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INITIAL COEFFICIENT BOUNDS AND SECOND HANKEL DETERMINANT FOR A CERTAIN CLASS OF BI-UNIVALENT FUNCTIONS USING CHEBYSHEV POLYNOMIALS

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Abstract. In this work, we define a new subclass $C_\Sigma(\beta,\gamma,n,t)$ of analytic and bi-univalent functions in the open unit disk. Thereafter, we investigated the coefficient bounds and the second Hankel determinant by means of Chebyshev polynomials.

1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of functions

$$f(z) = z + a_2z^2 + a_3z^3 + a_4z^4 + \cdots \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$ and satisfy the condition $f(0) = 0$ and $f'(0) = 1$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ that are univalent in $\mathcal{U}$.

Let $\mathcal{D}^n : \mathcal{A} \to \mathcal{A}$ be defined by

$$\mathcal{D}^0 f(z) = f(z),\ \mathcal{D}^1 f(z) = zf'(z), \ldots, \mathcal{D}^n f(z) = z[\mathcal{D}^{n-1} f(z)]' \tag{1.2}$$

which is equivalent to saying

$$\mathcal{D}^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k,\ n = \{0, 1, 2, \ldots\},\ z \in \mathcal{U}.$$

The operator $\mathcal{D}^n$ is known as Sălăgean differential operator (see [16]).

A function $f(z)$ is subordinate to $g(z)$ in $\mathcal{U}$ written as $f(z) \prec g(z),\ z \in \mathcal{U}$, if there exists a Schwarz function $\omega(z)$, analytic in $\mathcal{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathcal{U}$ such that $f(z) = g(\omega(z)),\ z \in \mathcal{U}$. If g is univalent, then $f(z) \prec g(z) \implies f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

Every function $f \in \mathcal{S}$ has inverse function f^{-1}, which is defined by

$$f^{-1}(f(z)) = z,\ z \in \mathcal{U}$$

and

$$f(f^{-1}(w)) = w \left(|w| < r_0(f) :\ r_0(f) \geq \frac{1}{4}\right).$$

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The inverse function is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \qquad (1.3)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathcal{U}$ if both f and f^{-1} are univalent in $\mathcal{U}$. Let Σ denote the class of bi-univalent functions defined in the unit disk $\mathcal{U}$.

Lewin [12] introduced the class Σ and showed that $|a_2| < 1.51$ for every $f \in \Sigma$. Later, Brannan and Clunie [4] conjectured that $|a_2| \leq \sqrt{2}$ for bi-starlike functions and $|a_2| \leq 1$ for bi-convex functions. Also, Brannan and Taha [5] introduced certain subclasses of bi-univalent functions called bi-starlike function of order α and bi-convex function of order α. For example see [1, 4, 10, 15, 18] for more background details.

Fekete and Szegö [8] introduced the generalised the functional $|a_3 - \lambda a_2^2|$, where λ is a real number. Some works on this functional can be found in [2, 3, 10, 17, 18].

Noonan and Thomas [14] defined the q^{th} Hankel determinant of $f \in \mathcal{A}$ for $q, n \in \mathbb{N}$ by

$$\mathcal{H}_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}$$

where $a_1 = 1$ for $f(z) \in \mathcal{S}$. The Hankel determinant $|\mathcal{H}_2(1)| = |a_3 - a_2^2|$ and $|\mathcal{H}_2(2)| = |a_2 a_4 - a_2^3|$ are respectively well-known as Fekete-Szegö functional ($\lambda = 1$) and second Hankel determinant. The upper estimates of $|\mathcal{H}_2(2)|$ of various subclasses of Σ have been obtained by researchers such as in [2, 11, 15].

It is well-known that the Chebyshev polynomials are of four kinds but we are only interested in making use of the second kinds in this paper. For any real variable $t \in [-1, 1]$ the first and second kinds of the Chebyshev polynomials are defined by

$$T_n(t) = \cos(n \cos^{-1} t) \qquad (1.4)$$

and

$$U_n(t) = \frac{\sin(n+1)\cos^{-1} t}{\sin(\cos^{-1} t)} \qquad (1.5)$$

Examples of Chebyshev polynomials of the second kind are

$$U_0(t) = 1,\ U_1(t) = 2t,\ U_2(t) = 4t^2 - 1, \ldots \qquad (1.6)$$

while the generating function for $U_n(t)$ is given by

$$F(z,t) = \frac{1}{1 - 2tz + z^2} = U_0(t) + U_1(t)z + U_2(t)z^2 + U_3(t)z^3 + \cdots . \qquad (1.7)$$

2. Lemmas

Now, we state the following lemmas necessary for proving our results.

Let $\mathcal{P}$ denote the class of analytic functions $p(z) \in \mathcal{P}$ which are analytic in the unit disk $\mathcal{U}$ such that $p(0) = 1$ and $\mathcal{R}ep(z) > 0$.

Lemma 2.1. [7]. *Let $p \in \mathcal{P}$, then $|p_k| \leq 2,\ k \in \mathbb{N}$.*

Lemma 2.2. [13]. *Let $p \in \mathcal{P}$, then*

$$2p_2 = p_1^2 + x\left(4 - p_1^2\right)$$
$$4p_3 = p_1^3 + 2p_1(4-p_1^2)x - p_1(4-p_1^2)x^2 + 2(4-p_1^2)(1-|x|^2)z$$

for some $x, z; |x| \leq 1$ and $|z| \leq 1$.

In this paper, we considered a subclass of bi-univalent functions and obtained its coefficient bounds and the second Hankel determinant for functions belonging to subclass $C_\Sigma(\beta, \gamma, n, t)$ using the Chebyshev polynomial expansion.

3. The Main Results

Definition 3.1. For $\beta \in [0,1]$, $\gamma \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, $n \in \mathbb{N}_0$ and $t \in (\frac{1}{2}, 1)$, a function $f(z) \in \Sigma$ given by (1.1) is said to be in the class $C_\Sigma(\beta, \gamma, n, t)$ if the following subordination conditions hold for all $z, \omega \in \mathcal{U}$.

$$\left(1 - e^{-2i\gamma}\beta^2 z^2\right)\frac{\mathcal{D}^{n+1}f(z)}{z} \prec F(z,t) = \frac{1}{1 - 2tz + z^2} \tag{3.1}$$

and

$$\left(1 - e^{-2i\gamma}\beta^2 w^2\right)\frac{\mathcal{D}^{n+1}g(w)}{w} \prec F(w,t) = \frac{1}{1 - 2tw + w^2} \tag{3.2}$$

where $g = f^{-1}$ is defined by (1.3).

If $t = \cos\lambda$ where $\lambda \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$, then from (3.1) we have

$$F(z,t) = (1 - 2\cos\lambda z + z^2)^{-1} = 1 + \sum_{n=1}^{\infty}\frac{\sin(n+1)\lambda}{\sin\lambda}z^n, \quad z \in \mathcal{U} \tag{3.3}$$

hence,

$$F(z,t) = 1 + U_1(t)z + U_2(t)z^2 + U_3(t)z^3 + \cdots. \tag{3.4}$$

Also in the same way, (3.2) becomes

$$F(w,t) = 1 + U_1(t)w + U_2(t)w^2 + U_3(t)w^3 + \cdots. \tag{3.5}$$

In what follows are the main results in this paper.

Theorem 3.2. *Let $f(z) \in C_\Sigma(\beta, \gamma, n, t)$, $\beta \in [0,1]$, $\gamma \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, $n \in \mathbb{N}_0$ and $t \in (\frac{1}{2}, 1)$. Then*

$$|a_2| \leq \sqrt{\frac{2t^3 + \beta^2 t^2}{|3^{n+1}\cdot t^2 + 2^{2n}(2t - 4t^2 + 1)|}} \tag{3.6}$$

$$|a_3| \leq \frac{t^2}{2^{2n}} + \frac{2t}{3^{n+1}} \tag{3.7}$$

Proof. Let the function $f(z) \in \Sigma$ given by (1.1) be in the class $C_\Sigma(\beta, \gamma, n, t)$ where $\beta \in [0,1]$, $\gamma \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, $n \in \mathbb{N}_0$, $t \in (\frac{1}{2}, 1)$ and $g = f^{-1}$, then there exists an analytic functions $\omega, \varpi : \mathcal{U} \to \mathcal{U}$ with $\omega(0) = 0 = \varpi(0)$, $|\omega(z)| < 1, |\varpi(\omega)| < 1$ and satisfying the following conditions

$$\left(1 - e^{-2i\gamma}\beta^2 z^2\right)\frac{\mathcal{D}^{n+1}f(z)}{z} = F(\omega(z), t), \quad z \in \mathcal{U} \tag{3.8}$$

and

$$\left(1 - e^{-2i\gamma}\beta^2 w^2\right) \frac{\mathcal{D}^{n+1} g(w)}{\omega} = F(\varpi(w), t), \quad w \in D \tag{3.9}$$

where

$$F(\omega(z), t) = 1 + U_1(t)\omega(z) + U_2(t)\omega(z)^2 + U_3(t)\omega(z)^3 + \cdots \tag{3.10}$$

and

$$F(\varpi(\omega), t) = 1 + U_1(t)\varpi(\omega) + U_2(t)\varpi(\omega)^2 + U_3(t)\varpi(\omega)^3 + \cdots . \tag{3.11}$$

Simplifying the LHS of (3.8) we obtain

$$\begin{aligned}(1 - e^{-2i\gamma}\beta^2 z^2)\frac{\mathcal{D}^{n+1} f(z)}{z} &= 1 + 2^{n+1}a_2 z + (3^{n+1}a_3 - e^{-2i\gamma}\beta^2)z^2 \\ &\quad + (4^{n+1}a_4 - 2^{n+1}e^{-2i\gamma}\beta^2 a_2)z^3 + \cdots . \end{aligned}\tag{3.12}$$

Similarly, simplifying the LHS of (3.9) gives

$$\begin{aligned}(1 - e^{-2i\gamma}\beta^2 w^2)\frac{\mathcal{D}^{n+1} g(w)}{w} &= 1 + 2^{n+1}a_2 w + (3^{n+1}a_3 - e^{-2i\gamma}\beta^2)w^2 \\ &\quad + (4^{n+1}a_4 - 2^{n+1}e^{-2i\gamma}\beta^2 a_2)w^3 + \cdots . \end{aligned}\tag{3.13}$$

Let $p, q \in \mathcal{P}$ be define as

$$p(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \implies \omega(z) = \frac{p(z) - 1}{p(z) + 1}, \ z \in \mathcal{U} \tag{3.14}$$

and

$$q(w) = \frac{1 + \varpi(w)}{1 - \varpi(w)} = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots \implies \varpi(w) = \frac{q(w) - 1}{q(w) + 1}, \ w \in \mathcal{U}. \tag{3.15}$$

Now by simplifying (3.14) and (3.15) we obtain

$$\omega(z) = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \left(p_3 - p_1 p_2 + \frac{p_1^3}{4}\right) z^3 + \cdots\right] \tag{3.16}$$

and

$$\varpi(w) = \frac{1}{2}\left[q_1 w + \left(q_2 - \frac{q_1^2}{2}\right) w^2 + \left(q_3 - q_1 q_2 + \frac{q_1^3}{4}\right) w^3 + \cdots\right]. \tag{3.17}$$

Also note that

$$\left.\begin{aligned} \omega(z)^2 &= \tfrac{p_1^2}{4} z^2 + \tfrac{p_1 p_2}{2} z^3 - \tfrac{p_1^3}{4} z^3 + \cdots \\ \omega(z)^3 &= \tfrac{p_1^3}{8} z^3 + \tfrac{p_1^2 p_2}{4} z^4 + \cdots \end{aligned}\right\} \tag{3.18}$$

and similarly,

$$\left.\begin{aligned} \varpi(w)^2 &= \tfrac{q_1^2}{4} w^2 + \tfrac{q_1 q_2}{2} w^3 - \tfrac{q_1^3}{4} w^3 + \cdots \\ \varpi(w)^3 &= \tfrac{q_1^3}{8} w^3 + \cdots \end{aligned}\right\}. \tag{3.19}$$

Using (3.18) and taking $F(\omega(z),t)$ and $F(\varpi(w),t)$ as given in (3.10) and (3.11) we obtain

$$F(\omega(z),t) = 1 + \frac{U_1(t)}{2}p_1 z + \left[\frac{U_1(t)}{2}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{U_2(t)}{4}p_1^2\right] z^2 + \left[\frac{U_1(t)}{2}\left(p_3 - p_1p_2 + \frac{p_1^3}{4}\right) + \frac{U_2(t)}{2}p_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{U_3(t)}{8}p_1^3\right] z^3 + \cdots \quad (3.20)$$

and

$$F(\varpi(w),t) = 1 + \frac{U_1(t)}{2}q_1 w + \left[\frac{U_1(t)}{2}\left(q_2 - \frac{q_1^2}{2}\right) + \frac{U_2(t)}{4}q_1^2\right] w^2 + \left[\frac{U_1(t)}{2}\left(q_3 - q_1q_2 + \frac{q_1^3}{4}\right) + \frac{U_2(t)}{2}q_1\left(q_2 - \frac{q_1^2}{2}\right) + \frac{U_3(t)}{8}q_1^3\right] w^3 + \cdots \quad (3.21)$$

From (3.12) and (3.13) we can easily obtain

$$2^{n+1}a_2 = \frac{U_1(t)}{2}p_1 \quad (3.22)$$

$$3^{n+1}a_3 - e^{-2i\gamma}\beta^2 = \frac{U_1(t)}{2}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{U_2(t)}{4}p_1^2 \quad (3.23)$$

$$4^{n+1}a_4 - 2^{n+1}e^{-2i\gamma}\beta^2 a_2 = \frac{U_1(t)}{2}\left(p_3 - p_1p_2 + \frac{p_1^3}{4}\right) + \frac{U_2(t)}{2}p_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{U_3(t)}{8}p_1^3 \quad (3.24)$$

and also from (3.13) and (3.21) we obtain

$$-2^{n+1}a_2 = \frac{U_1(t)}{2}q_1 \quad (3.25)$$

$$3^{n+1}(2a_2^2 - a_3) - e^{-2i\gamma}\beta^2 = \frac{U_1(t)}{2}\left(q_2 - \frac{q_1^2}{2}\right) + \frac{U_2(t)}{4}q_1^2 \quad (3.26)$$

$$-4^{n+1}(5a_2^3 - 5a_2a_3 + a_4) + 2^{n+1}e^{-2i\gamma}\beta^2 a_2 = \frac{U_1(t)}{2}\left(q_3 - q_1q_2 + \frac{q_1^3}{4}\right) + \frac{U_2(t)}{2}q_1\left(q_2 - \frac{q_1^2}{2}\right) + \frac{U_3(t)}{8}q_1^3. \quad (3.27)$$

Adding (3.22) and (3.25) give

$$\frac{U_1(t)}{2}p_1 + \frac{U_1(t)}{2}q_1 = 0 \implies p_1 = -q_1, \quad p_1^2 = q_1^2 \quad \text{and} \quad p_1^3 = -q_1^3. \quad (3.28)$$

Squaring (3.22) and (3.25) and adding the results give

$$2^{2n+2}a_2^2 = \frac{U_1(t)p_1^2}{4} \quad \text{and} \quad 2^{2n+2}a_2^2 = \frac{U_1(t)q_1^2}{4} \implies a_2^2 = \frac{U_1^2(t)(p_1^2+q_1^2)}{2^{2n+5}}. \quad (3.29)$$

Also, adding (3.23) and (3.26) and using (3.28) give

$$4\cdot 3^{n+1}a_2^2 - 4e^{-2i\gamma}\beta^2 = U_1(t)(p_2+q_2) - q_1^2(U_1(t) - U_2(t)). \quad (3.30)$$

Substituting for q_1^2 in (3.30) and simplifying give

$$a_2^2 = \frac{U_1^3(t)(p_2+q_2) + 4e^{-2i\gamma}\beta^2 U_1^2(t)}{4\cdot 3^{n+1}U_1^2(t) + 2^{2n+4}(U_1(t)-U_2(t))}. \tag{3.31}$$

Applying (1.6) and Lemma 2.1 we obtain (3.6).

Subtracting (3.26) from (3.23) and considering (3.29) give

$$a_3 = \frac{U_1^2(t)}{2^{2n+4}}p_1^2 + \frac{U_1(t)(p_2-q_2)}{2^2\cdot 3^{n+1}}. \tag{3.32}$$

Using (1.6) and Lemma 2.1 we obtain (3.7).

Also, subtracting (3.27) from (3.24) and considering (3.28) give

$$\begin{aligned} a_4 =& \frac{5\cdot 2^n U_1^2(t)(p_2-q_2)p_1}{2^{2n+5}\cdot 3^{n+1}} + \frac{U_1(t)(p_3-q_3)}{2^{2n+4}} + \frac{[U_2(t)-U_1(t)]p_1(p_2+q_2)}{2^{2n+4}} \\ &+ \frac{[U_1(t)-2U_2(t)+U_3(t)]p_1^3}{2^{2n+5}} + \frac{e^{-2i\sigma}\beta^2 U_1(t)p_1}{2^{2n+3}}. \end{aligned} \tag{3.33}$$

□

Theorem 3.3. *Let $f(z) \in C_\Sigma(\beta,\gamma,n,t)$ where $\beta \in [0,1]$, $\gamma \in \left(\frac{-\pi}{2},\frac{\pi}{2}\right)$, $n \in \mathbb{N}_0$ and $t \in (\frac{1}{2},1)$, then*

$$|\mathcal{H}_2(2)| \le \begin{cases} M(2,t) & : N_1 \ge 0, N_2 \ge 0 \\ \max\left\{\frac{4t^2}{3^{2n+2}}, M(2,t)\right\} & : N_1 > 0, N_2 < 0 \\ \frac{4t^2}{3^{2n+2}} & : N_1 \le 0, N_2 \le 0 \\ \max\left\{M(r_0,t), M(2,t)\right\} & : N_1 < 0, N_2 > 0 \end{cases}$$

where

$$\begin{aligned} M(2,t) =& \frac{U_1(t)|U_1(t)-2U_2(t)+U_3(t)|}{2^{3n+3}} \\ &+ \frac{U_1(t)|U_2-U_1(t)|}{2^{3n+2}} + \frac{U_1^2(t)}{2^{3n+2}} + \frac{U_1^4(t)}{2^{4n+4}} + \frac{\beta^2 U_1^2(t)}{2^{3n+3}}, \\ M(r_0,t) =& \frac{U_1^2(t)}{3^{2n+2}} + \frac{N_2^4}{2^4\cdot 3^{2n+2}N_1^3} + \frac{N_2^2}{2^{n+3}\cdot 3^{2n+2}N_1^2} \\ N_1 =& U_1(t)\Big\{2^{n+1}\cdot 3^{2n+2}|U_1(t)-2U_2(t)+U_3(t)| + U_1^3(t)3^{2n+2} \\ &- 2^{n+2}\cdot 3^{2n+2}U_1(t) + 2^{4n+4}U_1(t) - 2^{2n+1}\cdot 3^{n+1}U_1^2(t)\Big\}r^4 \end{aligned}$$

and

$$\begin{aligned} N_2 =& U_1(t)\frac{U_1(t)}{2^{3n+5}\cdot 3^{2n+2}}\Big\{2\cdot 3^{2n+2}|U_2(t)-U_1(t)| + 3^{2n+3}U_1(t) \\ &- 2^{3n+4}U_1(t) + 2^n\cdot 3^{n+1}U_1^2(t) + \beta^2 3^{2n+2}U_1(t)\Big\}r^2. \end{aligned}$$

Proof. From (3.22), (3.32) and (3.33) we obtain

$$a_2a_4-a_3^2=\frac{U_1^3(t)(p_2-q_2)}{2^{2n+7}\cdot 3^{n+1}}p_1^2+\frac{U_1^2(t)(p_3-q_3)}{2^{3n+6}}p_1+\frac{[U_2(t)-U_1(t)]U_1(t)(p_2+q_2)}{2^{3n+6}}p_1^2$$
$$+\frac{[U_1(t)-2U_2(t)+U_3(t)]}{2^{3n+7}}p_1^4+\frac{e^{-2i\gamma}\beta^2U_1^2(t)}{2^{3n+5}}p_1^2-\frac{U_1^4(t)}{2^{4n+8}}p_1^4-\frac{U_1^2(t)(p_2-q_2)}{2^4\cdot 3^{2n+2}}. \quad (3.34)$$

From Lemma 2.2 we get

$$\left.\begin{aligned} p_2-q_2&=\tfrac{4-p_1^2}{2}(x-y)\\ p_2+q_2&=p_1^2+\tfrac{4-p_1^2}{2}(x+y)\end{aligned}\right\} \quad (3.35)$$

and

$$p_3-q_3=\frac{p_1^3}{2}+\frac{4-p_1^2}{2}p_1(x+y)-\frac{4-p_1^2}{4}p_1(x^2+y^2)$$
$$+\frac{4-p_1^2}{2}\left[(1-|x|^2z)-(1-|y|^2)w\right] \quad (3.36)$$

for some x,y,z,w with $|x|\le 1$, $|y|\le 1$, $|z|\le 1$, $|w|\le 1$ and $|p_1|\in[0,2]$.

Now, putting (3.35) and (3.36) into (3.34), we obtain

$$a_2a_4-a_3^2=\frac{U_1^3(t)(4-p_1^2)(x-y)}{2^{2n+8}\cdot 3^{n+1}}p_1^2+\frac{U_1(t)[U_2(t)-U_1(t)]}{2^{3n+6}}p_1^4+\frac{U_1^2(t)}{2^{3n+7}}p_1^4$$
$$+\frac{U_1(t)\left[U_2(t)-U_1(t)\right](4-p_1^2)(x+y)}{2^{3n+7}}p_1^2+\frac{U_1^2(t)(4-p_1^2)(x+y)}{2^{3n+7}}p_1^2$$
$$-\frac{U_1^2(t)(4-p_1^2)(x^2+y^2)}{2^{3n+8}}p_1^2+\frac{U_1^2(t)(4-p_1^2)[(1-|x|^2)z-(1-|y|^2)w]}{2^{3n+7}}p_1$$
$$-\frac{U_1^2(t)(4-p_1^2)^2(x-y)^2}{2^6\cdot 3^{2n+2}}+\frac{e^{-2i\sigma}\alpha^2U_1^2(t)}{2^{3n+5}}p_1^2-\frac{U_1^4(t)}{2^{4n+8}}p_1^4$$
$$+\frac{U_1(t)\left[U_1(t)-2U_2(t)+U_3(t)\right]}{2^{3n+7}}p_1^4. \quad (3.37)$$

For $p(z)\in\mathcal{P}, |p_1|\le 2$, let $r=p_1$, assume without any restriction that $r\in[0,2]$, $\psi_1=|x|\le 1$ and $\psi_2=|y|\le 1$, then by triangle inequality,

$$|a_2a_4-a_3^2|\le\left\{\frac{U_1(t)|U_1(t)-2U_2(t)+U_3(t)|}{2^{3n+7}}r^4+\frac{U_1(t)|U_2(t)-U_1(t)|}{2^{3n+6}}r^4\right.$$
$$\left.+\frac{U_1^2(t)}{2^{3n+7}}r^4+\frac{U_1^2(t)(4-r^2)}{2^{3n+6}}r+\frac{U_1^4(t)}{2^{4n+8}}r^4+\frac{\alpha^2U_1^2(t)}{2^{3n+5}}r^2\right\}$$
$$+\left\{\frac{U_1(t)|U_2(t)-U_1(t)|(4-r^2)}{2^{3n+7}}r^2+\frac{U_1^2(t)(4-r^2)}{2^{3n+7}}r^2+\frac{U_1^3(t)(4-r^2)}{2^{2n+8}\cdot 3^{n+1}}r^2\right\}(\psi_1+\psi_2)$$
$$+\left\{\frac{U_1^2(t)(4-r^2)}{2^{3n+8}}r^2-\frac{U_1^2(t)(4-r^2)}{2^{3n+7}}r\right\}(\psi_1^2+\psi_2^2)$$
$$+\frac{U_1^2(t)(4-r^2)^2}{2^6\cdot 3^{2n+2}}(\psi_1+\psi_2)^2 \quad (3.38)$$

and equivalently

$$|a_2a_4 - a_3^2| \leq Q_1(t,r) + Q_2(t,r)(\psi_1+\psi_2) + Q_3(t,r)(\psi_1^2+\psi_2^2) + Q_4(t,r)(\psi_1+\psi_2)^2 = G(\psi_1,\psi_2) \quad (3.39)$$

where

$$Q_1(t,r) = \left\{\frac{U_1(t)|U_1(t)-2U_2(t)+U_3(t)|}{2^{3n+7}}r^4 + \frac{U_1(t)|U_2(t)-U_1(t)|}{2^{3n+6}}r^4 + \frac{U_1^2(t)}{2^{3n+7}}r^4 + \frac{U_1^2(t)(4-r^2)}{2^{3n+6}}r + \frac{U_1^4(t)}{2^{4n+8}}r^4 + \frac{\beta^2U_1^2(t)}{2^{3n+5}}r^2\right\} \geq 0$$

$$Q_2(t,r) = \left\{\frac{U_1(t)|U_2(t)-U_1(t)|(4-r^2)}{2^{3n+7}}r^2 + \frac{U_1^2(t)(4-r^2)}{2^{3n+7}}r^2 + \frac{U_1^3(t)(4-r^2)}{2^{2n+8}\cdot 3^{n+1}}r^2\right\} \geq 0$$

$$Q_3(t,r) = \left\{\frac{U_1^2(t)(4-r^2)}{2^{3n+8}}r^2 - \frac{U_1^2(t)(4-r^2)}{2^{3n+7}}r\right\} \leq 0$$

and

$$Q_4(t,r) = \frac{U_1^2(t)(4-r^2)^2}{2^6\cdot 3^{2n+2}} \geq 0$$

where $\frac{1}{2} < t < 1$, $0 \leq r \leq 2$.

Next is to maximize the function $G(\psi_1,\psi_2)$ in the closed square

$$\mathbb{T} = \{(\psi_1,\psi_2) : \psi_1 \in [0,1], \psi_2 \in [0,1]\} \;\; for \; r \in [0,2].$$

The coefficients in (3.39) depend on r for a fixed value of t. We now investigate the maximum of $G(\psi_1,\psi_2)$ with respect to r taking into consideration the cases when $r=0$, $r=2$ and $r \in (0,2)$.

When $r = 0$,

$$G(\psi_1,\psi_2) = Q_4(t,0) = \frac{U_1^2(t)}{2^2\cdot 3^{2n+2}}(\psi_1+\psi_2)^2 \quad (3.40)$$

The maximum of the function $G(\psi_1,\psi_2)$ occurs at (ψ_1,ψ_2) and

$$\max\{G(\psi_1,\psi_2) : \psi_1,\psi_2 \in [0,1]\} = G(1,1) = \frac{U_1^2(t)}{3^{2n+2}}. \quad (3.41)$$

When $r = 2$,

$$G(\psi_1,\psi_2) = Q_1(t,2) = \left\{\frac{U_1(t)|U_1(t)-2U_2(t)+U_3(t)|}{2^{3n+3}} + \frac{U_1(t)|U_2(t)-U_1(t)|}{2^{3n+2}} + \frac{U_1^2(t)}{2^{3n+3}} + \frac{U_1^4(t)}{2^{4n+4}} + \frac{\beta^2U_1^2(t)}{2^{3n+3}}\right\}. \quad (3.42)$$

When $r \in (0,2)$, in this case let $\psi_1+\psi_2 = u$ and $\psi_1 \times \psi_2 = v$ then (3.39) can be written as

$$G(\psi_1,\psi_2) = Q_1(t,r) + Q_2(t,r)u + (Q_3(t,r)+Q_4(t,r))u^2 - 2Q_3(t,r)v = K(u,v) \quad (3.43)$$

where $u \in [0,2]$ and $v \in [0,1]$. Next is to investigate the maximum of

$$K(u,v) \in \mathbb{V} = \{(u,v) : u \in [0,2], v \in [0,1]\} \tag{3.44}$$

so that

$$\frac{\partial K}{\partial u} = Q_2(t,r) + 2(Q_3(t,r) + Q_4(t,r))u = 0 \tag{3.45}$$

and

$$\frac{\partial K}{\partial v} = -2Q_3(t,r) = 0 \tag{3.46}$$

With this, $K(u,v)$ has no critical point in $\mathbb{V}$, hence, $G(\psi_1,\psi_2)$ has no critical point in the square $\mathbb{T}$. Thus, the function $G(\psi_1,\psi_2)$ can not have maximum value in the interior of $\mathbb{T}$. Next is to investigate the maximum of $G(\psi_1,\psi_2)$ on the boundary of the square $\mathbb{T}$.

For $\psi_1 = 0$, $\psi_2 \in [0,1]$ (similarly, for $\psi_2 = 0, \psi_1 \in [0,1]$) and

$$G(0,\psi_2) = Q_1(t,r) + Q_2(t,r)\psi_2 + (Q_3(t,r) + Q_4(t,r))\psi_2^2 = J(\psi_2). \tag{3.47}$$

Since $Q_3(t,r) + Q_4(t,r) \geq 0$, then

$$J'(\psi_2) = Q_2(t,r) + 2[Q_3(t,r) + Q_4(t,r)]\psi_2 > 0$$

implies that $J(\psi_2)$ is an increasing function. Therefore, for a fixed $r \in [0,2)$ and $t \in (\frac{1}{2},1)$, the maximum occurs at $\psi_2 = 1$. From (3.47),

$$\begin{aligned} \max\{G(0,\psi_2) : \psi_2 \in [0,1]\} &= G(0,1) \\ &= Q_1(t,r) + Q_2(t,r) + Q_3(t,r) + Q_4(t,r). \end{aligned} \tag{3.48}$$

Also, for $\psi_1 = 1$, $\psi_2 \in [0,1]$ (similarly, for $\psi_2 = 1$, $\psi_1 \in [0,1]$) and

$$\begin{aligned} G(1,\psi_2) = Q_1(t,r) &+ Q_2(t,r) + Q_3(t,r) + Q_4(t,r) + [Q_2(t,r) \\ &+ 2Q_4(t,r)]\psi_2 + [Q_3(t,r) + Q_4(t,r)]\psi_2^2 = T(\psi_2) \end{aligned} \tag{3.49}$$

$$T'(\psi_2) = [Q_2(t,r) + 2Q_4(t,r)] + 2[Q_3(t,r) + Q_4(t,r)]\psi_2 \tag{3.50}$$

and since $Q_3(t,r) + Q_4(t,r) \geq 0$, then

$$T'(\psi_2) = [Q_2(t,r) + 2Q_4(t,r)] + 2[Q_3(t,r) + Q_4(t,r)]\psi_2 > 0.$$

Therefore, the function $T(\psi_2)$ is an increasing function and the maximum occurs at $\psi_2 = 1$. From (3.49),

$$\begin{aligned} \max\{G(1,\psi_2) : \psi_2 \in [0,1]\} &= G(1,1) \\ &= Q_1(t,r) + 2[Q_2(t,r) + Q_3(t,r)] + 4Q_4(t,r) \end{aligned} \tag{3.51}$$

Hence, for every $r \in (0,2)$, it is clear that

$$Q_1(t,r)+2[Q_2(t,r)+Q_3(t,r)]+4Q_4(t,r) > Q_1(t,r)+Q_2(t,r)+Q_3(t,r)+Q_4(t,r).$$

Therefore,

$$\begin{aligned} \max\{G(\psi_1,\psi_2) &: \psi_1 \in [0,1], \psi_2 \in [0,1]\} \\ &= Q_1(t,r) + 2[Q_2(t,r) + Q_3(t,r)] + 4Q_4(t,r). \end{aligned} \tag{3.52}$$

Since

$$J(1) \le T(1) \text{ for } r \in [0,2] \text{ and } t \in (\frac{1}{2}, 1),$$

then

$$\max\{G(\psi_1, \psi_2)\} = G(1,1)$$

occurs on the boundary of square $\mathbb{T}$.

Let $M : (0,2) \to \mathbb{R}$ defined by

$$\begin{aligned} M(r,t) &= \max\{G(\psi_1,\psi_2)\} \\ &= G(1,1) = Q_1(t,r) + 2Q_2(t,r) + 2Q_3(t,r) + 4Q_4(t,r) \end{aligned} \tag{3.53}$$

for fixed value of t. Substituting the values of Q_1, Q_2, Q_3 and Q_4 in the function $M(r,t)$ we obtain

$$M(r,t) = \frac{U_1^2(t)}{3^{2n+2}} + \frac{N_1}{2^{4n+8} \cdot 3^{2n+2}} r^4 + \frac{N_2}{2^{3n+5} \cdot 3^{2n+2}} r^2 \tag{3.54}$$

where

$$\begin{aligned} N_1 = U_1(t)\Big\{ & 2^{n+1} \cdot 3^{2n+2} |U_1(t) - 2U_2(t) + U_3(t)| + U_1^3(t) 3^{2n+2} \\ & - 2^{n+2} \cdot 3^{2n+2} U_1(t) + 2^{4n+4} U_1(t) - 2^{2n+1} \cdot 3^{n+1} U_1^2(t) \Big\} r^4 \end{aligned}$$

and

$$\begin{aligned} N_2 = U_1(t) \frac{U_1(t)}{2^{3n+5} \cdot 3^{2n+2}} \Big\{ & 2 \cdot 3^{2n+2} |U_2(t) - U_1(t)| + 3^{2n+3} U_1(t) \\ & - 2^{3n+4} U_1(t) + 2^n \cdot 3^{n+1} U_1^2(t) + \beta^2 3^{2n+2} U_1(t) \Big\} r^2. \end{aligned}$$

Suppose $M(r)$ has a maximum value in the interior of $r \in [0,2]$, then by elementary calculus we obtain

$$M'(r,t) = \frac{4N_1}{2^{4n+8} \cdot 3^{2n+2}} r^3 + \frac{2N_2}{2^{3n+6} \cdot 3^{2n+2}} r. \tag{3.55}$$

Now, we will examine the sign of the function $M'(r,t)$ depending on the signs of N_1 and N_2 as follows.

(i) Let $N_1 \ge 0$ and $N_2 \ge 0$, then $M'(r,t) \ge 0$. This shows that $M(r,t)$ is an increasing function. Therefore,

$$\begin{aligned} \max\{M(r,t) : r \in (0,2)\} = & \frac{U_1(t)|U_1(t) - 2U_2(t) + U_3(t)|}{2^{3n+3}} \\ & + \frac{U_1(t)|U_2 - U_1(t)|}{2^{3n+2}} + \frac{U_1^2(t)}{2^{3n+3}} + \frac{U_1^2(t)}{2^{3n+3}} + \frac{U_1^4(t)}{2^{4n+4}} + \frac{\beta^2 U_1^2(t)}{2^{3n+3}}. \end{aligned} \tag{3.56}$$

(ii) Let $N_1 > 0$ and $N_2 < 0$, then

$$M'(r,t) = \frac{N_1}{2^{4n+6} \cdot 3^{2n+2}} r^3 + \frac{N_2}{2^{3n+5} \cdot 3^{2n+2}} r = \frac{N_1 r^3 + 2^{n+1} N_2 r}{2^{4n+6} \cdot 3^{2n+2}} = 0. \tag{3.57}$$

At critical point

$$r(N_1 r^2 + 2^{n+1} N_2) = 0 \implies r_0 = 0 \text{ or}$$

$$r_1 = \sqrt{\frac{-2^{n+1}N_2}{N_1}}. \tag{3.58}$$

Thus,

$$M''(r_0) = \frac{-N_2}{2^{3n+5}\cdot 3^{2n+1}} + \frac{N_2}{2^{3n+5}\cdot 3^{2n+2}} > 0. \tag{3.59}$$

Therefore, r_0 is the minimum point of the function $M(r,t)$. Hence, $M(r,t)$ can not have a maximum.

(iii) Let $N_1 \leq 0$ and $N_2 \leq 0$ then, $M'(r,t) \leq 0$. So, $M(r,t)$ is a decreasing function on the interval $(0,2)$. Therefore,

$$\max\{M(r,t) : r \in (0,2)\} = M(0) = \frac{U_1^2(t)}{3^{2n+2}} \tag{3.60}$$

(iv) Let$N_1 < 0$ and $N_2 > 0$

$$M''(r,t) = \frac{3N_1}{2^{4n+6}\cdot 3^{2n+2}}r^2 + \frac{N_2}{2^{3n+5}\cdot 3^{2n+2}} \tag{3.61}$$

$$M''(r_0,t) = \frac{-N_2}{2^{3n+5}\cdot 3^{2n+1}} + \frac{N_2}{2^{3n+5}\cdot 3^{2n+2}} < 0 \tag{3.62}$$

Therefore, $M''(r_0,t) < 0$. Hence, r_0 is the maximum point of the function $M(r,t)$ and the maximum value occurs at $r = r_0$. Thus,

$$\max\{M(r,t) : r \in (0,2)\} = M(r_0,t) \tag{3.63}$$

where

$$M(r_0,t) = \frac{U_1^2(t)}{3^{2n+2}} + \frac{N_2^4}{2^4\cdot 3^{2n+2}N_1^3} + \frac{N_2^2}{2^{n+3}\cdot 3^{2n+2}N_1^2}. \tag{3.64}$$

□

Corollary 3.4. *Let $f(z) \in C_\Sigma(\beta,\gamma,n,t)$ of the form (3.8) be in $C_\Sigma(\beta,t)$, then*

$$\mathcal{H}_2(2) \leq \begin{cases} M(2,t) & : N_3 \geq 0, N_4 \geq 0 \\ \max\left\{\frac{4t^2}{3^4}, M(2,t)\right\} & : N_3 > 0, N_4 < 0 \\ \frac{4t^2}{3^4} & : N_3 \leq 0, N_4 \leq 0 \\ \max\{M(r_0,t), M(2,t)\} & : N_3 < 0, N_4 > 0 \end{cases}$$

where

$$\begin{aligned} M(2,t) =& \frac{U_1(t)|U_1(t) - 2U_2(t) + U_3(t)|}{2^6} + \frac{U_1(t)|U_2 - U_1(t)|}{2^5} \\ &+ \frac{U_1^2(t)}{2^5} + \frac{U_1^4(t)}{2^8} + \frac{\beta^2 U_1^2(t)}{2^6} \\ M(r_0,t) =& \frac{U_1^2(t)}{3^4} + \frac{N_4^4}{2^4\cdot 3^4 N_3^3} + \frac{N_4^2}{2^4\cdot 3^4 N_3^2} \\ N_3 =& U_1(t)\{2^2\cdot 3^4|U_1(t) - 2U_2(t) + U_3(t)| + U_1^3(t)3^4 \\ &- 2^3\cdot 3^4 U_1(t) + 2^8 U_1(t) - 2^3\cdot 3^2 U_1^2(t)\}r^4 \\ N_4 =& U_1(t)\frac{U_1(t)}{2^8\cdot 3^4}\{2\cdot 3^4|U_2(t) - U_1(t)| + 3^5 U_1(t) - 2^7 U_1(t) + 2\cdot 3^2 U_1^2(t) \\ &+ \beta^2 3^4 U_1(t)\}r^2. \end{aligned}$$

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