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COEFFICIENT ESTIMATES FOR CERTAIN CLASS OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH κ-FIBONACCI NUMBERS

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Abstract. In the present work, we propose to introduce and investigate a new class $\mathcal{SL}_{\Sigma}^{q,\mu}(\gamma,\lambda,n,\tilde{p}_\kappa)$ of the function class Σ of bi-univalent functions related to κ-Fibonacci numbers defined in the open unit disk, which is associated with the Sălăgean type q-difference operator and satisfy some subordination conditions. We obtain coefficient bounds for the TaylorMaclaurin coefficients $|a_2|$ and $|a_3|$ of the functions in the new class. Furthermore, we solve the FeketeSzeg functional for functions in the class $\mathcal{SL}_{\Sigma}^{q,\mu}(\gamma,\lambda,n,\tilde{p}_\kappa)$.

1. Introduction and preliminaries

Let $\mathbb{C}$ be the complex plane and $\mathbb{U} = \{z : z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in $\mathbb{C}$. Also, let $\mathcal{A}$ denote the class of all analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k\, z^k \tag{1.1}$$

normalized by $f(0) = f'(0) - 1 = 0$ in the open unit disk $\mathbb{U}$. Further, $\mathcal{S} \subset \mathcal{A}$ be holomorphic functions in $\mathbb{U}$. If f and g are two analytic functions in $\mathbb{U}$, then f is subordinate to g, if there exists an analytic function w in $\mathbb{U}$ such that $w(0) = 0$ and $|w(0)| < 1$ for all $z \in \mathbb{U}$ that satisfies $f = g\,o\,w$ in $\mathbb{U}$. Symbolically, we write it as $f \prec g$. If $f \prec g$, then $f(0) = g(0)$ and $f(\mathbb{U}) \subseteq g(\mathbb{U})$. A lot of the classes have studied in geometric function theory can be described in terms subordination (for details, see [5], [15], [16], [28], [29], [25]). Using subordination, Ma and Minda [35] gave a unified representation of various geometric subclasses of $\mathcal{S}$ such as $\mathcal{S}_\alpha^*$ and $\mathcal{K}_\alpha$ which consisting of starlike and convex functions of order $\alpha\,(0 \leq \alpha < 1)$, respectively; which are usually characterized by the quantities $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ lying in a given domain in the right half-plane, for more details (see [21], [25]). Let us recall

$$\mathcal{S}^*(\varphi) = \left\{ f : f \in \mathcal{A} \ \text{ and } \ \frac{zf'(z)}{f(z)} \prec \varphi(z),\, z \in \mathbb{U} \right\}, \tag{1.2}$$

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$$\mathcal{K}(\varphi) = \left\{ f : f \in \mathcal{A} \text{ and } 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z),\, z \in \mathbb{U} \right\}. \tag{1.3}$$

The classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ which are the extensions of a well-known classical sets of S^*_γ and $\mathcal{K}_\gamma$. Many classes of functions have been defined by exchanging the function $\varphi(z)$ in (1.2) or in (1.3), for details reader can refer Table 1.

TABLE 1. Well-known Subclasses defined by exchanging $\varphi(z)$ in (1.2) or in (1.3)

$\varphi(z)$	Description
$\frac{1+z}{1-z}$	$\mathcal{S}^*, \mathcal{K}$; (*D. A. Brannan, W. E. kirwan* [2])
$\sqrt{1+cz}$; $(0 < c \leq 1)$	$\mathcal{S}^*(q_c)$; (*J. Sokół* [28])
$z + \sqrt{1+z}$; $(0 < c \leq 1)$	$\mathcal{S}^*(q)$; (*R. K. Raina, J. Sokół* [21])
$\frac{1+(1-2\alpha)z}{1-z}$; $(\alpha < 1)$	$\mathcal{S}^*(\alpha), \mathcal{K}(\alpha)$; (*M. S. Robertson* [23])
$\left(\frac{1+z}{1-z}\right)^\beta$; $(0 < \beta \leq 1)$	$\mathcal{S}^*(\beta)$; (*J. Stankiewicz* [29])
$1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$	$\mathcal{UCV}$; (*A. W. Goodman*[10], *F. Rønning* [24])
$\left(\frac{1+z}{1+(1-b)/bz}\right)^{1/\alpha}$	$\mathcal{S}^*(q_\alpha)$; (*E. Paprocki, J. Sokół* [20])

For any positive real number κ, the κ-Fibonacci number sequence $\{\mathcal{F}_{\kappa,n}\}_{n=0}^\infty$ is defined recursively by

$$\mathcal{F}_{\kappa,0} = 0, \mathcal{F}_{\kappa,1} = 1 \quad \text{and} \quad \mathcal{F}_{\kappa,n+1} = \kappa\mathcal{F}_{\kappa,n} + \mathcal{F}_{\kappa,n-1} \quad \text{for} \quad n \geq 1. \tag{1.4}$$

When $\kappa = 1$, we obtain the well-known Fibonacci numbers $\mathcal{F}_n$ defined as $\mathcal{F}_0 = 0$, $\mathcal{F}_1 = 1$ and $\mathcal{F}_{n+1} = \mathcal{F}_n + \mathcal{F}_{n-1}$, for $n \geq 1$ (refer [8], [19]). It is known that the nth κ-Fibonacci number is given by

$$\mathcal{F}_{\kappa,n} = \frac{(\kappa - \tau_\kappa)^n - \tau_\kappa^n}{\sqrt{\kappa^2+4}}, \tag{1.5}$$

where $\tau_\kappa = \dfrac{\kappa - \sqrt{\kappa^2+4}}{4}$. If $\tilde{p}_\kappa(z) = 1 + \sum\limits_{n=1}^{\infty} \tilde{p}_{\kappa,n}\, z^n$ then we have (see also [19])

$$\tilde{p}_{\kappa,n} = (\mathcal{F}_{\kappa,n-1} + \mathcal{F}_{\kappa,n+1})\, \tau_\kappa^n, n = 1, 2, 3, \cdots. \tag{1.6}$$

Recently, Özgür and Sokół [19] have studied the class $\mathcal{SL}^k$ of starlike functions connected with κ-Fibonacci numbers. Also, one can refer to ([9], [11]) and its reference for detail study of subclasses of analytic functions related to κ-Fibonacci numbers.

Koebe One-Quarter Theorem [6] ensures that every function $f \in \mathcal{S}$ has a inverse f^{-1}, defined by $f^{-1}(f(z)) = z,\ z \in \mathbb{U}$, $f^{-1}(f(w)) = w,\ |w| < r_0(f); r_0(f) \geq$

$\frac{1}{4}$ where $f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots$. A function $f \in \mathcal{S}$ is said to be bi-univalent in $\mathbb{U}$ if there exists a function $g \in \mathcal{S}$ such that $g(z)$ is a univalent extension of f^{-1} to $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$. Lewin [18] first introduced and studied bi-univalent function classes, and he proved that $|a_2| < 1.51$. Subsequently, Styer and Wright [32] showed $|a_2| > 4/3$. Ismail et al. [13] first introduced a class of generalized complex function on subclasses of the analytic function via basic (or q-) calculus. Recently many newsworthy results related to bi-univalent functions and basic (or q-) calculus are meticulously studied by Srivastava et al. [30], also refer ([7], [17], [31]) and the references therein.

For the convenience, we recall basic (or q-) calculus due to Jakson [14] which are used in this paper. For $0 < q < 1$, the basic (or q-) derivative of a function f is defined by

$$\mathcal{D}_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}; \qquad (z \neq 0), \tag{1.7}$$

from (1.7), we deduce that

$$\mathcal{D}_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q \, a_k \, z^{k-1}, \tag{1.8}$$

where

$$[k]_q = \frac{1-q^k}{1-q}. \tag{1.9}$$

In [10], Govindaraj and Sivasubramanian define Sălăgean q-differential operator as follows

$$\mathcal{D}_q^n f(z) = z + \sum_{k=2}^{\infty} [k]_q^n \, a_k \, z^k \qquad (n \in \mathbb{N}_0, \, z \in \mathbb{U}). \tag{1.10}$$

We note as $q \to 1^-$

$$\mathcal{D}^n f(z) = z + \sum_{k=2}^{\infty} k^n \, a_k z^k \qquad (n \in \mathbb{N}_0, \, z \in \mathbb{U}). \tag{1.11}$$

Motivated by the aforementioned works we define the following subclass of the function class Σ, as follows:

Definition 1.1. A function $f \in \Sigma$ is said to be in the class $\mathcal{SL}_{\Sigma}^{q,\mu}(\gamma, \lambda, n, \tilde{p}_\kappa)$ if the following subordination hold:

$$1+\frac{1}{\gamma}\left[(1-\lambda)\left(\frac{\mathcal{D}_q^n f(z)}{z}\right)^{\mu} + \lambda\left(\mathcal{D}_q^n f(z)\right)'\left(\frac{\mathcal{D}_q^n f(z)}{z}\right)^{\mu-1} - 1\right] \prec \tilde{p}_\kappa(z) = \frac{1+\tau_\kappa^2 z^2}{1-\kappa\tau_\kappa z + \tau_\kappa^2 z^2} \tag{1.12}$$

and

$$1+\frac{1}{\gamma}\left[(1-\lambda)\left(\frac{\mathcal{D}_q^n g(w)}{w}\right)^{\mu} + \lambda\left(\mathcal{D}_q^n g(w)\right)'\left(\frac{\mathcal{D}_q^n g(w)}{w}\right)^{\mu-1} - 1\right] \prec \tilde{p}_\kappa(w) = \frac{1+\tau_\kappa^2 w^2}{1-\kappa\tau_\kappa w + \tau_\kappa^2 w^2}, \tag{1.13}$$

where $0 < q < 1, \lambda \geq 1, \mu \geq 0,\ \gamma \in \mathbb{C} \setminus \{0\}, n \in \mathbb{N}_0$ and $g = f^{-1}$ and $\tau_\kappa = \frac{\kappa-\sqrt{\kappa^2+4}}{4}$.

Remark 1.2. A function $f \in \Sigma$ given by (1.1) and $\lambda = 0$, we note that $\mathcal{SL}_{\Sigma}^{q,\mu}(\gamma, \lambda, n, \tilde{p}_\kappa)$= $\mathcal{SL}_{\Sigma}^{q,\mu}(\gamma, n, \tilde{p}_\kappa)$ satisfies the following conditions respectively:

$$1 + \frac{1}{\gamma}\left[\left(\frac{\mathcal{D}_q^n f(z)}{z}\right)^{\mu} - 1\right] \prec \tilde{p}_\kappa(z) = \frac{1+\tau_\kappa^2 z^2}{1-\kappa\tau_\kappa z+\tau_\kappa^2 z^2}$$

and

$$1 + \frac{1}{\gamma}\left[\left(\frac{\mathcal{D}_q^n g(w)}{w}\right)^{\mu} - 1\right] \prec \tilde{p}_\kappa(w) = \frac{1+\tau_\kappa^2 w^2}{1-\kappa\tau_\kappa w+\tau_\kappa^2 w^2},$$

where $0 < q < 1, \mu \geq 0,\ \gamma \in \mathbb{C} \setminus \{0\}, n \in \mathbb{N}_0$ and $g = f^{-1}$ and $\tau_\kappa = \frac{\kappa-\sqrt{\kappa^2+4}}{4}$.

Remark 1.3. A function $f \in \Sigma$ given by (1.1) and $\lambda = 1$, we note that $\mathcal{SL}_{\Sigma}^{q,\mu}(\gamma, \lambda, n, \tilde{p}_\kappa)$= $\mathcal{SL}_{\Sigma}^{q,\mu}(\gamma, 1, n, \tilde{p}_\kappa)$ satisfies the following conditions respectively:

$$1 + \frac{1}{\gamma}\left[\left(\mathcal{D}_q^n f(z)\right)'\left(\frac{\mathcal{D}_q^n f(z)}{z}\right)^{\mu-1} - 1\right] \prec \tilde{p}_\kappa(z) = \frac{1+\tau_\kappa^2 z^2}{1-\kappa\tau_\kappa z+\tau_\kappa^2 z^2}$$

and

$$1 + \frac{1}{\gamma}\left[\left(\mathcal{D}_q^n g(w)\right)'\left(\frac{\mathcal{D}_q^n g(w)}{w}\right)^{\mu-1} - 1\right] \prec \tilde{p}_\kappa(w) = \frac{1+\tau_\kappa^2 w^2}{1-\kappa\tau_\kappa w+\tau_\kappa^2 w^2},$$

where $0 < q < 1, \mu \geq 0,\ \gamma \in \mathbb{C} \setminus \{0\}, n \in \mathbb{N}_0$ and $g = f^{-1}$ and $\tau_\kappa = \frac{\kappa-\sqrt{\kappa^2+4}}{4}$.

For suitable choice of $\gamma,\ \lambda,\ n,\ \mu$ and special functions, as $q \to 1^-$ in Definition 1.1, we obtain the following well known subclasses

(i) $\mathcal{SL}_{\Sigma}^{1,\mu}(1, \lambda, 0, H(z,t)) = \mathcal{N}_\sigma(\lambda, t)$ consist of bi-univalent functions $f \in \Sigma$ satisfying the condition

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \prec H(z,t) = \frac{1}{1-2tz+z^2}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} \prec H(w,t) = \frac{1}{1-2tw+w^2},$$

where the function $g = f^{-1}$. This class was introduced and studied by Bulut et al. [3] and Orhan et al. [20].

(ii) $\mathcal{SL}_{\Sigma}^{1,1}(1, \lambda, 0, H(z,t)) = \mathcal{B}_\sigma(\lambda, t)$ consist of bi-univalent functions $f \in \Sigma$ satisfying the condition

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \prec H(z,t) = \frac{1}{1-2tz+z^2}$$

and

$$(1-\lambda)\frac{g(w)}{w} + \lambda g'(z) \prec H(w,t) = \frac{1}{1-2tw+w^2},$$

where the function $g = f^{-1}$. This class was studied by Bulut et al. [4].

(iii) $\mathcal{SL}_{\Sigma}^{1,\mu}(1,1,0,H(z,t)) = \mathcal{B}_{\sigma}^{\mu}(t)$ consist of Bi-Bazilevič functions satisfying the condition

$$f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \prec H(z,t) = \frac{1}{1-2tz+z^2}$$

and

$$g'(z)\left(\frac{g(w)}{w}\right)^{\mu-1} \prec H(w,t) = \frac{1}{1-2tw+w^2},$$

where the function $g = f^{-1}$. This class was studied by Altnkaya and Yalçn [1].

(iv) $\mathcal{SL}_{\Sigma}^{1,0}(1,1,0,H(z,t)) = \mathcal{S}_{\sigma}^{*}(t)$ consist of functions f satisfying the condition

$$\frac{zf'(z)}{f(z)} \prec H(z,t) = \frac{1}{1-2tz+z^2}$$

and

$$\frac{wg'(w)}{g(w)} \prec H(w,t) = \frac{1}{1-2tw+w^2},$$

where the function $g = f^{-1}$.

(v) $\mathcal{SL}_{\Sigma}^{1,1}(1,1,0,H(z,t)) = \mathcal{B}_{\sigma}(t)$ consist of functions f satisfying the condition

$$f'(z) \prec H(z,t) = \frac{1}{1-2tz+z^2}$$

and

$$g'(w) \prec H(w,t) = \frac{1}{1-2tw+w^2},$$

where the function $g = f^{-1}$.

In order to derive our main results, we have to recall the following lemma.

Lemma 1.4. [22] *If $p \in \mathcal{P}$ then $|d_k| \leq 2$ for each k, where $\mathcal{P}$ is the family of all functions p analytic in $\mathbb{U}$ for which $Re\{p(z)\} > 0$*

$$p(z) = 1 + d_1 z + d_2 z^2 + d_3 z^3 + \ldots \qquad for\ z \in \mathbb{U}.$$

The object of the present paper is to introduce a new subclass $\mathcal{SL}_{\Sigma}^{q,\mu}(\gamma,\lambda,n,\tilde{p}_{\kappa})$ of bi-univalent function and obtain estimates of two coefficients as well as the Fekete-Szeg functional for functions in aforementioned class.

2. Coefficient estimates for the function class $\mathcal{SL}_{\Sigma}^{q,\mu}(\gamma,\lambda,n,\tilde{p}_{\kappa})$

In this section we find the coefficient bounds for certain class of bi-univalent functions in the class $\mathcal{SL}_{\Sigma}^{q,\mu}(\gamma,\lambda,n,\tilde{p}_{\kappa})$.

Theorem 2.1. *Let $f(z)$ be in the form (1.1). If $f \in \mathcal{SL}_{\Sigma}^{q,\mu}(\gamma,\lambda,n,\tilde{p}_{\kappa})$, then*

$$|a_2| \leq \frac{|\gamma||\kappa\tau_{\kappa}|\sqrt{\kappa}}{\sqrt{\Psi}} \tag{2.1}$$

and

$$|a_3| \leq \frac{|\gamma||\kappa\tau_{\kappa}|\Big(\Psi + \gamma(2\lambda+\mu)[3]_q^n\kappa^2\tau_{\kappa}\Big)}{(2\lambda+\mu)\Psi[3]_q^n}, \tag{2.2}$$

where

$$\Psi = (2\lambda+\mu)\gamma\kappa^2\tau_{\kappa}\left(2[3]_q^n + \frac{(\mu-1)}{2}[2]_q^{2n}\right) + \Big(\kappa - (\kappa^2+2)\tau_{\kappa}\Big)(\lambda+\mu)^2[2]_q^{2n}.$$

Proof. Since $f \in \mathcal{SL}_{\Sigma}^{q,\mu}(\gamma,\lambda,n,\tilde{p}_{\kappa})$. From the Definition 1.1, we have

$$1 + \frac{1}{\gamma}\left[(1-\lambda)\left(\frac{\mathcal{D}_q^n f(z)}{z}\right)^{\mu} + \lambda\left(\mathcal{D}_q^n f(z)\right)'\left(\frac{\mathcal{D}_q^n f(z)}{z}\right)^{\mu-1} - 1\right] = \tilde{p}_{\kappa}(u(z)) \tag{2.3}$$

and

$$1 + \frac{1}{\gamma}\left[(1-\lambda)\left(\frac{\mathcal{D}_q^n g(w)}{w}\right)^{\mu} + \lambda\left(\mathcal{D}_q^n g(w)\right)'\left(\frac{\mathcal{D}_q^n g(w)}{w}\right)^{\mu-1} - 1\right] = \tilde{p}_{\kappa}(v(w)). \tag{2.4}$$

Define $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$. If $p \prec \tilde{p}_{\kappa}$, then there exists an analytic function u such that $|u(z)| < 1$ in $\mathbb{U}$ and $p(z) = \tilde{p}_{\kappa}(u(z))$. Therefore, the function

$$s(z) := \frac{1+u(z)}{1-u(z)} = 1 + b_1 z + b_2 z^2 + \cdots$$

is in the class $\mathcal{P}$. It follows that

$$u(z) = \frac{b_1}{2}z + \left(b_2 - \frac{b_1^2}{2}\right)\frac{z^2}{2} + \left(b_3 - b_1 b_2 + \frac{b_1^3}{4}\right)\frac{z^3}{2} + \cdots$$

and

$$\begin{aligned}\tilde{p}_{\kappa}(u(z)) =& 1 + \frac{\tilde{p}_{\kappa,1}b_1}{2}z + \left(\frac{1}{2}\left(b_2 - \frac{b_1^2}{2}\right)\tilde{p}_{\kappa,1} + \frac{b_1^2}{4}\tilde{p}_{\kappa,2}\right)z^2 \\ &+ \left(\frac{1}{2}\left(b_3 - b_1 b_2 + \frac{b_1^3}{4}\right)\tilde{p}_{\kappa,1} + \frac{1}{2}b_1\left(b_2 - \frac{b_1^2}{2}\right)\tilde{p}_{\kappa,2} + \frac{b_1^3}{8}\tilde{p}_{\kappa,3}\right)z^3 + \cdots .\end{aligned} \tag{2.5}$$

Similarly, there exists an analytic function v such that $|v(z)| < 1$ in $\mathbb{U}$ and $p(w) = \tilde{p}_{\kappa}(v(w))$. Therefore, the function

$$t(w) := \frac{1+v(w)}{1-v(w)} = 1 + c_1 w + c_2 w^2 + \cdots$$

is in the class $\mathcal{P}$. It follows that

$$v(w) = \frac{c_1}{2}w + \left(c_2 - \frac{c_1^2}{2}\right)\frac{w^2}{2} + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)\frac{w^3}{2} + \cdots$$

and

$$\tilde{p}_\kappa(v(w)) = 1 + \frac{\tilde{p}_{\kappa,1}c_1}{2}w + \left(\frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)\tilde{p}_{\kappa,1} + \frac{c_1^2}{4}\tilde{p}_{\kappa,2}\right)w^2 + \left(\frac{1}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)\tilde{p}_{\kappa,1} + \frac{1}{2}c_1\left(c_2 - \frac{c_1^2}{2}\right)\tilde{p}_{\kappa,2} + \frac{c_1^3}{8}\tilde{p}_{\kappa,3}\right)w^3 + \cdots . \tag{2.6}$$

From (1.6), we find the coefficients $\tilde{p}_{\kappa,n}$ of the function $\tilde{p}_\kappa$ is given by

$$\tilde{p}_{\kappa,n} = (\mathcal{F}_{\kappa,n-1} + \mathcal{F}_{\kappa,n+1})\,\tau_\kappa^n, n = 1, 2, 3, \cdots . \tag{2.7}$$

This shows the relevant connection $\tilde{p}_\kappa$ with the sequence of κ-Fibonacci numbers

$$\begin{aligned}\tilde{p}_\kappa(z) &= 1 + \sum_{n=1}^{\infty} \tilde{p}_{\kappa,n}\, z^n \\ &= 1 + (\mathcal{F}_{\kappa,0} + \mathcal{F}_{\kappa,2})\,\tau_\kappa z + (\mathcal{F}_{\kappa,0} + \mathcal{F}_{\kappa,2})\,\tau_\kappa^2 z^2 + \cdots \\ &= 1 + k\tau_\kappa z + \left(k^2 + 2\right)\tau_\kappa^2 z^2 + \left(k^3 + 3k\right)\tau_\kappa^3 z^3 + \cdots .\end{aligned} \tag{2.8}$$

By virtue of (2.3), (2.4), (2.5), (2.6) and (2.8), we have

$$\frac{1}{\gamma}(\lambda + \mu)[2]_q^n a_2 = \frac{b_1\kappa\tau_\kappa}{2}, \tag{2.9}$$

$$\frac{(2\lambda + \mu)}{\gamma}\frac{(\mu - 1)}{2}[2]_q^{2n}a_2^2 + \frac{(2\lambda + \mu)}{\gamma}[3]_q^n a_3 = \frac{1}{2}\left(b_2 - \frac{b_1^2}{2}\right)\kappa\tau_\kappa + \frac{b_1^2}{4}(\kappa^2 + 2)\tau_\kappa^2, \tag{2.10}$$

$$-\frac{1}{\gamma}(\lambda + \mu)[2]_q^n a_2 = \frac{c_1\kappa\tau_\kappa}{2} \tag{2.11}$$

and

$$-\frac{(2\lambda + \mu)}{\gamma}[3]_q^n a_3 + \frac{(2\lambda + \mu)}{\gamma}\left(2[3]_q^n + \frac{(\mu - 1)}{2}[2]_q^{2n}\right)a_2^2 = \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)\kappa\tau_\kappa + \frac{c_1^2}{4}(\kappa^2 + 2)\tau_\kappa^2. \tag{2.12}$$

In view of (2.9) and (2.11), we get

$$b_1 = -c_1$$

and

$$a_2^2 = \frac{\gamma^2(b_1^2 + c_1^2)\kappa^2\tau_\kappa^2}{8(\lambda + \mu)^2[2]_q^{2n}}. \tag{2.13}$$

Adding (2.10) and (2.12), as a result of subsequent computations performed by using (2.13), we find

$$\frac{(2\lambda + \mu)}{\gamma}\left(2[3]_q^n + (\mu - 1)[2]_q^{2n}\right)a_2^2 = \frac{1}{2}(b_2 + c_2)\kappa\tau_\kappa - \frac{1}{4}(b_1^2 + c_1^2)\kappa\tau_\kappa + \frac{1}{4}(b_1^2 + c_1^2)(\kappa^2 + 2)\tau_\kappa^2.$$

$$a_2^2 = \frac{\gamma^2(b_2 + c_2)\kappa^3\tau_\kappa^2}{4\sqrt{\Psi}}, \tag{2.14}$$

where

$$\Psi = (2\lambda + \mu)\gamma\kappa^2\tau_\kappa\left(2[3]_q^n + \frac{(\mu - 1)}{2}[2]_q^{2n}\right) + \left(\kappa - (\kappa^2 + 2)\tau_\kappa\right)(\lambda + \mu)^2[2]_q^{2n}.$$

By applying Lemma 1.4, we get

$$|a_2| \le \frac{|\gamma||\kappa\tau_\kappa|\sqrt{\kappa}}{\sqrt{\Psi}}. \tag{2.15}$$

Subtracting (2.12) from (2.10), we obtain

$$|a_3| \le \frac{|\gamma||\kappa\tau_\kappa|}{(2\lambda+\mu)[3]_q^n} + |a_2|^2. \tag{2.16}$$

Then, applying Lemma 1.4, we get

$$|a_3| \le \frac{|\gamma||\kappa\tau_\kappa|\Big(\Psi + \gamma(2\lambda+\mu)[3]_q^n\kappa^2\tau_\kappa\Big)}{(2\lambda+\mu)\Psi[3]_q^n}. \tag{2.17}$$

This completes the proof. $\square$

Corollary 2.2. *Let $f(z)$ be in the form (1.1). If $f \in \mathcal{SL}_\Sigma^{q,\mu}(\gamma, n, \tilde{p}_\kappa)$, then*

$$|a_2| \le \frac{|\gamma||\kappa\tau_\kappa|\sqrt{\kappa}}{\sqrt{\Psi}} \tag{2.18}$$

and

$$|a_3| \le \frac{|\gamma||\kappa\tau_\kappa|\Big(\Psi + \gamma\mu[3]_q^n\kappa^2\tau_\kappa\Big)}{\mu\Psi[3]_q^n},$$

where

$$\Psi = \mu\gamma\kappa^2\tau_\kappa\left(2[3]_q^n + \frac{(\mu-1)}{2}[2]_q^{2n}\right) + \Big(\kappa - (\kappa^2+2)\tau_\kappa\Big)\mu^2[2]_q^{2n}.$$

Corollary 2.3. *Let $f(z)$ be in the form (1.1). If $f \in \mathcal{SL}_\Sigma^{q,\mu}(\gamma, 1, n, \tilde{p}_\kappa)$, then*

$$|a_2| \le \frac{|\gamma||\kappa\tau_\kappa|\sqrt{\kappa}}{\sqrt{\Psi}}$$

and

$$|a_3| \le \frac{|\gamma||\kappa\tau_\kappa|\Big(\Psi + \gamma(2+\mu)[3]_q^n\kappa^2\tau_\kappa\Big)}{(2+\mu)\Psi[3]_q^n},$$

where

$$\Psi = (2+\mu)\gamma\kappa^2\tau_\kappa\left(2[3]_q^n + \frac{(\mu-1)}{2}[2]_q^{2n}\right) + \Big(\kappa - (\kappa^2+2)\tau_\kappa\Big)(1+\mu)^2[2]_q^{2n}.$$

3. Fekete-Szeg problem for the function class $\mathcal{SL}_\Sigma^{q,\mu}(\gamma, \lambda, n, \tilde{p}_\kappa)$

In this section we find the Fekete-Szeg problem for certain classes of bi-univalent functions in the class $\mathcal{SL}_\Sigma^{q,\mu}(\gamma, \lambda, n, \tilde{p}_\kappa)$.

Theorem 3.1. *Let $f(z)$ be in the form (1.1). If $f \in \mathcal{SL}_\Sigma^{q,\mu}(\gamma, \lambda, n, \tilde{p}_\kappa)$, then*

$$|a_3 - \sigma a_2^2| \le \begin{cases} \frac{\gamma|\kappa\tau_\kappa|}{(2\lambda+\mu)[3]_q^n}; & 0 \le |h(\sigma)| \le \frac{\gamma|\kappa\tau_\kappa|}{4(2\lambda+\mu)[3]_q^n} \\ 4|h(\sigma)|; & |h(\sigma)| \ge \frac{\gamma|\kappa\tau_\kappa|}{4(2\lambda+\mu)[3]_q^n}. \end{cases} \tag{3.1}$$

Proof. By virtue of (2.16), we have

$$a_3 - \sigma a_2^2 = \frac{\gamma(b_2 - c_2)\kappa\tau_\kappa}{4(2\lambda+\mu)[3]_q^n} + (1-\sigma)a_2^2. \tag{3.2}$$

From (3.2) and (2.14) it follows that

$$\begin{aligned} a_3 - \sigma a_2^2 &= \frac{\gamma(b_2 - c_2)\kappa\tau_\kappa}{4(2\lambda+\mu)[3]_q^n} + (1-\sigma)\frac{\gamma^2(b_2+c_2)\kappa^3\tau_\kappa^2}{4\Psi} \\ &= \left(h(\sigma) + \frac{\gamma|\kappa\tau_\kappa|}{4(2\lambda+\mu)[3]_q^n}\right)b_2 + \left(h(\sigma) + \frac{\gamma|\kappa\tau_\kappa|}{4(2\lambda+\mu)[3]_q^n}\right)c_2, \end{aligned}$$

where

$$h(\sigma) = \frac{(1-\sigma)\gamma^2\kappa^3\tau_\kappa^2}{4\Psi}.$$

This enables us to conclude that

$$|a_3 - \sigma a_2^2| \leq \begin{cases} \frac{\gamma|\kappa\tau_\kappa|}{(2\lambda+\mu)[3]_q^n}; & 0 \leq |h(\sigma)| \leq \frac{\gamma|\kappa\tau_\kappa|}{4(2\lambda+\mu)[3]_q^n} \\ 4|h(\sigma)|; & |h(\sigma)| \geq \frac{\gamma|\kappa\tau_\kappa|}{4(2\lambda+\mu)[3]_q^n} \end{cases} \tag{3.3}$$

which yields the desired inequality. □

Corollary 3.2. *Let $f(z)$ be in the form (1.1). If $f \in \mathcal{SL}_\Sigma^{q,\mu}(\gamma, n, \tilde{p}_\kappa)$, then*

$$|a_3 - \sigma a_2^2| \leq \begin{cases} \frac{\gamma|\kappa\tau_\kappa|}{\mu[3]_q^n}; & 0 \leq |h(\sigma)| \leq \frac{\gamma|\kappa\tau_\kappa|}{4\mu[3]_q^n} \\ 4|h(\sigma)|; & |h(\sigma)| \geq \frac{\gamma|\kappa\tau_\kappa|}{4\mu[3]_q^n}. \end{cases} \tag{3.4}$$

Corollary 3.3. *Let $f(z)$ be in the form (1.1). If $f \in \mathcal{SL}_\Sigma^{q,\mu}(\gamma, 1, n, \tilde{p}_\kappa)$, then*

$$|a_3 - \sigma a_2^2| \leq \begin{cases} \frac{\gamma|\kappa\tau_\kappa|}{(2+\mu)[3]_q^n}; & 0 \leq |h(\sigma)| \leq \frac{\gamma|\kappa\tau_\kappa|}{4(2+\mu)[3]_q^n} \\ 4|h(\sigma)|; & |h(\sigma)| \geq \frac{\gamma|\kappa\tau_\kappa|}{4(2+\mu)[3]_q^n}. \end{cases} \tag{3.5}$$

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