

EXISTENCE OF A WEAK SOLUTIONS TO A CLASS OF NONLINEAR PARABOLIC PROBLEMS VIA TOPOLOGICAL DEGREE METHOD

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ABSTRACT. In this paper, we study the existence of "weak solutions" for a class of nonlinear parabolic problems of the type

$$\frac{\partial u}{\partial t} - \operatorname{div} \mathcal{A}(x, t, \nabla u) = \phi(x, t) + \operatorname{div} \mathcal{B}(x, t, u, \nabla u).$$

Using Berkovits and Mustonen's topological degree theory, we demonstrate the existence of a weak solutions to the problems under consideration in the space $L^p(0, T; W_0^{1,p}(\Omega))$, where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$ and $p \geq 2$.

1. INTRODUCTION

During this whole work, we will assume that Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a Lipschitz boundary denoted by $\partial\Omega$, $\Omega_T = \Omega \times (0, T)$ is a cylinder, and that $\Gamma = \partial\Omega \times (0, T)$ is its lateral surface, where $T > 0$ is a fixing time.

In this paper, we consider the following nonlinear parabolic initial boundary value problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div} \mathcal{A}(x, t, \nabla u) = \phi(x, t) + \operatorname{div} \mathcal{B}(x, t, u, \nabla u) & \text{in } \Omega_T, \\ u(x, 0) = u_0(x) & \text{in } \Omega, \\ u(x, t) = 0 & \text{in } \Gamma, \end{cases} \quad (1.1)$$

where $\mathcal{A} : \Omega_T \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ and $\mathcal{B} : \Omega_T \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ are Carathéodory's functions that satisfy some conditions.

The study of nonlinear partial differential problems is highly motivated by various physics phenomena, specifically problems related to non-Newtonian fluids with strongly inhomogeneous behavior and a high ability to increase their viscosity under a different stimulus, such as shear rate, magnetic or electric field [21].

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In [15] Lions proved that the following problem

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div} a(x, t, u, \nabla u) = f(x, t) & \text{in } \Omega_T, \\ u(x, 0) = 0, & \text{in } \Omega, \\ u(x, t) = 0, & \text{in } \Gamma, \end{cases}$$

admits at least one solutions $u \in \mathcal{W}$.

In the case when $\mathcal{A} \equiv 0$, we know that the problem has a weak solution solutions from [12] (also, we mention some works in this direction [4, 10, 13, 20]).

Motivated by above, we study (1), on the one hand as a kind of generalisation of the previous work, and on the other hand, using another approach based on the topological degree theory for operators of the type $\mathcal{L} + \mathcal{S}$, where $\mathcal{L}$ is a linear densely defined maximal monotone map and $\mathcal{S}$ is a bounded demicontinuous map of type (S_+) with respect to a domain of $\mathcal{L}$. For more information on the history of this theory, the reader should consult [1, 2, 3, 6, 7, 9, 16, 17, 19].

Let us rapidly summarize the work's contents. In the next section, we recall some basic preliminaries. Section 3 introduces some classes of mapping (S_+) type, followed by the Berkovits and Mustonen topological degree. Section 4 is devoted to basic assumptions, some necessary lemmas and the main result.

2. MATHEMATICAL BACKGROUND

In this section, we recall some necessary definitions and basic results required for further developments.

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set and $p \geq 2$. We denote by $L^p(\Omega)$ the space of all measurable functions $v \in \Omega$ such that

$$\|v\|_{L^p(\Omega)} = \left(\int_{\Omega} |v(x)|^p dx \right)^{1/p} < \infty.$$

We define the Sobolev Space $W^{1,p}(\Omega) = \{v \in L^p(\Omega) : \nabla v \in L^p(\Omega)\}$ equipped with the norm

$$\|v\|_{W^{1,p}(\Omega)} = \left(\|v\|_{L^p(\Omega)}^p + \|\nabla v\|_{L^p(\Omega)}^p \right)^{1/p}.$$

Let the space $W_0^{1,p}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in the Sobolev space $W^{1,p}(\Omega)$. Note that, according to the Poincaré inequality, the norm $\|\cdot\|_{W^{1,p}(\Omega)}$ on $W^{1,p}(\Omega)$ is equivalent to the norm $\|\cdot\|_{W_0^{1,p}(\Omega)}$ defined as

$$\|v\|_{W_0^{1,p}(\Omega)} = \|\nabla v\|_{L^p(\Omega)} \text{ for all } v \in W_0^{1,p}(\Omega).$$

Again, the Sobolev space $W_0^{1,p}(\Omega)$ is a uniformly convex Banach space and the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact (see [22]).

In this work, we consider the following space

$$\mathcal{W} := L^p(0, T; W_0^{1,p}(\Omega)),$$

in this space, we defined the norm

$$|v|_{\mathcal{W}} = \left(\int_0^T \|v\|_{W^{1,p}(\Omega)}^p dt \right)^{1/p}.$$

Thanks to Poincaré inequality, the expression

$$\|v\|_{\mathcal{W}} = \left(\int_0^T \|\nabla v\|_{L^p(\Omega)}^p dt \right)^{1/p},$$

is a norm defined on $\mathcal{W}$ and is equivalent to the norm $|v|_{\mathcal{W}}$.

Note that $(\mathcal{W}, \|\cdot\|_{\mathcal{W}})$ is a separable and reflexive Banach space.

In this work, we use the following results.

Theorem 2.1. [11, 18] *Let $1 < p < \infty$. If $u_n \rightarrow u$ in $L^p(\Omega_T)$, then there exist a subsequence (u_k) and $\psi \in L^p(\Omega_T)$ such that*

- (i): $u_k(y) \rightarrow u(y)$, a.e. on Ω_T .
- (ii): $|u_k(y)| \leq \psi(y)$, a.e. on Ω_T .

Lemma 2.2. [5] *Let $1 < p < \infty$, $(f_n)_n \subset L^p(\Omega_T)$ and $f \in L^p(\Omega_T)$ such that $\|f_n\|_{L^p(\Omega_T)} \leq C$. If $f_n(y) \rightarrow f(y)$ a.e. in Ω_T then $f_n \rightharpoonup f$ in $L^p(\Omega_T)$.*

3. SOME CLASSES OF MAPPINGS AND TOPOLOGICAL DEGREE THEORY

Now we'll look at some topological degree mappings, results, and properties.

In what follows, let Y is a real reflexive and separable Banach space with dual Y^* and continuous pairing $\langle \cdot, \cdot \rangle$, and given a nonempty subset Ω of Y , $\partial\Omega$ and $\overline{\Omega}$ represent the boundary and the closure of Ω in Y , respectively. Strong (weak) convergence is represented by the symbol $\rightarrow$ ($\rightharpoonup$).

Definition 3.1. We consider a mapping $\mathcal{T}$ defined from Y to Y^* and its graph is given by

$$G(\mathcal{T}) = \{(u, v) \in Y \times Y^* : v \in \mathcal{T}(u)\}.$$

- (1) $\mathcal{T}$ is said to be *monotone* if for all $(u_1, v_1), (u_2, v_2)$ in $G(\mathcal{T})$, we get that

$$\langle v_1 - v_2, u_1 - u_2 \rangle \geq 0.$$

- (2) $\mathcal{T}$ is said to be *maximal monotone* if it is monotone and maximal in the sense of graph inclusion among monotone mappings from Y to Y^* , or for any $(u_0, v_0) \in Y \times Y^*$ for which $\langle v_0 - v, u_0 - u \rangle \geq 0$, for all $(u, v) \in G(\mathcal{T})$, we have $(u_0, v_0) \in G(\mathcal{T})$.

Definition 3.2. Let Z be a real Banach space. A operator $\mathcal{T} : \Omega \subset Y \rightarrow Z$ is said to be :

- (1) bounded, if it takes any bounded set into a bounded set.
- (2) demicontinuous, if for any sequence $(u_n) \subset \Omega$, $u_n \rightarrow u$ implies $\mathcal{T}(u_n) \rightharpoonup \mathcal{T}(u)$.
- (3) compact, if it is continuous and the image of any bounded set is relatively compact.

Definition 3.3. A mapping $\mathcal{S} : D(\mathcal{S}) \subset Y \rightarrow Y^*$ is said to be :

- (1) of type (S_+) , if for any $(u_n) \subset D(\mathcal{S})$ with $u_n \rightharpoonup u$ and $\limsup_{n \rightarrow \infty} \langle \mathcal{S}u_n, u_n - u \rangle \leq 0$, it follows that $u_n \rightarrow u$.
- (2) quasimonotone, if for any sequence $(u_n) \subset D(\mathcal{S})$ with $u_n \rightharpoonup u$, we have

$$\limsup_{n \rightarrow \infty} \langle \mathcal{S}u_n, u_n - u \rangle \geq 0.$$

Definition 3.4. Let $\mathcal{T} : \Omega_1 \subset Y \rightarrow Y^*$ be a bounded operator such that $\Omega \subset \Omega_1$. For any operator $\mathcal{S} : \Omega \subset Y \rightarrow Y$, we say that $\mathcal{S}$ of type $(S_+)_{\mathcal{T}}$, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, $y_n := \mathcal{T}u_n \rightharpoonup y$ and $\limsup_{n \rightarrow \infty} \langle \mathcal{S}u_n, y_n - y \rangle \leq 0$, we have $u_n \rightarrow u$.

In the sequel, let $\mathcal{L}$ be a linear maximal monotone map from $D(\mathcal{L}) \subset Y$ to Y^* , and we consider the following classes of operators for each open and bounded subset G on Y :

$$\begin{aligned} \mathcal{F}_1(\Omega) &:= \left\{ F : \Omega \rightarrow Y^* : F \text{ is bounded, demicontinuous and of type } (S_+) \right\}, \\ \mathcal{F}_{\mathcal{T}}(\Omega) &:= \left\{ F : \Omega \rightarrow X : F \text{ is demicontinuous and of type } (S_+)_{\mathcal{T}} \right\}, \\ \mathcal{F}_G &:= \left\{ \mathcal{L} + \mathcal{S} : \overline{G} \cap D(\mathcal{L}) \rightarrow Y^* : \mathcal{S} \text{ is bounded, demicontinuous map of type } (S_+) \right. \\ &\quad \left. \text{with respect to } D(\mathcal{L}) \text{ from } \overline{G} \text{ to } Y^* \right\}, \\ \mathcal{H}_G &:= \left\{ \mathcal{L} + \mathcal{S}(t) : \overline{G} \cap D(\mathcal{L}) \rightarrow Y^* : \mathcal{S}(t) \text{ is a bounded homotopy map of type } (S_+) \right. \\ &\quad \left. \text{with respect to } D(\mathcal{L}) \text{ from } \overline{G} \text{ to } Y^* \right\}. \end{aligned}$$

Definition 3.5. Let E be a bounded open subset of a real reflexive Banach space Y , $\mathcal{T} \in \mathcal{F}_1(\overline{E})$ be continuous and let $F, \mathcal{S} \in \mathcal{F}_{\mathcal{T}}(\overline{E})$. The affine homotopy $\Pi : [0, 1] \times \overline{E} \rightarrow Y$ defined by

$$\Pi(t, u) := (1 - t)Fu + t\mathcal{S}u, \quad \text{for all } (t, u) \in [0, 1] \times \overline{E}$$

is called an admissible affine homotopy.

Remark 3.6. Note that the class $\mathcal{H}_G$ includes all affine homotopies

$$\mathcal{L} + (1 - t)\mathcal{S}_1 + t\mathcal{S}_2, \quad \text{with } (\mathcal{L} + \mathcal{S}_i) \in \mathcal{F}_G, \quad i = 1, 2.$$

Now, we introduce the Berkovits and Mustonen topological degree for the class $\mathcal{F}_G$, and see [6, 7] for more informations.

Theorem 3.7. Let $\mathcal{L}$ be a linear maximal monotone densely defined map from $D(\mathcal{L}) \subset Y$ to Y^* , and let

$$\mathcal{E} = \{(F, G, \phi) : F \in \mathcal{F}_G, \quad G \text{ an open bounded subset in } Y, \quad \phi \notin F(\partial G \cap D(\mathcal{L}))\}.$$

Then, there exists a topological degree function $d : \mathcal{E} \rightarrow \mathbb{Z}$ satisfying the following properties:

- (1) *(Existence)* if $d(F, G, \phi) \neq 0$, then the equation $Fu = \phi$ has a solutions in $G \cap D(\mathcal{L})$.
- (2) *(Additivity)* If G_1 and G_2 are two disjoint open subsets of G such that $\phi \notin F[(\overline{G} \setminus (G_1 \cup G_2)) \cap D(\mathcal{L})]$, then we have

$$d(F, G, \phi) = d(F, G_1, \phi) + d(F, G_2, \phi).$$

- (3) *(Homotopy invariance)* If $F(t) \in \mathcal{H}_G$ and $\phi(t) \notin F(t)(\partial G \cap D(\mathcal{L}))$ for all $t \in [0, 1]$, where $\phi(t)$ is a continuous curve in Y^* , then

$$d(F(t), G, \phi(t)) = \text{const}, \quad \forall t \in [0, 1].$$

- (4) (Normalization) $\mathcal{L} + \mathcal{J}$ is a normalising map, where $\mathcal{J}$ is the duality mapping of Y into Y^* , that is,

$$d(\mathcal{L} + \mathcal{J}, G, \phi) = 1, \text{ for all } \phi \in (\mathcal{L} + \mathcal{J})(G \cap D(\mathcal{L})).$$

Lemma 3.8. *Let $\mathcal{L} + \mathcal{S} \in \mathcal{F}_Y$ and $\phi \in Y^*$ and assume that there exists a radius $r > 0$ such that*

$$\langle \mathcal{L}u + \mathcal{S}u - \phi, u \rangle > 0, \quad (3.1)$$

for all $u \in \partial B_r(0) \cap D(\mathcal{L})$. Then the equation $\mathcal{L}u + \mathcal{S}u = \phi$ has a solutions u in $D(\mathcal{L})$.

Proof. To show this lemma, it suffices to prove that $(\mathcal{L} + \mathcal{S})(D(\mathcal{L})) = Y^*$.

Let $F_\varepsilon(t, u) = \mathcal{L}u + (1 - t)\mathcal{J}u + t(\mathcal{S}u + \varepsilon\mathcal{J}u - \phi)$, for all $\varepsilon > 0$ and $t \in [0, 1]$.

From (3.1) and since $0 \in \mathcal{L}(0)$, we obtain

$$\begin{aligned} \langle F_\varepsilon(t, u), u \rangle &= \langle t(\mathcal{L}u + \mathcal{S}u - \phi, u) + \langle (1 - t)\mathcal{L}u + (1 - t + \varepsilon)\mathcal{J}u, u \rangle \\ &\geq \langle (1 - t)\mathcal{L}u + (1 - t + \varepsilon)\mathcal{J}u, u \rangle \\ &= (1 - t)\langle \mathcal{L}u, u \rangle + (1 - t + \varepsilon)\langle \mathcal{J}u, u \rangle \\ &\geq (1 - t + \varepsilon)\|u\|^2 \\ &= (1 - t + \varepsilon)r^2 > 0, \end{aligned}$$

then $0 \notin F_\varepsilon(t, u)$.

We are $\mathcal{J}$ and $\mathcal{S} + \varepsilon\mathcal{J}$ are continuous, bounded and of type (S_+) , then $\{F_\varepsilon(t, \cdot)\}_{t \in [0, 1]}$ is an admissible homotopy. Therefore, applying the homotopy invariance and normalisation property of the degree d stated in Theorem 3.7, we obtain

$$d(F_\varepsilon(t, \cdot), B_r(0), 0) = d(\mathcal{L} + \mathcal{J}, B_r(0), 0) = 1 \neq 0.$$

Consequently, by existence property of the degree d there exists a point $u_\varepsilon \in D(\mathcal{L})$ such that $0 \in F_\varepsilon(t, \cdot)$. In particular, by setting $\varepsilon \rightarrow 0^+$ and $t = 1$, we get $\phi \in (\mathcal{L} + \mathcal{S})(D(\mathcal{L}))$ for some $u \in D(\mathcal{L})$ and that for all $\phi \in Y^*$ (ϕ is arbitrary). Which implies that $(\mathcal{L} + \mathcal{S})(D(\mathcal{L})) = Y^*$. $\square$

4. MAIN RESULT

4.1. Hypotheses and technical lemmas. In this subsection, we concentrate on the fundamental assumptions and operators associated with our problem, as well as some useful technical lemma for proving existence result.

We suppose in this paper that the operators $\mathcal{A} : \Omega_T \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ and $\mathcal{B} : \Omega_T \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ are Carathéodory's functions, such that there exist $c_1, c_2, \alpha_1, \alpha_2$ positive constants and $k_1, k_2 \in L^{p'}(\Omega_T)$ such that

$$|\mathcal{A}(x, t, \zeta)| \leq k_1(x, t) + c_1|\zeta|^{p-1}, \quad |\mathcal{B}(x, t, \eta, \zeta)| \leq k_2(x, t) + c_2(|\eta|^{p-1} + |\zeta|^{p-1}), \quad (4.1)$$

$$(\mathcal{A}(x, t, \zeta) - \mathcal{A}(x, t, \zeta'))(\zeta - \zeta') > 0, \quad (\mathcal{B}(x, t, \eta, \zeta) - \mathcal{B}(x, t, \eta, \zeta'))(\zeta - \zeta') > 0, \quad (4.2)$$

$$\mathcal{A}(x, t, \zeta) \cdot \zeta \geq \alpha_1|\zeta|^p, \quad \mathcal{B}(x, t, \eta, \zeta) \cdot \zeta \geq \alpha_2|\zeta|^p, \quad (4.3)$$

for all $(x, t) \in \Omega_T, \eta \in \mathbb{R}$ and $(\zeta', \zeta) \in \mathbb{R}^N \times \mathbb{R}^N$ with $\zeta' \neq \zeta$.

Now, we give the propertie of the related operator which will be used later.

Lemma 4.1. *Assume that the assumptions (4.1) – (4.3) hold. Then the operator $\mathcal{S} : \mathcal{W} \rightarrow \mathcal{W}^*$ defined by*

$$\langle \mathcal{S}u, v \rangle_{\mathcal{W}^*, \mathcal{W}} = \int_{\Omega_T} \left(\mathcal{A}(x, t, \nabla u) + \mathcal{B}(x, t, u, \nabla u) \right) \cdot \nabla v dx dt,$$

is bounded, continuous and of type (S_+) .

Proof. Firstly, let's show that the operator $\mathcal{S}$ is bounded.

By using the Hölder's inequality, we have for all $u, v \in \mathcal{W}$

$$\begin{aligned} |\langle \mathcal{S}u, v \rangle| &= \left| \int_0^T \left(\int_{\Omega} \left(\mathcal{A}(x, t, \nabla u) + \mathcal{B}(x, t, u, \nabla u) \right) \cdot \nabla v dx \right) dt \right| \\ &\leq \int_0^T \left(\int_{\Omega} |\mathcal{A}(x, t, \nabla u)| \cdot |\nabla v| dx \right) dt + \int_0^T \left(\int_{\Omega} |\mathcal{B}(x, t, u, \nabla u)| \cdot |\nabla v| dx \right) dt \\ &\leq \int_0^T \|\mathcal{A}(x, t, \nabla u)\|_{L^{p'}(\Omega)} \|\nabla v\|_{L^p(\Omega)} dt + \int_0^T \|\mathcal{B}(x, t, u, \nabla u)\|_{L^{p'}(\Omega)} \|\nabla v\|_{L^p(\Omega)} dt \\ &\leq \left[\|\mathcal{A}(x, t, \nabla u)\|_{L^{p'}(\Omega_T)} + \|\mathcal{B}(x, t, u, \nabla u)\|_{L^{p'}(\Omega_T)} \right] \|v\|_{L^1(0, T; W_0^{1,p}(\Omega))}. \end{aligned}$$

Thanks to (4.1), we can easily prove that $\|\mathcal{A}(x, t, \nabla u)\|_{L^{p'}(\Omega_T)}$ and $\|\mathcal{B}(x, t, u, \nabla u)\|_{L^{p'}(\Omega_T)}$ are bounded for all $u \in W_0^{1,p}(\Omega)$. Therefore

$$|\langle \mathcal{S}u, v \rangle| \leq \text{const} \|v\|_{L^1(0, T; W_0^{1,p}(\Omega))}.$$

By the continuous embedding $\mathcal{W} \hookrightarrow L^1(0, T; W_0^{1,p}(\Omega))$, we concludes that

$$|\langle \mathcal{S}u, v \rangle| \leq \text{const} \|v\|_{\mathcal{W}},$$

which means that the operator $\mathcal{S}$ is bounded.

Secondly, we show that $\mathcal{S}$ is continuous. Let $u_n \rightarrow u$ in $\mathcal{W}$ and $v \in \mathcal{W}$. By using the Hölder's inequality, we have

$$\begin{aligned} |\langle \mathcal{S}u_n - \mathcal{S}u, v \rangle| &\leq \int_0^T \left(\int_{\Omega} |\mathcal{A}(x, t, \nabla u_n) - \mathcal{A}(x, t, \nabla u)| \cdot |\nabla v| dx \right) dt \\ &\quad + \int_0^T \left(\int_{\Omega} |\mathcal{B}(x, t, u, \nabla u_n) - \mathcal{B}(x, t, u, \nabla u)| \cdot |\nabla v| dx \right) dt \\ &\leq \int_0^T \|\mathcal{A}(x, t, \nabla u_n) - \mathcal{A}(x, t, \nabla u)\|_{L^{p'}(\Omega)} \|\nabla v\|_{L^p(\Omega)} dt \\ &\quad + \int_0^T \|\mathcal{B}(x, t, u, \nabla u_n) - \mathcal{B}(x, t, u, \nabla u)\|_{L^{p'}(\Omega)} \|\nabla v\|_{L^p(\Omega)} dt \\ &\leq \left[\|\mathcal{A}(x, t, \nabla u_n) - \mathcal{A}(x, t, \nabla u)\|_{L^{p'}(\Omega_T)} \right. \\ &\quad \left. + \|\mathcal{B}(x, t, u, \nabla u_n) - \mathcal{B}(x, t, u, \nabla u)\|_{L^{p'}(\Omega_T)} \right] \|v\|_{\mathcal{W}}, \end{aligned}$$

so, we need to show that

$$\|\mathcal{A}(x, t, \nabla u_n) - \mathcal{A}(x, t, \nabla u)\|_{L^{p'}(\Omega_T)} \rightarrow 0,$$

and

$$\|\mathcal{B}(x, t, u_n, \nabla u_n) - \mathcal{B}(x, t, u, \nabla u)\|_{L^{p'}(\Omega_T)} \rightarrow 0.$$

However, notice that if $u_n \rightarrow u$ in $\mathcal{W}$, then $u_n \rightarrow u$ and $\nabla u_n \rightarrow \nabla u$ in $(L^p(\Omega_T))^N$. Hence, by Theorem 2.1, there exist a subsequence (u_k) of (u_n) and φ in $L^p(\Omega_T)$ and ψ in $(L^p(\Omega_T))^N$ such that

$$u_k \rightarrow u \text{ and } \nabla u_k \rightarrow \nabla u,$$

$$|u_k(x, t)| \leq \varphi(x, t) \text{ and } |\nabla u_k(x, t)| \leq |\psi(x, t)|, \quad (4.4)$$

for a.e. $(x, t) \in \Omega_T$ and all $k \in \mathbb{N}$.

Then, given that $\mathcal{A}$ and $\mathcal{B}$ are Carathéodory functions we can deduce

$$\mathcal{A}(x, t, \nabla u_k(x, t)) \rightarrow \mathcal{A}(x, t, \nabla u(x, t)) \text{ a.e. } (x, t) \in \Omega_T, \quad (4.5)$$

$$\mathcal{B}(x, t, u_k, \nabla u_k(x, t)) \rightarrow \mathcal{B}(x, t, u, \nabla u(x, t)) \text{ a.e. } (x, t) \in \Omega_T. \quad (4.6)$$

On the other hand, in view of (4.1), we get

$$|\mathcal{A}(x, t, \nabla u_k(x, t))| \leq k_1(x, t) + c_1|\psi(x, t)|^{p-1},$$

$$|\mathcal{B}(x, t, u_k, \nabla u_k(x, t))| \leq k_2(x, t) + c_2\left(|\varphi(x, t)|^{p-1} + |\psi(x, t)|^{p-1}\right)$$

for a.e. $(x, t) \in \Omega_T$.

Or

$$k_1 + c_1|\psi|^{p-1} \in L^{p'}(\Omega_T) \text{ and } k_2 + c_2\left(|\varphi|^{p-1} + |\psi|^{p-1}\right) \in L^{p'}(\Omega_T),$$

therefore, thanks to (4.5), (4.6) and the dominated convergence theorem, we obtain

$$\mathcal{A}(x, t, \nabla u_k(x, t)) \rightarrow \mathcal{A}(x, t, \nabla u(x, t)) \text{ in } (L^{p'}(\Omega_T))^N,$$

$$\mathcal{B}(x, t, u_k, \nabla u_k(x, t)) \rightarrow \mathcal{B}(x, t, u, \nabla u(x, t)) \text{ in } (L^{p'}(\Omega_T))^N.$$

Thus, in view of the convergence principle in Banach spaces, we conclude that

$$\mathcal{A}(x, t, \nabla u_n(x, t)) \rightarrow \mathcal{A}(x, t, \nabla u(x, t)) \text{ in } (L^{p'}(\Omega_T))^N, \quad (4.7)$$

$$\mathcal{B}(x, t, u_n, \nabla u_n(x, t)) \rightarrow \mathcal{B}(x, t, u, \nabla u(x, t)) \text{ in } (L^{p'}(\Omega_T))^N. \quad (4.8)$$

According to (4.7) and (4.8), we deduce that

$$\langle \mathcal{S}u_n - \mathcal{S}u, v \rangle \rightarrow 0, \text{ for all } v \in \mathcal{W},$$

that means, the operator $\mathcal{S}$ is continuous.

Next, we will show that the operator $\mathcal{S}$ is of type (S_+) . Let $(u_n)_n \subset \mathcal{W}$ such that

$$\begin{cases} u_n \rightharpoonup u \text{ in } \mathcal{W}, \\ \limsup_{n \rightarrow \infty} \langle \mathcal{S}u_n, u_n - u \rangle \leq 0. \end{cases} \quad (4.9)$$

We will prove that $u_n \rightarrow u$ in $\mathcal{W}$. Since $u_n \rightharpoonup u$ in $\mathcal{W}$ and $\mathcal{W}$ embeds compactly in $L^p(\Omega_T)$

([15, Theorem 5.1]), then there exist a subsequence still denoted by (u_n) such that

$$u_n \rightarrow u \quad \text{in } L^p(\Omega_T).$$

On the other hand, we have

$$\begin{aligned}
 \limsup_{n \rightarrow \infty} \langle \mathcal{S}u_n, u_n - u \rangle &= \limsup_{n \rightarrow \infty} \langle \mathcal{S}u_n - \mathcal{S}u, u_n - u \rangle \\
 &= \limsup_{n \rightarrow \infty} \left[\int_{\Omega_T} \left(\mathcal{A}(x, t, \nabla u_n(x, t)) - \mathcal{A}(x, t, \nabla u(x, t)) \right) \cdot (\nabla u_n - \nabla u) dx dt \right. \\
 &\quad \left. + \int_{\Omega_T} \left(\mathcal{B}(x, t, u_n, \nabla u_n(x, t)) - \mathcal{B}(x, t, u, \nabla u(x, t)) \right) \cdot (\nabla u_n - \nabla u) dx dt \right] \\
 &\leq 0.
 \end{aligned}$$

From (4.2) and (4.9), we obtain

$$\lim_{n \rightarrow \infty} \langle \mathcal{S}u_n, u_n - u \rangle = \lim_{n \rightarrow \infty} \langle \mathcal{S}u_n - \mathcal{S}u, u_n - u \rangle = 0. \quad (4.10)$$

Let

$$\Theta_n(x, t) = \left(\mathcal{A}(x, t, \nabla u_n) - \mathcal{A}(x, t, \nabla u) \right) \cdot (\nabla u_n - \nabla u),$$

Under (4.10), we have

$$\Theta_n \rightarrow 0 \text{ in } L^1(\Omega_T).$$

Or $\Theta_n \rightarrow 0$ a.e. in Ω_T , then there exists a subset B of Ω_T ($\text{mes}(B) = 0$) such that for all $(x, t) \in \Omega_T \setminus B$,

$$|u(x, t)| < \infty, \quad |\nabla u(x, t)| < \infty, \quad u_n \rightarrow u, \quad \Theta_n \rightarrow 0.$$

Thanks to (4.1) and (4.3), if we pose $\zeta_n = \nabla u_n$ and $\zeta = \nabla u$, we get

$$\begin{aligned}
 \Theta_n(x, t) &= \left(\mathcal{A}(x, t, \zeta_n) - \mathcal{A}(x, t, \zeta) \right) \cdot (\zeta_n - \zeta) \\
 &= \mathcal{A}(x, t, \zeta_n) \cdot \zeta_n + \mathcal{A}(x, t, \zeta) \cdot \zeta - \mathcal{A}(x, t, \zeta_n) \cdot \zeta - \mathcal{A}(x, t, \zeta) \cdot \zeta_n \\
 &\geq \alpha_1 |\zeta_n|^p + \alpha_1 |\zeta|^p - \sum_{i=1}^N \left(k_1(x, t) + c_1 |\zeta_n|^{p-1} \right) |\zeta^i| \\
 &\quad - \sum_{i=1}^N \left(k_1(x, t) + c_1 |\zeta|^{p-1} \right) |\zeta_n^i| \\
 &\geq \alpha_1 |\zeta_n|^p - C \left[1 + |\zeta_n|^{p-1} + |\zeta_n| \right],
 \end{aligned}$$

where C is a constant which depends only on x .

Then by a standard argument $(\zeta_n)_n$ is bounded a.e. Ω_T , we deduce that

$$\Theta_n(x, t) \geq |\zeta_n|^p \left(\alpha_1 - \frac{C}{|\zeta_n|^p} - \frac{C}{|\zeta_n|} - \frac{C}{|\zeta_n|^{p-1}} \right).$$

Hence, if $|\zeta_n| \rightarrow \infty$, then $\Theta_n \rightarrow \infty$; what is contradiction, because $\Theta_n \rightarrow 0$ in $L^1(\Omega_T)$.

Next, for ζ^* to be an accumulation point of ζ_n , we have $|\zeta^*| < \infty$ and the continuity of $\mathcal{A}$, with respect to the last two variables, we will obtain

$$\left(\mathcal{A}(x, t, \zeta^*) - \mathcal{A}(x, t, \zeta) \right) (\zeta^* - \zeta) = 0. \quad (4.11)$$

Analogously, if we choose

$$A_n(x, t) = \left(\mathcal{B}(x, t, u_n, \nabla u_n) - \mathcal{B}(x, t, u, \nabla u) \right) \cdot (\nabla u_n - \nabla u),$$

and we take $\zeta_n = \nabla u_n$ and $\zeta = \nabla u$, then, by the same arguments used above, we obtain

$$\left(\mathcal{B}(x, t, u, \zeta^*) - \mathcal{B}(x, t, u, \zeta) \right) (\zeta^* - \zeta) = 0. \quad (4.12)$$

Then, according to (4.11), (4.12) and (4.2) we get $\zeta^* = \zeta$. Hence, by the uniqueness of the accumulation point, we deduce that

$$\nabla u_n \longrightarrow \nabla u \quad \text{a.e. in } \Omega_T. \quad (4.13)$$

On the other hand, seeing that $\mathcal{A}(x, t, \nabla u_n)$ and $\mathcal{B}(x, t, u_n, \nabla u_n)$ are bounded in $(L^{p'}(\Omega_T))^N$, and

$$\mathcal{A}(x, t, \nabla u_n) \longrightarrow \mathcal{A}(x, t, \nabla u) \quad \text{a.e. in } \Omega_T,$$

$$\mathcal{B}(x, t, u_n, \nabla u_n) \longrightarrow \mathcal{B}(x, t, u, \nabla u) \quad \text{a.e. in } \Omega_T,$$

then, thanks to Lemma 2.2, we obtain

$$\mathcal{A}(x, t, \nabla u_n) \rightharpoonup \mathcal{A}(x, t, \nabla u) \quad \text{in } (L^{p'}(\Omega_T))^N,$$

$$\mathcal{B}(x, t, u, \nabla u_n) \rightharpoonup \mathcal{B}(x, t, u, \nabla u) \quad \text{in } (L^{p'}(\Omega_T))^N.$$

If we pose

$$\begin{aligned} \bar{\rho}_n &= \left(\mathcal{A}(x, t, \nabla u_n) + \mathcal{B}(x, t, u_n, \nabla u_n) \right) \cdot \nabla u_n, \\ \bar{\rho} &= \left(\mathcal{A}(x, t, \nabla u) + \mathcal{B}(x, t, u, \nabla u) \right) \cdot \nabla u, \end{aligned}$$

we can write

$$\bar{\rho}_n \rightarrow \bar{\rho} \quad \text{in } L^1(\Omega_T).$$

Thanks to (4.3), we obtain

$$\bar{\rho}_n \geq (\alpha_1 + \alpha_2) |\nabla u_n|^p \quad \text{and} \quad \bar{\rho} \geq (\alpha_1 + \alpha_2) |\nabla u|^p.$$

In view of $\tau_n = |\nabla u_n|^p$, $\tau = |\nabla u|^p$, $\rho_n = \frac{\bar{\rho}_n}{(\alpha_1 + \alpha_2)}$ and $\rho = \frac{\bar{\rho}}{(\alpha_1 + \alpha_2)}$, we have

$$\rho_n \geq \tau_n \quad \text{and} \quad \rho \geq \tau.$$

Then by Fatou's lemma, we get

$$\int_0^T \left(\int_{\Omega} 2\rho \, dx \right) dt \leq \int_0^T \left(\liminf_{n \rightarrow \infty} \int_{\Omega} \rho + \rho_n - |\tau_n - \tau| \, dx \right) dt,$$

i.e.,

$$0 \leq - \int_0^T \left(\limsup_{n \rightarrow \infty} \int_{\Omega} |\tau_n - \tau| \, dx \right) dt.$$

So

$$0 \leq \int_0^T \left(\liminf_{n \rightarrow \infty} \int_{\Omega} |\tau_n - \tau| \, dx \right) dt \leq \int_0^T \left(\limsup_{n \rightarrow \infty} \int_{\Omega} |\tau_n - \tau| \, dx \right) dt \leq 0,$$

then

$$\tau_n \longrightarrow \tau \quad \text{in } (L^p(\Omega_T))^N,$$

consequently

$$\nabla u_n \longrightarrow \nabla u \quad \text{in} \quad (L^p(\Omega_T))^N. \quad (4.14)$$

According to (4.13) and (4.14), we have

$$u_n \rightarrow u \text{ in } \mathcal{W},$$

what implies that $\mathcal{S}$ is of type (S_+) . This concludes the proof. $\square$

4.2. Main result. We are now in the position to get existence result of weak solutions for (1).

Theorem 4.2. *Let $\phi \in \mathcal{W}^*$, $u_0 \in L^2(\Omega)$ and assume that the assumptions (4.1) – (4.3) hold, then the problem (1) admits at least one weak solution $u \in D(\mathcal{L})$, where*

$$D(\mathcal{L}) = \{u \in \mathcal{W} : \frac{du}{dt} \in \mathcal{W}^*, u(0) = 0\}$$

.

Proof. First, let us define the operator $\mathcal{L} := \frac{d}{dt}$ with domain $D(\mathcal{L})$ given by

$$D(\mathcal{L}) = \{u \in \mathcal{W} : \frac{du}{dt} \in \mathcal{W}^* \text{ and } u(0) = 0\},$$

where the time derivative $\frac{du}{dt}$ is understood in the sense of vector-valued distributions, i.e.,

$$\langle \mathcal{L}u, v \rangle_{\mathcal{W}^*, \mathcal{W}} = \int_0^T \langle u'(t), v(t) \rangle dt, \quad \forall v \in \mathcal{W}.$$

with $\langle \cdot, \cdot \rangle_{\mathcal{W}^*, \mathcal{W}}$ the duality pairing between $\mathcal{W}^*$ and $\mathcal{W}$, and $\langle \cdot, \cdot \rangle$ the duality pairing between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$.

Defining the operator $\mathcal{S} : \mathcal{W} \rightarrow \mathcal{W}^*$ by

$$\langle \mathcal{S}u, v \rangle_{\mathcal{W}^*, \mathcal{W}} = \int_{\Omega_T} \left(\mathcal{A}(x, t, \nabla u) + \mathcal{B}(x, t, u, \nabla u) \right) \cdot \nabla v dx dt.$$

Consequently, the weak formulation of the problem (1) is given by the operator equation

$$u \in D(\mathcal{L}) : \mathcal{L}u + \mathcal{S}u = \phi.$$

Next, it follows from lemma 4.1 that $\mathcal{S}$ is bounded, continuous and of type (S_+) , and the operator $\mathcal{L}$ is well known to be closed, densely defined, and maximal monotone [22, Theorem 32.L, pp.897-899].

Let $u \in \mathcal{W}$. Using the monotonicity of $\mathcal{L}$ ($\langle \mathcal{L}u, u \rangle \geq 0$ for all $u \in D(\mathcal{L})$) and the condition (4.2), we deduce that

$$\begin{aligned} \langle \mathcal{L}u + \mathcal{S}u, u \rangle &\geq \langle \mathcal{S}u, u \rangle \\ &= \int_{\Omega_T} \left(\mathcal{A}(x, t, \nabla u) + \mathcal{B}(x, t, u, \nabla u) \right) \cdot \nabla u \, dxdt \\ &\geq \int_{\Omega_T} \alpha_1 |\nabla u|^p \, dxdt + \int_{\Omega_T} \alpha_2 |\nabla u|^p \, dxdt \\ &\geq \min(\alpha_1, \alpha_2) \int_{\Omega_T} |\nabla u|^p \, dxdt \\ &= \min(\alpha_1, \alpha_2) \|u\|_{\mathcal{W}}^p. \end{aligned}$$

Because the right-hand side of the previous inequality approximates to ∞ when $\|u\|_{\mathcal{W}} \rightarrow \infty$, then for every $\phi \in \mathcal{W}^*$ there is a radius $r = r(\phi) > 0$ such that

$$\langle \mathcal{L}u + \mathcal{S}u - \phi, u \rangle > 0, \quad \text{for each } u \in B_r(0) \cap D(\mathcal{L}).$$

So all the conditions of Lemma 3.8 are satisfied. Consequently, Lemma 3.8 leads us to the conclusion that the equation $\mathcal{L}u + \mathcal{S}u = \phi$ has a weak solution in $D(\mathcal{L})$, which implies that the problem (1) admits at least one weak solution in $u \in D(\mathcal{L})$. This completes the proof. $\square$

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