T_{q-1,n-1})\wr S_q}(Ind_{Aut(T_{q-1,n-2})\wr(S_1)^{\ell}}^{Aut(T_{q-1,n-2})\wr S_{\ell_1}}B_1^{'}$$
$$\boxtimes Ind_{Aut(T_{q-1,n-2})\wr(S_{\alpha_1^1}\times\ldots\times S_{\alpha_{\ell_1}^1})}^{Aut(T_{q-1,n-2})\wr S_{q-\ell_1-1}}B_1^{''})\boxtimes\ldots\boxtimes($$
$$Ind_{Aut(T_{q-1,1})\wr(S_1)^{\ell}}^{Aut(T_{q-1,1})\wr S_{\ell_{n-2}}}B_{n-2}^{'}\boxtimes$$
$$Ind_{Aut(T_{q-1,1})\wr(S_{\alpha_1^{n-2}}\times\ldots\times S_{\alpha_{\ell_{n-2}}^{n-2}})}^{Aut(T_{q-1,1})\wr S_{q-\ell_{n-2}-1}}B_{n-2}^{''})\boxtimes D.$$

Let us set:

$$E = Aut(T_{q-1,n-2})\wr(S_1\times..\times S_1\times S_{\alpha_1^1}\times..\times S_{\alpha_{\ell_1}^1})\times..\times Aut(T_{q-1,1})\wr(S_1\times..\times$$
$$S_1\times S_{\alpha_1^{n-2}}\times..\times S_{\alpha_{\ell_{n-2}}^{n-2}})\times S_{q-1}.$$

Then we have

$$Ind_{H_1}^{Aut(T_{q,n})}(C_1 \boxtimes \ldots \boxtimes C_{n-2} \boxtimes D) = Ind_E^{Aut(T_{q-1,n-1}) \wr S_q}(B_1^{'} \boxtimes B_1^{''}) \boxtimes .. \boxtimes (B_{n-2}^{'} \boxtimes B_{n-2}^{''}) \boxtimes D.$$
$$= Ind_{H_3}^{Aut(T_{q-1,n-1}) \wr S_q} Ind_{H_4}^{H_3}(B_1^{'} \boxtimes B_1^{''}) \boxtimes \ldots \boxtimes (B_{n-2}^{'} \boxtimes B_{n-2}^{''}) \boxtimes D.$$

that is

$$Ind_{H_1}^{Aut(T_{q,n})}(C_1 \boxtimes \ldots \boxtimes C_{n-2} \boxtimes D) = Ind_{H_3}^{Aut(T_{q-1,n-1}) \wr S_q}(Ind_{Aut(T_{q-1,n-2}) \wr (S_1 \times S_{\alpha_1^1} \times \ldots \times S_1 \times S_{\alpha_{\ell_1}^1})}^{Aut(T_{q-1,n-2}) \wr (S_{1+\alpha_1^1} \times \ldots \times S_{1+\alpha_{\ell_1}^1})} B_1^{'} \boxtimes B_1^{''}) \boxtimes \ldots \boxtimes (Ind_{Aut(T_{q-1,1}) \wr (S_1 \times S_{\alpha_1^{n-2}} \times \ldots \times S_1 \times S_{\alpha_{\ell_{n-2}}^{n-2}})}^{Aut(T_{q-1,1}) \wr (S_{1+\alpha_1^{n-2}} \times \ldots \times S_{1+\alpha_{\ell_{n-2}}^{n-2}})} B_{n-2}^{'} \boxtimes B_{n-2}^{''}) \boxtimes D.$$

Moreover, if we set:
$F_1 = Aut(T_{q-1,n-2}) \wr (S_{1+\alpha_1^1} \times \ldots \times S_{1+\alpha_{\ell_1}^1})$ and
$F_2 = Aut(T_{q-1,n-2}) \wr (S_1 \times S_{\alpha_1^1} \times \ldots \times S_1 \times S_{\alpha_{\ell_1}^1})$ then,

$$Ind_{F_2}^{F_1} B_1^{'} \boxtimes B_1^{''} = Ind_{F_2}^{F_1}((\widetilde{\sigma_1^{1\boxtimes 1}} \otimes \overline{S^1}) \boxtimes \ldots \boxtimes (\widetilde{\sigma_{\ell_1}^{1\boxtimes 1}} \otimes \overline{S^1})) \boxtimes ((\widetilde{\sigma_1^{1\boxtimes \alpha_1^1}} \otimes \overline{S^{r_1^1}}) \boxtimes \ldots \boxtimes (\widetilde{\sigma_{\ell_1}^{1\boxtimes \alpha_{\ell_1}^1}} \otimes \overline{S^{r_{\ell_1}^1}}))$$
$$= Ind_{F_2}^{F_1}((\widetilde{\sigma_1^{1\boxtimes 1}} \otimes \overline{S^1}) \boxtimes (\widetilde{\sigma_1^{1\boxtimes \alpha_1^1}} \otimes \overline{S^{r_1^1}}) \boxtimes \ldots \boxtimes (\widetilde{\sigma_{\ell_1}^{1\boxtimes 1}} \otimes \overline{S^1}) \boxtimes (\widetilde{\sigma_{\ell_1}^{1\boxtimes \alpha_{\ell_1}^1}} \otimes \overline{S^{r_{\ell_1}^1}}))$$

which can be written as:

$$Ind_{F_2}^{F_1} B_1^{'} \boxtimes B_1^{''} = Ind_{Aut(T_{q-1,n-2}) \wr (S_1 \times S_{\alpha_1^1})}^{Aut(T_{q-1,n-2}) \wr (S_{1+\alpha_1^1})}(\widetilde{\sigma_1^{1\boxtimes 1}} \otimes \overline{S^1}) \boxtimes (\widetilde{\sigma_1^{1\boxtimes \alpha_1^1}} \otimes \overline{S^{r_1^1}}) \boxtimes \ldots \boxtimes Ind_{Aut(T_{q-1,n-2}) \wr (S_1 \times S_{\alpha_{\ell_1}^1})}^{Aut(T_{q-1,n-2}) \wr (S_{1+\alpha_{\ell^1}^1})}(\widetilde{\sigma_{\ell_1}^{1\boxtimes 1}} \otimes \overline{S^1}) \boxtimes (\widetilde{\sigma_{\ell_1}^{1\boxtimes \alpha_{\ell_1}^1}} \otimes \overline{S^{r_{\ell_1}^1}})$$

or

$$Ind_{F_2}^{F_1} B_1^{'} \boxtimes B_1^{''} = Ind_{Aut(T_{q-1,n-2}) \wr (S_1 \times S_{\alpha_1^1})}^{Aut(T_{q-1,n-2}) \wr (S_{1+\alpha_1^1})}(\widetilde{\sigma_1^{1\boxtimes 1+\alpha_1^1}} \otimes \overline{S^1 \boxtimes S^{r_1^1}}) \boxtimes \ldots \boxtimes Ind_{Aut(T_{q-1,n-2}) \wr (S_1 \times S_{\alpha_{\ell_1}^1})}^{Aut(T_{q-1,n-2}) \wr (S_{1+\alpha_{\ell_1}^1})}(\widetilde{\sigma_{\ell_1}^{1\boxtimes 1+\alpha_{\ell_1}^1}} \otimes \overline{S^1 \boxtimes S^{r_{\ell_1}^1}})$$
$$= (\widetilde{\sigma_1^{1\boxtimes 1+\alpha_1^1}} \otimes \overline{Ind_{S_1 \times S_{\alpha_1^1}}^{S_{1+\alpha_1^1}} S^1 \boxtimes S^{r_1^1}}) \boxtimes \ldots \boxtimes (\widetilde{\sigma_1^{1\boxtimes 1+\alpha_{\ell_1}^1}} \otimes \overline{Ind_{S_1 \times S_{\alpha_{\ell_1}^1}}^{S_{1+\alpha_{\ell_1}^1}} S^1 \boxtimes S^{r_{\ell_1}^1}}).$$

Also,

$Ind_{S_1\times S_{\alpha_1^1}}^{S_{1+\alpha_1^1}} S^1 \boxtimes S^{r_1^1} = \bigoplus_{\beta_1^1\vdash(1+\alpha_1^1)} S^{\beta_1^1}$ and $Ind_{S_1\times S_{\alpha_{\ell_1}^1}}^{S_{1+\alpha_{\ell_1}^1}} S^1 \boxtimes S^{r_{\ell_1}^1} = \bigoplus_{\beta_{\ell_1}^1\vdash(1+\alpha_{\ell_1}^1)} S^{\beta_{\ell_1}^1}$

that is $\widetilde{\sigma_1^{1\boxtimes 1+\alpha_1^1}} \otimes \overline{Ind_{S_1\times S_{\alpha_1^1}}^{S_{1+\alpha_1^1}} S^1 \boxtimes S^{r_1^1}} = \widetilde{\sigma_1^{1\boxtimes 1+\alpha_1^1}} \otimes \bigoplus_{\beta_1^1\vdash(1+\alpha_1^1)} \overline{S^{\beta_1^1}}$ and

$Ind_{S_1\times S_{\alpha_{\ell_1}^1}}^{S_{1+\alpha_{\ell_1}^1}} S^1 \boxtimes S^{r_{\ell_1}^1} = \widetilde{\sigma_1^{1\boxtimes 1+\alpha_{\ell_1}^1}} \otimes \bigoplus_{\beta_{\ell_1}^1\vdash(1+\alpha_{\ell_1}^1)} \overline{S^{\beta_{\ell_1}^1}}$.

Therefore:

$$\begin{aligned}
Ind_{F_2}^{F_1} B_1' \boxtimes B_1'' &= \bigoplus_{\beta_1^1\vdash(1+\alpha_1^1)} (\widetilde{\sigma_1^{1\boxtimes 1+\alpha_1^1}} \otimes \overline{S^{\beta_1^1}}) \boxtimes ... \boxtimes \bigoplus_{\beta_{\ell_1}^1\vdash(1+\alpha_{\ell_1}^1)} (\widetilde{\sigma_1^{1\boxtimes 1+\alpha_{\ell_1}^1}} \otimes \overline{S^{\beta_{\ell_1}^1}}) \\
&= \bigoplus_{\beta_1^1\vdash(1+\alpha_1^1)} ... \bigoplus_{\beta_{\ell_1}^1\vdash(1+\alpha_{\ell_1}^1)} (\widetilde{\sigma_1^{1\boxtimes 1+\alpha_1^1}} \otimes \overline{S^{\beta_1^1}}) \boxtimes ... \boxtimes (\widetilde{\sigma_1^{1\boxtimes 1+\alpha_{\ell_1}^1}} \otimes \overline{S^{\beta_{\ell_1}^1}}) \\
&= \bigoplus_{(\beta_1^1,...,\beta_{\ell_1}^1)\Vdash q-1} \Lambda_1
\end{aligned}$$

where $\Lambda_1 = (\widetilde{\sigma_1^{1\boxtimes 1+\alpha_1^1}} \otimes \overline{S^{\beta_1^1}}) \boxtimes ... \boxtimes (\widetilde{\sigma_1^{1\boxtimes 1+\alpha_{\ell_1}^1}} \otimes \overline{S^{\beta_{\ell_1}^1}})$.

Using the same method, we obtain:

$$Ind_{Aut(T_{q-1,1})\wr(S_1\times S_{\alpha_1^{n-2}}\times...\times S_1\times S_{\alpha_{\ell_{n-2}}^{n-2}})}^{Aut(T_{q-1,1})\wr(S_{1+\alpha_1^{n-2}}\times...\times S_{1+\alpha_{\ell_{n-2}}^{n-2}})} B_{n-2}' \boxtimes B_{n-2}'' = \bigoplus_{(\beta_1^{n-2},...,\beta_{\ell_{n-2}}^{n-2})\Vdash q-1} \Lambda_{n-2}$$

where $\Lambda_{n-2} = (\widetilde{\sigma_1^{n-2\boxtimes 1+\alpha_1^{n-2}}} \otimes \overline{S^{\beta_1^{n-2}}}) \boxtimes ... \boxtimes (\widetilde{\sigma_{\ell_{n-2}}^{n-2\boxtimes 1+\alpha_{\ell_{n-2}}^{n-2}}} \otimes \overline{S^{\beta_{\ell_{n-2}}^{n-2}}})$.

Consequently, we have

$$\begin{aligned}
Ind_{Stab_{Aut(T_{q,n})}(0^n)}^{Aut(T_{q,n})}(C_1 \boxtimes ... \boxtimes C_{n-2} \boxtimes D) &= Ind_{H_1}^{Aut(T_{q,n})}(C_1 \boxtimes ... \boxtimes C_{n-2} \boxtimes D) \\
&= Ind_{H_3}^{Aut(T_{q-1,n-1})\wr S_q}(\bigoplus_{(\beta_1^1,...,\beta_{\ell_1}^1)\Vdash q-1} \Lambda_1 \boxtimes \\
&... \boxtimes \bigoplus_{(\beta_1^{n-2},...,\beta_{\ell_{n-2}}^{n-2})\Vdash q-1} \Lambda_{n-2} \boxtimes \bigoplus_{\lambda\vdash q-1} S^\lambda) \\
&= \bigoplus_{(\beta_1^1,...,\beta_{\ell_1}^1)\Vdash q-1} ... \bigoplus_{(\beta_1^{n-2},...,\beta_{\ell_{n-2}}^{n-2})\Vdash q-1} \bigoplus_{\lambda\vdash q-1} \\
&Ind_{H_3}^{Aut(T_{q-1,n-1})\wr S_q}(\Lambda_1 \boxtimes ... \boxtimes \Lambda_{n-2} \boxtimes S^\lambda).
\end{aligned}$$

□

We deduce from this result, that the induced representation of $Stab_{Aut(T_{q,n})}(0^n)$ on $Aut(T_{q,n})$ is multiplicity-free, so the triple $(Aut(T_{q,n}),\ Stab_{Aut(T_{q,n})}(0^n),\ C_1 \boxtimes ... \boxtimes C_{n-2} \boxtimes D)$ is commutative.

We are now interested in the triple $(Aut(T_{q,n}), Stab_{Aut(T_{q,n})}(\Sigma^1), \tau)$ where $\tau = (\widetilde{\sigma^{\boxtimes 1}} \otimes \overline{S^1}) \boxtimes ... \boxtimes (\widetilde{\sigma^{\boxtimes 1}} \otimes \overline{S^1})$.

Theorem 2.11. *Let Σ^1 be the first level of the homogeneous rooted tree $T_{q,n}$. Then, $Stab_{Aut(T_{q,n})}(\Sigma^1) \cong (Aut(T_{q-1,n-1}))^q$.*

Proof. The vertices of Σ^1 are $0; 1; 2; ...; q-1$. Thanks to theorem 2.7, $Stab_{Aut(T_{q,n})}(x_0) = Aut(T_{q-1,n-1})$ for all $x_0 \in \Sigma^1$. Moreover, $g_\emptyset$ is the permutation induced by g on the first level of the homogeneous tree $T_{q,n}$. Hence $g_\emptyset$ is an automorphism of the subtree $T_{q-1,n-1}$. It is straightforward to see that the map

$$\begin{array}{cccc} \psi & : & Stab_{Aut(T_{q,n})}(\Sigma^1) & \to & Aut(T_{q-1,n-1}) \times ... \times Aut(T_{q-1,n-1}) \\ & & g & \mapsto & (g_\emptyset, g_\emptyset, ..., g_\emptyset) \end{array}$$

is bijective. So, $Stab_{Aut(T_{q,n})}(\Sigma^1) \cong (Aut(T_{q-1,n-1}))^q$. □

Theorem 2.12. *Let $Aut(T_{q,n})$ be the group of automorphisms of the homogeneous rooted tree $T_{q,n}$, let $Stab_{Aut(T_{q,n})}(\Sigma^1)$ be a subgroup of $Aut(T_{q,n})$ and let $\tau = (\widetilde{\sigma^{\boxtimes 1}} \otimes \overline{S^1}) \boxtimes ... \boxtimes (\widetilde{\sigma^{\boxtimes 1}} \otimes \overline{S^1})$ be a representation of $Stab_{Aut(T_{q,n})}(\Sigma^1)$. Then,*

1. *if $q = 1$ or $q = 2$, $Ind^{Aut(T_{q-1,n-1}) \wr S_q}_{Stab_{Aut(T_{q,n})}(\Sigma^1)} \tau$ is multiplicity-free.*
2. *if $q > 2$, $Ind^{Aut(T_{q-1,n-1}) \wr S_q}_{Stab_{Aut(T_{q,n})}(\Sigma^1)} \tau$ is not multiplicity-free.*

Proof. We have proved in theorem 2.11 that $Stab_{Aut(T_{q,n})}(\Sigma^1) \cong (Aut(T_{q-1,n-1}))^q$. Also, we have $Aut(T_{q-1,n-1}) = Aut(T_{q-1,n-1}) \wr S_1$. So it follows that $(Aut(T_{q-1,n-1}))^q = (Aut(T_{q-1,n-1}) \wr S_1)^q$. Therefore, we have

$$\begin{aligned} (Aut(T_{q-1,n-1}) \wr S_1)^q &= Aut(T_{q-1,n-1}) \wr S_1 \times ... \times Aut(T_{q-1,n-1}) \wr S_1 \\ &= Aut(T_{q-1,n-1}) \wr (S_1 \times ... \times S_1) \\ &= Aut(T_{q-1,n-1}) \wr (\prod_{i=1}^{q} S_1). \end{aligned}$$

In the following, we set $K = Aut(T_{q-1,n-1}) \wr (\prod_{i=1}^{q} S_1)$. Let σ be an irreducible representation of $Aut(T_{q-1,n-1})$. We have

$$\begin{aligned} Ind_K^{Aut(T_{q-1,n-1}) \wr S_q} (\widetilde{\sigma^{\boxtimes 1}} \otimes \overline{S^1}) \boxtimes ... \boxtimes (\widetilde{\sigma^{\boxtimes 1}} \otimes \overline{S^1}) &= Ind_K^{Aut(T_{q-1,n-1}) \wr S_q} (\widetilde{\sigma^{\boxtimes 1}} \boxtimes ... \\ &\quad \boxtimes \widetilde{\sigma^{\boxtimes 1}}) \otimes (\overline{S^1} \boxtimes ... \boxtimes \overline{S^1}) \\ &= Ind_K^{Aut(T_{q-1,n-1}) \wr S_q} \widetilde{\sigma^{\boxtimes q}} \otimes (\\ &\quad \overline{S^1 \boxtimes ... \boxtimes S^1}) \\ &= \widetilde{\sigma^{\boxtimes q}} \otimes Ind_K^{Aut(T_{q-1,n-1}) \wr S_q} (\\ &\quad \overline{S^1 \boxtimes ... \boxtimes S^1}) \\ &= \widetilde{\sigma^{\boxtimes q}} \otimes \overline{Ind^{S_q}_{\prod_{i=1}^{q} S_1} S^1 \boxtimes ... \boxtimes S^1} \end{aligned}$$

Moreover, $Ind^{S_q}_{\prod_{i=1}^{q} S_1}\ S^1 \boxtimes ... \boxtimes S^1 = Ind^{S_q}_{\{id\}}\ \mathbf{1}_{\{id\}} \cong \mathbb{C}[S_q]$ and $\mathbb{C}[S_q] = \bigoplus_{\lambda \vdash q} (dim S^\lambda)\ S^\lambda$. Therefore we have:

$$\begin{aligned}\widetilde{\sigma^{\boxtimes q}} \otimes \overline{Ind^{S_q}_{\prod_{i=1}^{q} S_1} S^1 \boxtimes ... \boxtimes S^1} &= \widetilde{\sigma^{\boxtimes q}} \otimes \Big(\bigoplus_{\lambda \vdash q} (dim S^\lambda) \overline{S^\lambda}\Big) \\ &= \bigoplus_{\lambda \vdash q} (dim S^\lambda)(\widetilde{\sigma^{\boxtimes q}} \otimes \overline{S^\lambda}).\end{aligned}$$

Hence, $Ind_K^{Aut(T_{q-1,n-1}) \wr S_q}(\widetilde{\sigma^{\boxtimes 1}} \otimes \overline{S^1}) \boxtimes ... \boxtimes (\widetilde{\sigma^{\boxtimes 1}} \otimes \overline{S^1}) = \bigoplus_{\lambda \vdash q} (dim S^\lambda)(\widetilde{\sigma^{\boxtimes q}} \otimes \overline{S^\lambda})$.

Thus,

(1) if $q = 1$ or $q = 2$ then $dim S^\lambda \leq 1$ for any $\lambda \vdash q$, so $Ind^{Aut(T_{q-1,n-1}) \wr S_q}_{Aut(T_{q-1,n-1}) \wr (\prod_{i=1}^{q} S_1)} \tau$ $= \bigoplus_{\lambda \vdash q} (\widetilde{\sigma^{\boxtimes q}} \otimes \overline{S^\lambda}\ )$ and $Ind^{Aut(T_{q-1,n-1}) \wr S_q}_{Stab_{Aut(T_{q,n})}(\Sigma^1)} \tau$ is multiplicity-free.

(2) if q > 2, then $dim S^\lambda > 1$ for some $\lambda \vdash q$, so $Ind^{Aut(T_{q-1,n-1}) \wr S_q}_{Stab_{Aut(T_{q,n})}(\Sigma^1)} \tau$ is not multiplicity-free.

□

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