

ESSENTIALLY GENERALIZED λ -SLANT HANKEL OPERATORS

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ABSTRACT. The notion of an essentially generalized λ -slant Hankel operator of k^{th} -order on the space L^2 is introduced and its algebraic properties are discussed. The study is further carried to compression and essentially compression of generalized λ -slant Hankel operator of k^{th} -order on the space H^2 .

1. INTRODUCTION

The notion of Toeplitz operators was introduced by O. Toeplitz[14] in the year 1911. Hankel operators are the formal companions of Toeplitz operators which have occurred in realization problem for certain discrete time linear systems and in determining which systems are exactly controllable [12]. The theory of Hankel operators admits of vast applications. In 1881, Kronecker[9] obtained first theorem about Hankel matrices that characterizes Hankel matrices of finite rank. This development led to different generalizations of the original concept, like, slant Hankel operators, λ -Hankel operators and (λ, μ) -Hankel operators (see [1,2]). A lot of progress has taken place in the study of Hankel operators on Bergman spaces on the disk, Dirichlet type spaces, Bergman and Hardy spaces on the unit ball on symmetric domains, etc[11]. We refer to [7,8,11] and the references therein for the basic study of Hankel operators on these spaces.

Meanwhile, a new class of operators was introduced which behaves essentially in the same manner as Hankel operators do. In the year 2002, Avendano [2] introduced the notion of λ -Hankel operators as those operators X which satisfy the operator equation $S^*X - XS = \lambda X$, where S denotes the unilateral forward shift on H^2 . Avendano also described the solution of the equation $\lambda S^*X = XS$ as λ -Hankel operators in a different approach. For the definition and basic facts about the spaces L^2 , H^2 and L^∞ , we refer to [7,8,11,14] and the references therein.

Motivated by the work of Avendano [2] and Barria and Halmos[4], another class of operators was discussed by G. Datt and R. Aggarwal which involved the study of operator equation $\lambda M_{\bar{z}}X = XM_{z^k}$ for $\lambda \in \mathbb{C}$ and $k \geq 2$ along with some spectral properties of the solutions of this equation (see [5]). We call the solution of this equation as generalized λ -slant Hankel operators of k^{th} -order. For $k=2$,

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the solutions of equation $\lambda M_{\bar{z}}X = XM_{z^2}$ are simply called generalized λ -slant Hankel operators.

Motivated by these developments, in this paper we introduce and study the notion of essentially generalized λ -slant Hankel operators of k^{th} -order on the space L^2 along with their compressions on the space H^2 . We also study compressions of generalized λ -slant Hankel operators of k^{th} -order on the space H^2 and their spectrum.

2. ESSENTIALLY GENERALIZED λ -SLANT HANKEL OPERATORS

Throughout the paper, k is assumed to be an integer greater than or equal to 2. We begin with the following definitions, the detailed study of which can be seen in [3,6].

Definition 2.1. A bounded linear operator A on the space H^2 is said to be essentially Hankel if $T_z^*A - AT_z$ is a compact operator on H^2 . Since T_z is unitary in Calkin algebra we equivalently have $AT_z^* - T_zA$ is a compact operator on H^2 as the required condition. The set of all essentially Hankel operators on H^2 is denoted by $essHank$.

Definition 2.2. A bounded linear operator A on the space H^2 is said to be an essentially compression of a generalized λ -slant Toeplitz operator if it satisfies the operator equation $\lambda T_z A - AT_{z^k} = K$, for some compact operator K on H^2 . We denote by $k-essToep_\lambda(H^2)$, the set of all essentially compressions of generalized λ -slant Toeplitz operators on H^2 .

We introduce the following:

Definition 2.3. For a fixed integer $k \geq 2$, a bounded linear operator A on the space L^2 is said to be an essentially generalized λ -slant Hankel operator of k^{th} -order if it satisfies the operator equation $\lambda M_{\bar{z}}A - AM_{z^k} = K$, for some compact operator K on L^2 . We denote the set of all essentially generalized λ -slant Hankel operators of k^{th} -order on the space L^2 by $k-essHank_\lambda(L^2)$.

It is easy to see that every generalized λ -slant Hankel operator of k^{th} -order as well as every compact operator is in $k-essHank_\lambda(L^2)$. Also, from the definition itself, it is clear that every compact perturbation of generalized λ -slant Hankel operator (i.e., the sum of a generalized λ -slant Hankel operator and a compact operator) is in $k-essHank_\lambda(L^2)$. However, we find the strict inclusion of $\mathcal{K}(L^2)$, the ideal of all compact operators on L^2 and the class of all generalized λ -slant Hankel operators in $k-essHank_\lambda(L^2)$. For, we present an example of a non-compact operator in the class $k-essHank_\lambda(L^2)$ which is not a generalized λ -slant Hankel operator.

Example 2.4. Let $\lambda \in \mathbb{C}$ be such that $|\lambda| = 1$. Define A on L^2 as

$$Ae_n = \begin{cases} e_1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0, n+1 \text{ is not a multiple of } k \\ (\bar{\lambda})^m e_m & \text{where } m = -(\frac{n+1}{k}), \text{ if } n+1 \text{ is a multiple of } k \end{cases},$$

where $\{e_n\}_{n \in \mathbb{Z}}$ denotes the standard orthonormal basis of L^2 . Then A is a non-compact operator satisfying

$$A = D_\lambda W_k J M_z + C,$$

where W_k is defined on L^2 as

$$W_k(z^i) = \begin{cases} z^{i/k} & \text{if } i \text{ is divisible by } k \\ 0 & \text{if } i \text{ is not divisible by } k \end{cases},$$

D_λ is the composition operator on L^2 defined as $D_\lambda f(z) = f(\bar{\lambda}z)$ for all $f \in L^2$ with a property that $D_\lambda M_{\bar{z}} = \lambda M_{\bar{z}} D_\lambda$ (see [3]), J denotes the unitary flip operator on L^2 defined as $J(f(z)) = f(\bar{z})$ and C on L^2 is defined as

$$C e_n = \begin{cases} e_1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

for all $n \in \mathbb{Z}$.

If we write $A_0 = D_\lambda W_k J M_z$, without much efforts, we find that $A = A_0 + C$ is a compact perturbation of A_0 which satisfies $\lambda M_{\bar{z}} A_0 - A_0 M_{z^k} = 0$. Therefore, $A \in \text{k-essHank}_\lambda(L^2)$. Further, A is not a generalized λ -slant Hankel operator of k^{th} -order since $\lambda M_{\bar{z}} A - A M_{z^k} = \lambda M_{\bar{z}} C - C M_{z^k}$ which is non-zero as $\lambda M_{\bar{z}} C - C M_{z^k}(e_0) = \lambda \neq 0$.

The set $\text{k-essHank}_\lambda(L^2)$ has following basic properties:

- (1) $\text{k-essHank}_\lambda(L^2)$ is a norm-closed vector subspace of $\mathcal{B}(L^2)$, the space of all bounded linear operators on the space L^2 .
- (2) $\text{k-essHank}_\lambda(L^2) \cap \mathcal{K}(L^2) = \mathcal{K}(L^2)$.
- (3) If $T_1, T_2 \in \text{k-essHank}_\lambda(L^2)$ then $T_1 T_2 \in \text{k-essHank}_\lambda(L^2)$ iff $T_1 M_{z^k} T_2 = \lambda T_1 M_{\bar{z}} T_2 \pmod{\mathcal{K}}$.
- (4) Let $T \in \text{k-essHank}_\lambda(L^2)$ and $p \in \mathbb{N}, p > 1$. If $n(p)$ denotes number of partitions of p as $p = m_i + n_i; i = 1, 2, \dots, n(p); m_i, n_i \in \mathbb{N}$ then the following are equivalent:
 - (a) $T^p \in \text{k-essHank}_\lambda(L^2)$.
 - (b) $\lambda T^{m_i} M_{\bar{z}} T^{n_i} = T^{m_i} M_{z^k} T^{n_i} \pmod{\mathcal{K}}, i = 1, 2, \dots, n(p)$.
 - (c) $\lambda T^{n_i} M_{\bar{z}} T^{m_i} = T^{n_i} M_{z^k} T^{m_i} \pmod{\mathcal{K}}, i = 1, 2, \dots, n(p)$.
- (5) $\text{k-essHank}_\lambda(L^2)$ is neither a self-adjoint set nor an algebra of operators on L^2 , which we verify through the following example.

Example 2.5. Let $A = B = D_\lambda W_k J M_z + C$. Then $A, B \in \text{k-essHank}_\lambda(L^2)$ by Example 2.4. Put $H = AB$, then we find that $\lambda M_{\bar{z}} H - H M_{z^k} = G \pmod{\mathcal{K}}$, where $G = \lambda M_{\bar{z}} (D_\lambda W_k J M_z)^2 - (D_\lambda W_k J M_z)^2 M_{z^k}$ satisfies

$$G e_n = \begin{cases} -(\bar{\lambda})^{-km+m-1} e_m & \text{if } n = k^2 m - 1 \\ (\bar{\lambda})^{-km+m-2} e_{m-1} & \text{if } n = k^2 m + k - 1 \\ 0 & \text{otherwise} \end{cases}.$$

Therefore, $\lambda M_{\bar{z}} H - H M_{z^k} \notin \mathcal{K}$. Hence, $H = AB \notin \text{k-essHank}_\lambda(L^2)$.

On the similar lines, we have

$$\begin{aligned}\lambda M_{\bar{z}} A^* - A^* M_{z^k} &= \lambda M_{\bar{z}} (D_{\lambda} W_k J M_z + C)^* - (D_{\lambda} W_k J M_z + C)^* M_{z^k} \\ &= (\lambda M_{\bar{z}^2} J W_k^* D_{\bar{\lambda}} - M_{\bar{z}} J W_k^* D_{\bar{\lambda}} M_{z^k}) \pmod{\mathcal{K}}.\end{aligned}$$

Also, we find that $(\lambda M_{\bar{z}^2} J W_k^* D_{\bar{\lambda}} - M_{\bar{z}} J W_k^* D_{\bar{\lambda}} M_{z^k}) e_n = \lambda^{n+1} e_{-(kn+2)} - \lambda^{k+n} e_{-(k^2+kn+1)}$ for every integer n , which ensures that A^* does not belong to the class $k\text{-essHank}_{\lambda}(L^2)$.

For the product of two operators to be an essentially generalized λ -slant Hankel operator of k^{th} -order, we obtain the following:

Theorem 2.6. *Let $T \in k\text{-essHank}_{\lambda}(L^2)$. Then we have the following:*

- (1) *If T_1 commutes essentially with $M_{\bar{z}}$ then $T_1 T \in k\text{-essHank}_{\lambda}(L^2)$.*
- (2) *If T_2 commutes essentially with M_{z^k} then $T T_2 \in k\text{-essHank}_{\lambda}(L^2)$.*

Proof. For (1), suppose that T_1 commutes essentially with $M_{\bar{z}}$. This implies that $T_1 M_{\bar{z}} = M_{\bar{z}} T_1 \pmod{\mathcal{K}}$. Now $T \in k\text{-essHank}_{\lambda}(L^2)$ implies $\lambda M_{\bar{z}} T - T M_{z^k} \in \mathcal{K}$. Using these, it is easy to see that $\lambda M_{\bar{z}} T_1 T - T_1 T M_{z^k} \in \mathcal{K}$. Along the similar lines, we can proof (ii). $\square$

As a consequence of Theorem 2.6, we obtain the following.

Corollary 2.7. *If M_{ϕ} is multiplication operator on L^2 and $T \in k\text{-essHank}_{\lambda}(L^2)$ then $T M_{\phi}$, $M_{\phi} T \in k\text{-essHank}_{\lambda}(L^2)$.*

Theorem 2.8. *If $A, A^* \in k\text{-essHank}_{\lambda}(L^2)$ then $T A^* = A^* T^* \pmod{\mathcal{K}}$, where $T = M_{z^k} + \lambda M_{\bar{z}}$.*

Proof. Let $A, A^* \in k\text{-essHank}_{\lambda}(L^2)$. Then $\lambda M_{\bar{z}} A - A M_{z^k}$ and $\lambda M_{\bar{z}} A^* - A^* M_{z^k}$ are compact operators and hence we obtain that $(M_{z^k} + \lambda M_{\bar{z}}) A^* - A^* (M_{z^k} + \lambda M_{\bar{z}})$ is a compact operator. Therefore, $T A^* = A^* T^* \pmod{\mathcal{K}}$, where $T = M_{z^k} + \lambda M_{\bar{z}}$. $\square$

This yields the following.

Corollary 2.9. *A necessary condition for any operator A of the class $k\text{-essHank}_{\lambda}(L^2)$ to be self adjoint is that $T A$ is essentially self adjoint for $T = M_{z^k} + \lambda M_{\bar{z}}$.*

Simple computations provide that two classes of essentially generalized λ -slant Hankel operators of k^{th} -order for different values of λ share the compact operators.

Theorem 2.10. *If λ and μ are distinct complex numbers, then*

$$k\text{-essHank}_{\lambda}(L^2) \cap k\text{-essHank}_{\mu}(L^2) = \mathcal{K}(L^2).$$

We further extend this result and obtain the following.

Theorem 2.11. *Let $0 \neq \lambda \neq \mu$ be such that $|\frac{\mu}{\lambda}| \neq 1$ and $k_1 \neq k_2$. Then*

$$k_1\text{-essHank}_{\lambda}(L^2) \cap k_2\text{-essHank}_{\mu}(L^2) = \mathcal{K}(L^2).$$

Proof. Let $A \in k_1\text{-essHank}_{\lambda}(L^2) \cap k_2\text{-essHank}_{\mu}(L^2)$. Then $(\lambda M_{\bar{z}} A - A M_{z^{k_1}})$ and $(\mu M_{\bar{z}} A - A M_{z^{k_2}})$ are compact operators on L^2 . This gives that $(\mu M_{\bar{z}} A - A M_{z^{k_2}}) - (\lambda M_{\bar{z}} A - A M_{z^{k_1}}) M_{z^{k_2-k_1}}$ is a compact operator. It means that $\lambda M_{\bar{z}} A (M_{z^{k_2-k_1}} - \frac{\mu}{\lambda} I) \in \mathcal{K}(L^2)$. Now $(M_{z^{k_2-k_1}} - \frac{\mu}{\lambda} I)$ is an invertible operator as $|\frac{\mu}{\lambda}| \neq 1$ and hence we conclude that $A \in \mathcal{K}(L^2)$. This proves that $k_1\text{-essHank}_{\lambda}(L^2) \cap k_2\text{-essHank}_{\mu}(L^2) \subseteq \mathcal{K}(L^2)$. The reverse inclusion is obvious. This completes the proof. $\square$

3. ESSENTIALLY COMPRESSION OF A GENERALIZED λ -SLANT HANKEL OPERATOR OF k^{th} -ORDER

A generalized λ -slant Hankel operator of k^{th} -order $X_{\phi,\lambda}$, $\phi \in L^\infty$, $|\lambda| = 1$ on the space L^2 is of the form $X_{\phi,\lambda} = D_\lambda X_\phi$, where $X_\phi (= JW_k M_\phi)$ denotes the k^{th} -order slant Hankel operator on L^2 and D_λ is the unitary operator given by $D_\lambda f(z) = f(\bar{\lambda}z)$ (see [5]). We denote the compression of a generalized λ -slant Hankel operator of k^{th} -order to H^2 by $Y_{\phi,\lambda}$. Then by the definition of compression, we have $Y_{\phi,\lambda} = PX_{\phi,\lambda}|_{H^2}$, that is, $Y_{\phi,\lambda} = PD_\lambda X_\phi|_{H^2}$. Since $PD_\lambda = D_\lambda P$, we find that $Y_{\phi,\lambda} = D_\lambda Y_\phi$, where $Y_\phi = PX_\phi|_{H^2} (= PJW_k M_\phi|_{H^2})$ is the compression of k^{th} -order slant Hankel operator X_ϕ to H^2 . It is interesting to obtain the following.

Theorem 3.1. *An operator Y on H^2 is the compression of a generalized λ -slant Hankel operator of k^{th} -order if and only if $\lambda T_{\bar{z}} Y = Y T_{z^k}$.*

Proof. Suppose Y is compression of a generalized λ -slant Hankel operator of k^{th} -order. Then $Y = D_\lambda Y_\phi$ for some $\phi \in L^\infty$. Now $Y T_{z^k} = D_\lambda Y_\phi T_{z^k} = D_\lambda T_{\bar{z}} Y_\phi = \lambda T_{\bar{z}} D_\lambda Y_\phi = \lambda T_{\bar{z}} Y$.

Conversely, if Y is an operator satisfying $\lambda T_{\bar{z}} Y = Y T_{z^k}$ then $\lambda D_{\bar{\lambda}} T_{\bar{z}} Y = D_{\bar{\lambda}} Y T_{z^k}$, that is, $T_{\bar{z}} D_{\bar{\lambda}} Y = D_{\bar{\lambda}} Y T_{z^k}$. So $D_{\bar{\lambda}} Y$ is compression of a k^{th} -order slant Hankel operator. Thus $Y = D_\lambda Y_\phi$ for some ϕ in L^∞ . $\square$

Theorem 3.2. *The point spectrum, $\sigma_p(Y_{\bar{z}^j,\lambda})$ of the operator $Y_{\bar{z}^j,\lambda}$, for $j = 1, 2, 3, \dots$ is as follows:*

$$\sigma_p(Y_{\bar{z}^j,\lambda}) = \begin{cases} \{0, \bar{\lambda}^m\} & \text{if } j = m(k+1), \quad m = 1, 2, 3, \dots \\ \{0\} & \text{if } j \neq m(k+1), \quad m = 1, 2, 3, \dots \end{cases}$$

Proof. Let $\mu \in \sigma_p(Y_{\bar{z}^j,\lambda})$. Then there exists $f \in H^2$, $f \neq 0$ such that $Y_{\bar{z}^j,\lambda} f = \mu f$. Let the fourier expansion of f be $\sum_{i=0}^{\infty} a_i z^i$. Then, we have $D_\lambda P J W_k M_{\bar{z}^j} f(z) = \mu f(z)$, that is, $D_\lambda P W_k (\sum_{i=0}^{\infty} a_i z^{j-i}) = \sum_{i=0}^{\infty} (\mu a_i) z^i$. If $j = 1$, then $D_\lambda P W_k (\sum_{i=0}^{\infty} a_i z^{1-i}) = \sum_{i=0}^{\infty} (\mu a_i) z^i$. From this, we get $D_\lambda (a_1 z^0) = \sum_{i=0}^{\infty} (\mu a_i) z^i$, that is, $a_1 z^0 = \sum_{i=0}^{\infty} (\mu a_i) z^i$. Thus, $\mu a_0 = a_1$ and $\mu a_i = 0$ for all $i \geq 1$.

If $\mu \neq 0$ then each $a_i = 0$, which contradicts the fact that $f \neq 0$. Thus, $\mu = 0$ is the only eigenvalue of $Y_{\bar{z}^1,\lambda}$ with corresponding eigen functions $f(z) = \sum_{i=0}^{\infty} a_i z^i$ with $a_1 = 0$. Proceeding this way, it can be easily seen that 0 is the only eigen value of $Y_{\bar{z}^j,\lambda}$, $j = 2, 3, \dots, k$.

If $j = k+1$, then $D_\lambda P W_k (\sum_{i=0}^{\infty} a_i z^{k+1-i}) = \sum_{i=0}^{\infty} (\mu a_i) z^i$. Similar calculations give $a_1 \bar{\lambda} z^1 + a_{k+1} z^0 = \sum_{i=0}^{\infty} (\mu a_i) z^i$. Thus, $\mu a_0 = a_{k+1}$, $a_1 \bar{\lambda} = \mu a_1$ and $\mu a_i = 0$ for all $i \geq 2$. This helps to conclude that $\mu = 0$ and $\mu = \bar{\lambda}$ are the only two eigen values

of $Y_{\bar{z}^{k+1}, \lambda}$. The eigenfunctions corresponding to $\mu = 0$ are $f(z) = \sum_{i=0}^{\infty} a_i z^i$ with $a_1 = a_{k+1} = 0$ and the eigenfunctions corresponding to $\mu = \bar{\lambda}$ are $f(z) = a_1 z$.

On proceeding along the similar lines, we attain the result. $\square$

For any operator T on a Hilbert space H , $\sigma(T) = \sigma_{ap}(T) \cup \sigma_c(T)$ and $\sigma_c(T) = \overline{\sigma_p(T^*)}$, where $\sigma(T)$, $\sigma_c(T)$ and $\sigma_{ap}(T)$ denote the spectrum, compression spectrum and approximate point spectrum of T respectively. We use these facts along with Theorem 3.2 to obtain that the spectrum of the operator $Y_{\bar{z}^j, \lambda}$ is equivalent to its approximate point spectrum.

Corollary 3.3. *For each natural number j , $\sigma(Y_{\bar{z}^j, \lambda}) = \sigma_{ap}(Y_{\bar{z}^j, \lambda})$.*

Proof. Now its evident that for each j , $\sigma_c(Y_{\bar{z}^j, \lambda}) = \overline{\sigma_p(Y_{\bar{z}^j, \lambda}^*)}$, where $\overline{\sigma_p(Y_{\bar{z}^j, \lambda}^*)} = \{\bar{\mu} : \mu \in \sigma_p(Y_{\bar{z}^j, \lambda}^*)\}$. On proceeding along the lines of proof of Theorem 3.2, we get that $\overline{\sigma_p(Y_{\bar{z}^j, \lambda}^*)} = \sigma_p(Y_{\bar{z}^j, \lambda})$, which implies $\sigma(Y_{\bar{z}^j, \lambda}) = \sigma_{ap}(Y_{\bar{z}^j, \lambda}) \cup \sigma_p(Y_{\bar{z}^j, \lambda})$. $\square$

We now define an essentially compression of a generalized λ -slant Hankel operator of k^{th} -order as follows.

Definition 3.4. For a fixed integer $k \geq 2$, an operator A on the space H^2 is said to be an essentially compression of a generalized λ -slant Hankel operator of k^{th} -order if it satisfies the operator equation $\lambda T_{\bar{z}} A - A T_{z^k} = K$, for some compact operator K on H^2 . We denote the set of all such operators by $k - essHank_{\lambda}(H^2)$.

The following findings can be stated for essentially compressions of generalized λ -slant Hankel operators of k^{th} -order on H^2 without an extra effort as the proof either holds by using definition or along the same lines for the respective results for their counterpart on L^2 .

Proposition 3.5. *For any fixed integer $k \geq 2$,*

- (1) $k - essHank_{\lambda}(H^2)$ contains all the compressions of generalized λ -slant Hankel operator of k^{th} -order to H^2 .
- (2) The sum of a compression of generalized λ -slant Hankel operator of k^{th} -order and a compact operator on H^2 is in $k - essHank_{\lambda}(H^2)$.
- (3) $k - essHank_{\lambda}(H^2)$ is a norm-closed vector subspace of $\mathcal{B}(H^2)$, the space of all bounded linear operators on the space H^2 .
- (4) $k - essHank_{\lambda}(H^2) \cap \mathcal{K}(H^2) = \mathcal{K}(H^2)$, where $\mathcal{K}(H^2)$ denotes the ideal of all compact operators on H^2 .
- (5) $k - essHank_{\lambda}(H^2)$ is neither a self-adjoint set nor an algebra of operators on H^2 . (This can be easily seen through Example 2.4 by defining operators on the space H^2).
- (6) If $A_1, A_2 \in k - essHank_{\lambda}(H^2)$ then $A_1 A_2 \in k - essHank_{\lambda}(H^2)$ if and only if $A_1 T_{z^k} A_2 = \lambda A_1 T_{\bar{z}} A_2 \pmod{\mathcal{K}(H^2)}$.
- (7) The product $A_1 A_2 \in k - essHank_{\lambda}(H^2)$ under either of the condition holds:
 - (a) A_1 commutes essentially with $T_{\bar{z}}$ and $A_2 \in k - essHank_{\lambda}(H^2)$
 - (b) $A_1 \in k - essHank_{\lambda}(H^2)$ and A_2 commutes essentially with T_{z^k}

(c) If one of the A_i ($i = 1, 2$) is a Toeplitz operator on H^2 and other is in the class $k - \text{essHank}_\lambda(H^2)$.

(8) If $A_1 \in \text{essHank}$ and $A_2 \in k - \text{essHank}_\lambda(H^2)$ then $A_1 A_2 \in k - \text{essToep}_\lambda(H^2)$.

Further, a simple computation provide the following.

Theorem 3.6. *The set $k - \text{essHank}_\lambda(H^2)$ contains no Fredholm operator.*

Proof. Suppose set $k - \text{essHank}_\lambda(H^2)$ contains a Fredholm operator A of index n say. Now $\lambda T_{\bar{z}} A = K + AT_{z^k}$ for some $K \in \mathcal{K}(H^2)$. Since A is Fredholm of index n , we have $\lambda T_{\bar{z}} A$ is Fredholm of index $n + 1$ and $K + AT_{z^k}$ is Fredholm of index $n - k$, which is absurd. Hence we have the desired result. $\square$

In order to find the Toeplitz operators in the class $k - \text{essHank}_\lambda(H^2)$, we first obtain the following.

Lemma 3.7. *A non-zero Toeplitz operator on H^2 cannot be compression of a generalized λ -slant Hankel operator of k^{th} -order.*

Proof. On the contrary, if we assume that a non-zero Toeplitz operator A is compression of a generalized λ -slant Hankel operator of k^{th} -order then we have $T_{\bar{z}} A T_z = A$ and $\lambda T_{\bar{z}} A = AT_{z^k}$. These jointly provide that $\lambda A(e_n) = A(e_{n+k+1})$ for each $n \geq 0$. This implies A is a finite rank operator and hence a compact operator on H^2 . This is a contradiction as non-zero Toeplitz operators are never compact. Hence conclusion follows. $\square$

We use this lemma to prove the following.

Theorem 3.8. $k - \text{essHank}_\lambda(H^2) \cap \mathcal{T} = \{0\}$, where $\mathcal{T}$ denotes the set of all Toeplitz operators on H^2 .

Proof. Let $A \in k - \text{essHank}_\lambda(H^2) \cap \mathcal{T}$. Then $\lambda T_{\bar{z}} A - AT_{z^k} = K$, for some compact operator K on H^2 . Since A is a Toeplitz operator, $\lambda T_{\bar{z}} A - AT_{z^k}$ is also a Toeplitz operator on H^2 . But a non-zero Toeplitz operator cannot be compact, which gives $\lambda T_{\bar{z}} A - AT_{z^k} = 0$. That is, A is a compression of generalized λ -slant Hankel operator of k^{th} -order to H^2 . Using Lemma (3.7), we get $A = 0$. Hence the result. $\square$

Theorem 3.9. *If $X \in k - \text{essHank}_\lambda(H^2)$ then $T_{z^k}(\ker X) \subseteq \ker(X + K)$ for some compact operator K on H^2 .*

Proof. Let $X \in k - \text{essHank}_\lambda(H^2)$. Then $\lambda T_{\bar{z}} X - XT_{z^k} = K'$ for some compact operator K' on H^2 . Now if $f \in \ker X$ then $(X + K' T_{z^k}^*) T_{z^k} f = (X T_{z^k} + K') f = (\lambda T_{\bar{z}} X) f = 0$. Thus $T_{z^k} f \in \ker(X + K)$, where $K = K' T_{z^k}^* \in \mathcal{K}(H^2)$. Hence the result. $\square$

It is known that a Rhaly matrix induced by a complex sequence $\{a_n\}_{n=0}^\infty$ with $\sum_{n=0}^\infty |a_n|^2 < \infty$ determines a bounded operator if $\{na_n\}$ is bounded. We refer [10] and [13] for the properties of Rhaly matrices. Let the set of all Rhaly operators on the space H^2 be denoted by $\mathcal{R}$. It is interesting to know the following about the Rhaly operators.

Theorem 3.10. *For a fixed integer $k \geq 2$, $k - \text{essToep}_\lambda(H^2) \cap \mathcal{R} = k - \text{essHank}_\lambda(H^2) \cap \mathcal{R}$.*

Proof. Let R be a Rhaly operator on H^2 corresponding to the complex sequence $\{a_n\}_{n=0}^\infty$. With easy calculations, it can be seen that the matrices of the operators $\lambda T_{\bar{z}}R - RT_{z^k}$ and $\lambda T_zR - RT_{z^k}$ with respect to standard orthonormal basis $\{e_n\}_{n \geq 0}$ of H^2 are respectively given by

$$\begin{bmatrix} \lambda a_1 & \lambda a_1 & 0 & \cdots \\ \lambda a_2 & \lambda a_2 & \lambda a_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \lambda a_k & \lambda a_k & \lambda a_k & \cdots \\ \lambda a_{k+1} - a_k & \lambda a_{k+1} & \lambda a_{k+1} & \cdots \\ \lambda a_{k+2} - a_{k+1} & \lambda a_{k+2} - a_{k+1} & \lambda a_{k+2} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 0 & 0 & \cdots \\ \lambda a_0 & 0 & 0 & \cdots \\ \lambda a_1 & \lambda a_1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \lambda a_{k-2} & \lambda a_{k-2} & \lambda a_{k-2} & \cdots \\ \lambda a_{k-1} - a_k & \lambda a_{k-1} & \lambda a_{k-1} & \cdots \\ \lambda a_k - a_{k+1} & \lambda a_k - a_{k+1} & \lambda a_k & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

Now define infinite matrices A and B respectively on H^2 as

$$\begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots \\ a_k & 0 & 0 & \cdots \\ a_{k+1} & a_{k+1} & 0 & \cdots \\ a_{k+2} & a_{k+2} & a_{k+2} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots \\ a_{k+1} & 0 & 0 & \cdots \\ a_{k+2} & a_{k+2} & 0 & \cdots \\ a_{k+3} & a_{k+3} & a_{k+3} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

Then

$$\lambda T_{\bar{z}}R - RT_{z^k} = [T_z^* \text{diag}\{0, \lambda a_1, \lambda a_2, \cdots\} + \text{diag}\{\lambda a_1, \lambda a_2, \cdots\} + T_z \text{diag}\{\lambda a_2, \lambda a_3, \cdots\} + T_{z^2} \text{diag}\{\lambda a_3, \lambda a_4, \cdots\} + \cdots + T_{z^{k-1}} \text{diag}\{\lambda a_k, \lambda a_{k+1}, \cdots\} + \lambda B - A];$$

$$\lambda T_zR - RT_{z^k} = [T_z \text{diag}\{\lambda a_0, \lambda a_1, \cdots\} + T_{z^2} \text{diag}\{\lambda a_1, \lambda a_2, \cdots\} + \cdots + T_{z^{k-1}} \text{diag}\{\lambda a_{k-2}, \lambda a_{k-1}, \cdots\} + T_{z^k} \text{diag}\{\lambda a_{k-1} - a_k, \lambda a_k - a_{k+1}, \cdots\} + T_z(\lambda A - B)].$$

Since all the diagonal operators involved in above equations are compact, it follows that $\lambda T_{\bar{z}}R - RT_{z^k}$ is compact if and only if both A and B are compact if and only if $\lambda T_zR - RT_{z^k}$ is compact. Hence the result. $\square$

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