ON sfp-INJECTIVE AND sfp-FLAT MODULES

C. SELVARAJ∗ AND P. PRABAKARAN

ABSTRACT. Let $R$ be a ring. A left $R$-module $M$ is said to be sfp-injective if, for every exact sequence $0 \to K \to L$ with $K$ and $L$ super finitely presented left $R$-modules, the induced sequence $\text{Hom}(L, M) \to \text{Hom}(K, M) \to 0$ is exact. A right $R$-module $N$ is called sfp-flat if, for every exact sequence $0 \to K \to L$ with $K$ and $L$ super finitely presented left $R$-modules, the induced sequence $0 \to N \otimes K \to N \otimes L$ is exact. We study precovers and preenvelopes by sfp-injective and sfp-flat modules, including their properties under (almost) excellent extensions of rings.

1. Introduction and preliminaries

Throughout this paper, $R$ denotes an associative ring with identity and all modules are unitary. As usual, $pd_R(M)$, $id_R(M)$, and $fd_R(M)$ will denote the projective, injective and flat dimensions of an $R$-module $M$, respectively. The character module $\text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by $M^+$. Let $M$, $N$ be $R$-modules, then we use $\text{Hom}(M, N)$ (resp. $M \otimes N$) to denote $\text{Hom}_R(M, N)$ (resp. $M \otimes_R N$). For unexplained concepts and notations, we refer the reader to [3, 11, 16].

We first recall some known notions and facts needed in the sequel.

Let $C$ be a class of right $R$-modules and $M$ a right $R$-module. Following [3], we say that a map $f \in \text{Hom}_R(C, M)$ with $C \in C$ is a $C$-precover of $M$, if the group homomorphism $\text{Hom}_R(C', f) : \text{Hom}_R(C', C) \to \text{Hom}_R(C', M)$ is surjective for each $C' \in C$. Let $M$, $N$ be $R$-modules, then we use $\text{Hom}(M, N)$ (resp. $M \otimes N$) to denote $\text{Hom}_R(M, N)$ (resp. $M \otimes_R N$). For unexplained concepts and notations, we refer the reader to [3, 11, 16].

A right $R$-module $M$ is called $FP$-injective (or absolutely pure) [12] if $\text{Ext}_R^1(F, M) = 0$ for all finitely presented right $R$-modules $F$.

A left $R$-module $A$ is said to be $n$-presented if there exists an exact sequence of left $R$-modules: $F_n \to F_{n-1} \to \cdots \to F_0 \to A \to 0$, in which every $F_i$ is a finitely generated free or projective left $R$-module. Clearly, a module is 0-presented (resp.
1-presented) if and only if it is finitely generated (resp. finitely presented), and every \( m \)-presented module is \( n \)-presented for \( m \geq n \). For any two non-negative integers \( n, d \), a left \( R \)-module \( M \) is called \((n, d)\)-injective \([17]\) if \( \text{Ext}_R^{d+1}(P, M) = 0 \) for all \( n \)-presented right \( R \)-modules \( P \); a right \( R \)-module \( M \) is called \((n, d)\)-flat \([17]\) if \( \text{Tor}_R^{d+1}(M, P) = 0 \) for all \( n \)-presented left \( R \)-modules \( P \).

A left \( R \)-module \( M \) is called super finitely presented \([5]\) if there exists an exact sequence of left \( R \)-modules: \( \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0 \), where each \( F_i \) is finitely generated and projective. Recently, Gao and Wang introduced the notion of \( \text{l.sp.gldim} \) by introducing the notion of \( \text{sfp-projective} \) modules and give some equivalent characterization to \( \text{l.sp.gldim}(R) = 0 \) in terms of \( \text{sfp-projective} \) modules.

In section 2, \( \text{sfp-injective} \) modules and \( \text{sfp-flat} \) modules are defined and studied. We prove that every left \( R \)-module has a \( \text{sfp-injective} \) cover (resp. preenvelope) and every right \( R \)-module has a \( \text{sfp-flat} \) cover (resp. preenvelope). Also we introduced the notion of \( \text{sfp-projective} \) modules and give some equivalent characterization to \( \text{l.sp.gldim}(R) = 0 \) in terms of \( \text{sfp-projective} \) modules.

In Section 3, we further study the exchange properties of \( \text{sfp-injective} \) and \( \text{sfp-flat} \) modules, as well as \( \text{sfp-injective} \) precovers (resp. preenvelopes) and \( \text{sfp-flat} \) precovers (resp. preenvelopes) under (almost) excellent extensions of rings.
2. sfp-INJECTIVE AND sfp-FLAT Modules

We begin the section with the following lemma.

**Lemma 2.1.** [13] Let $0 \to A \to B \to C \to 0$ be an exact sequence of left $R$-modules. If any two of $A, B, C$ are super finitely presented, then so is the third one.

Now we give the concepts of sfp-injective modules and sfp-flat modules.

**Definition 2.2.**

(1) A left $R$-module $M$ is said to be sfp-injective if for every exact sequence $0 \to K \to L$ with $K$ and $L$ super finitely presented left $R$-modules, the induced sequence $\text{Hom}(L, M) \to \text{Hom}(K, M) \to 0$ is exact.

(2) A right $R$-module $N$ is called sfp-flat if, for every exact sequence $0 \to K \to L$ with $K$ and $L$ super finitely presented left $R$-modules, the induced sequence $0 \to N \otimes K \to N \otimes L$ is exact.

**Remark 2.3.**

(1) Note that every injective (resp. flat) module is sfp-injective (resp. sfp-flat) module. Also we have the following implications:

\[
\begin{align*}
\text{FP-injective} & \Rightarrow (2,0)-\text{injective} \Rightarrow \cdots \Rightarrow \text{weak injective} \\
& \downarrow \quad \downarrow \quad \downarrow \\
\text{fp-injective} & \Rightarrow \text{fp}_2-\text{injective} \Rightarrow \cdots \Rightarrow \text{sfp-injective}
\end{align*}
\]

and

\[
\begin{align*}
\text{Flat} & \Rightarrow (2,0)\text{-flat} \Rightarrow \cdots \Rightarrow \text{weak flat} \\
& \downarrow \quad \downarrow \quad \downarrow \\
\text{fp-flat} & \Rightarrow \text{fp}_2\text{-flat} \Rightarrow \cdots \Rightarrow \text{sfp-flat}
\end{align*}
\]

(2) If $R$ is a coherent ring then all the modules in the first diagram coincide with $FP$-injective modules and all the modules in the second diagram coincide with flat modules by [6, Remark 2.2(1)] and [7, Theorem 2.4].

**Proposition 2.4.** Let $\{M_i\}_I$ be a class of left $R$-modules, and $\{N_i\}_I$ a class of right $R$-modules. Then,

(1) $\prod_I M_i$ (resp. $\bigoplus_I M_i$) is sfp-injective if and only if each $M_i$ is a sfp-injective module.

(2) $\prod_I N_i$ (resp. $\bigoplus_I N_i$) is sfp-flat if and only if each $N_i$ is a sfp-flat module.

**Proof.** (1). We know that $\text{Hom}(A, \prod_I M_i) \cong \prod_I \text{Hom}(A, M_i)$ for any left $R$-module $A$ and in addition that $\text{Hom}(A, \bigoplus_I M_i) \cong \bigoplus_I \text{Hom}(A, M_i)$ for any finitely presented left $R$-module $A$. Since every super finitely presented module is finitely presented, the assertion holds.

(2). Since $(\bigoplus_I N_i) \otimes A \cong \bigoplus_I (N_i \otimes A)$ for any left $R$-module $A$ and $(\prod_I N_i) \otimes A \cong \prod_I (N_i \otimes A)$ for any finitely presented left $R$-module $A$, the assertion holds. \qed

In what follows let us denote the class of all sfp-injective left $R$-modules by $\mathcal{SI}$ and the class of all sfp-flat right $R$-modules by $\mathcal{SF}$. 
Lemma 2.5. The following are true for any ring $R$:

1. A left $R$-module $M$ is sfp-injective if and only if $M^+$ is sfp-flat.
2. A right $R$-module $N$ is sfp-flat if and only if $N^+$ is sfp-injective.
3. The class $SI$ (resp., $SF$) is closed under pure submodules, pure quotients and direct limits.
4. A super finitely presented left $R$-module is sfp-injective if and only if it is weak injective.

Proof. (1). Let $0 \to K \to L$ be any exact sequence with $K$ and $L$ are super finitely presented left $R$-modules. Consider the following commutative diagram:

$$
\begin{array}{ccc}
M^+ \otimes K & \to & M^+ \otimes L \\
\downarrow \sigma_K & & \downarrow \sigma_L \\
\text{Hom}(K,M)^+ & \to & \text{Hom}(L,M)^+.
\end{array}
$$

Since any super finitely presented left $R$-module is finitely presented, $\sigma_K$ and $\sigma_L$ are isomorphisms by [11, Lemma 3.60]. Then, $M$ is sfp-injective if and only if $\text{Hom}(K,M) \to \text{Hom}(K,M)^+ \to \text{Hom}(L,M)^+$ is exact if and only if $0 \to M^+ \otimes K \to M^+ \otimes L$ is exact if and only if $M^+$ is sfp-flat by the definition.

(2). Let $0 \to A \to B$ be any exact sequence with $A$ and $B$ super finitely presented left $R$-modules. Then $N$ is sfp-flat if and only if $0 \to N \otimes A \to N \otimes B$ is exact if and only if $(N \otimes B)^+ \to (N \otimes A)^+ \to 0$ is exact if and only if $\text{Hom}(B,N^+) \to \text{Hom}(A,N^+) \to 0$ is exact if and only if $N^+$ is sfp-injective by the definition.

(3). Let $0 \to A \to B \to C \to 0$ be a pure exact sequence with $B$ sfp-injective. Then, the sequence $0 \to C^+ \to B^+ \to A^+ \to 0$ is split. Since $B^+$ is sfp-flat by (1), it follows that $A^+$ and $C^+$ are sfp-flat by Proposition 2.4. So $A$ and $C$ are sfp-injective by (1) again. Thus, the class $SI$ is closed under pure submodules and pure quotients. Similarly, we get that class $SF$ is closed under pure submodules and pure quotients. Since for every finitely presented left $R$-module $A$, we have

$$\text{Hom}(A, \lim_{\to} M_i) \cong \lim_{\to} \text{Hom}(A, M_i)$$

and for any left $R$-module $A$, we have

$$\lim_{\to}(N_i \otimes A) \cong \lim_{\to}(N_i \otimes A)$$

the class $SI$ (resp., $SF$) is closed under direct limits.

(4). It is enough to show that every super finitely presented sfp-injective module is weak injective. Let $A$ be a super finitely presented sfp-injective module and let $0 \to A \to B \to C \to 0$ be any exact sequence of left $R$-modules with $C$ super finitely presented. Then, $B$ is super finitely presented by Lemma 2.1, and $0 \to \text{Hom}(C,A) \to \text{Hom}(B,A) \to \text{Hom}(A,A) \to 0$ is exact by the definition of sfp-injective modules. Thus, $0 \to A \to B \to C \to 0$ is split, that is, $\text{Ext}^1(C,A) = 0$. So $A$ is weak injective. \qed
Theorem 2.6. The following hold for any ring $R$:

1. Every left (resp. right) $R$-module has a sfp-injective (resp. sfp-flat) cover.
2. Every left (resp. right) $R$-module has a sfp-injective (resp. sfp-flat) preenvelope.

Proof. (1). From Proposition 2.4 and Lemma 2.5, we get $SI$ and $SF$ are closed under direct sums and pure quotients. Then, every left (resp. right) $R$-module has a sfp-injective (resp. sfp-flat) cover by [8, Theorem 2.5].

(2). Since $SI$ and $SF$ are closed under direct products and pure submodules by Proposition 2.4 and Lemma 2.5, every left (resp. right) $R$-module has a sfp-injective (resp. sfp-flat) preenvelope by [10, Corollary 3.5]. □

Proposition 2.7. The following hold for any ring $R$:

1. If $M \to N$ is a sfp-injective preenvelope of a left $R$-module $M$, then $N^+ \to M^+$ is a sfp-flat precover of $M^+$.
2. If $M \to N$ is a sfp-flat preenvelope of a right $R$-module $M$, then $N^+ \to M^+$ is a sfp-injective precover of $M^+$.

Proof. By Lemma 2.5, we have $SF^+ \subseteq SI$ and $SI^+ \subseteq SF$. Now both the assertions follows immediately from [4, Theorem 3.1]. □

Proposition 2.8. For any ring $R$, the following are equivalent:

1. $R$ is weak injective as a left $R$-module;
2. Every left $R$-module has an epic sfp-injective cover;
3. Every right $R$-module has a monic sfp-flat preenvelope.

Proof. (1) $\Rightarrow$ (2). Let $M$ be a left $R$-module. Then there exists an exact sequence $F \to M \to 0$ with $F$ free. Note that $R$ is sfp-injective as a left $R$-module. From Proposition 2.4, we get that $F$ is sfp-injective. Since $M$ has a sfp-injective cover $f : I \to M$ by Theorem 2.6, then $f$ is epic.

(2) $\Rightarrow$ (1). Assume that $f : M \to _RR$ is an epic sfp-injective cover of $_RR$. Then, $_RR$ is isomorphic to a direct summand of $M$, and so it is sfp-injective by Proposition 2.4. Since $_RR$ is super finitely presented, $_RR$ is weak injective by Lemma 2.5.

(1) $\Rightarrow$ (3). Let $M$ be a right $R$-module. By Theorem 2.6 $M$ has a sfp-flat preenvelope $g : M \to F$. Since $(RR)^+$ is an injective cogenerator in the category of right $R$-modules, there exists an exact sequence $0 \to M \to \prod(RR)^+$. By assumption, $_RR$ is weak injective module, and thus sfp-injective, and so $(RR)^+$ is sfp-flat by Lemma 2.5. Thus, $\prod(RR)^+$ is sfp-flat by Proposition 2.4. Therefore, $g$ is monic.

(3) $\Rightarrow$ (1). Since the injective right $R$-module $(RR)^+$ has a monic sfp-flat preenvelope $h : (RR)^+ \to F$ by assumption, then $(RR)^+$ is a direct summand of $F$ and so it is sfp-flat by Proposition 2.4, which implies that $_RR$ is sfp-injective from Lemma 2.5. Since $_RR$ is super finitely presented, $_RR$ is weak injective by Lemma 2.5. It follows that $R$ is weak injective as a left $R$-module. □

Theorem 2.9. The following are equivalent for a ring $R$:

1. Every quotient of any sfp-injective left $R$-module is sfp-injective;
(2) Every left $R$-module has a monic sfp-injective cover;
(3) Every submodule of any sfp-flat right $R$-module is sfp-flat;
(4) Every right $R$-module has an epic sfp-flat envelope.

Proof. (1) $\Rightarrow$ (2). For any left $R$-module $M$, there is a sfp-injective cover $f : E \to M$ by Theorem 2.6(1). Note that $\text{im}(f)$ is sfp-injective by (1). It is easy to verify that $\text{im}(f) \to M$ is a monic sfp-injective cover.

(2) $\Rightarrow$ (3). Let $N$ be a submodule of a sfp-flat right $R$-module $M$. Then the exact sequence $0 \to N \to M$ induces the exact sequence $M^+ \to N^+ \to 0$. Note that $M^+$ is sfp-injective and $N^+$ has a monic sfp-injective cover by (2). Thus $N^+$ is sfp-injective, and so $N$ is sfp-flat.

(3) $\Rightarrow$ (4). For any right $R$-module $M$, there is a sfp-flat preenvelope $f : M \to F$ by Theorem 2.6(2). Since $\text{im}(f)$ is sfp-flat by (3), we have $M \to \text{im}(f)$ is an epic sfp-flat preenvelope, equivalently, an epic sfp-flat envelope.

(4) $\Rightarrow$ (1). Let $M$ be any sfp-injective left $R$-module and $N$ any submodule of $M$. Then we get a short exact sequence $0 \to N \to M \to M/N \to 0$ which induces the exact sequence $0 \to (M/N)^+ \to M^+$. Since $(M/N)^+$ has an epic sfp-flat envelope and $M^+$ is sfp-flat, we have $(M/N)^+$ is sfp-flat, and so $M/N$ is sfp-injective.

\begin{theorem}
The following are equivalent for a ring $R$:
\begin{enumerate}
\item Every left $R$-module is sfp-injective;
\item Every right $R$-module is sfp-flat;
\item Every super finitely presented left $R$-module is sfp-injective;
\item Every super finitely presented submodule of any super finitely presented left $R$-module is its direct summands;
\item $R$ is weak injective as a left $R$-module and every submodule of any sfp-flat right $R$-module is sfp-flat.
\end{enumerate}
\end{theorem}

Proof. (1) $\Rightarrow$ (2). Let $N$ be any right $R$-module. Since $N^+$ is sfp-injective, $N$ is sfp-flat by Lemma 2.5.

(2) $\Rightarrow$ (1). For any left $R$-module $M$, since $M^+$ is sfp-flat, $M$ is sfp-injective by Lemma 2.5.

(1) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (4). Let $A$ be a super finitely presented submodule of any super finitely presented left $R$-module $B$. Then we get an exact sequence $0 \to A \to B \to C \to 0$. Note that $C$ is super finitely presented by Lemma 2.1. Since $A$ is weak injective by (3) and Lemma 2.5(4), $\text{Ext}^1(C, A) = 0$. So the exact sequence $0 \to A \to B \to C \to 0$ is split. Thus $A$ is a direct summand of $B$.

(4) $\Rightarrow$ (5). Let $0 \to A \to B \to C \to 0$ be any exact sequence with $A$ and $B$ super finitely presented left $R$-modules. Then, this sequence is split. So every left $R$-module is sfp-injective and every right $R$-module is sfp-flat. Hence the assertions holds.

(5) $\Rightarrow$ (1). Let $M$ be a left $R$-module. Then there is an exact sequence $F \to M \to 0$ with $F$ free. Since $R$ is weak injective as a left $R$-module, $F$ is weak injective, and so is sfp-injective. Thus $M$ is sfp-injective by Theorem 2.9.
Theorem 2.11. The following are equivalent for a ring $R$:

1. Every sfp-injective left $R$-module is weak injective;
2. Every sfp-flat right $R$-module is weak flat;
3. The character module of a sfp-injective left $R$-module is weak flat;
4. A left $R$-module sfp-injective if and only if its character module is weak flat.

Proof. By using [6, Theorem 2.10, Remark 2.2(2)] and Lemma 2.5 it is easy to verify the equivalence. □

Next we introduce the concept of sfp-projective modules, which may be viewed as a dual of sfp-injective modules.

Definition 2.12. A left $R$-module $M$ is said to be sfp-projective if for every exact sequence $A \rightarrow B \rightarrow 0$ with $A$ and $B$ super finitely presented left $R$-modules, the induced sequence $\text{Hom}(M, A) \rightarrow \text{Hom}(M, B) \rightarrow 0$ is exact.

Theorem 2.13. The following are equivalent for a ring $R$:

1. $l.sp.gldim(R) = 0$;
2. Every left $R$-module is sfp-projective;
3. Every super finitely presented left $R$-module is sfp-projective.

Proof. (1) $\Rightarrow$ (2). Since every epimorphism $K \rightarrow L$ with $K$ and $L$ super finitely presented left $R$-modules is split by (1), every left $R$-module is sfp-projective.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). Let $M$ be any super finitely presented left $R$-module. There is an epimorphism $P \rightarrow M$ with $P$ finitely generated projective. Since $M$ is sfp-projective, $P \rightarrow M$ is a split epimorphism. Thus $M$ is projective. So $l.sp.gldim(R) = 0$. □

Remark 2.14.

1. Note that every fp-projective module (see [9, Definition 4.1]) is sfp-projective. If $R$ is a coherent ring then sfp-projective modules are coincide with fp-projective modules.

2. As Gao mentioned in [6, Remark 3.11] there exists a ring with $l.sp.gldim(R) = 0$ which is not von Neumann regular. By Theorem 2.13 and [9, Theorem 4.2] there exists sfp-projective module which is not fp-projective.

3. SFP-INJECTIVE AND SFP-FLAT MODULES UNDER (ALMOST) EXCELLENT EXTENSIONS OF RINGS

Recall that a ring $S$ is said to be an almost excellent extension of a subring $R$ if the following conditions are satisfied:

1. $S$ is a finite normalizing extension of $R$, that is, $R$ and $S$ have the same identity and there are elements $s_1, \ldots, s_n \in S$ such that $S = Rs_1 + \cdots + Rs_n$ and $Rs_i = s_i R$ for all $i = 1, \ldots, n$,
2. $S$ is flat and $R S$ is projective,
3. $S$ is left $R$-projective, that is, if $S M$ is a submodule of $S N$ and $R M$ is a direct summand of $R N$, then $S M$ is a direct summand of $S N$. 


Further, $S$ is an excellent extension of $R$ if $S$ is an almost excellent extension of $R$ and $S$ is free with basis $s_1, \ldots, s_n$ as both a right and a left $R$-module with $s_1 = 1_R$.

**Lemma 3.1.** Let $S$ be a ring extension of a ring $R$.

1. If $S_R$ (resp. $rS$) is flat, then an $R$-module $RA$ (resp. $A_R$) is super finitely presented implies that $S \otimes_R A$ (resp. $A \otimes_R S$) is super finitely presented $S$-module.
2. If $S_R$ (resp. $rS$) is finitely generated projective, then an $S$-module $AS$ (resp. $SA$) is super finitely presented implies that $A_R$ (resp. $rA$) is super finitely presented $R$-module.

**Proof.** (1). Since $S_R$ is flat, the assertion holds by the standard isomorphisms $S \otimes_R R^n \cong S^n$ for all $n \geq 1$.

(2). If $rS$ is a finitely generated projective left $R$-module, then we have a isomorphism $\text{Hom}_R(S,S^n) \cong rS^n$ for all $n \geq 1$. The assertion easily follows from the definition of super finitely presented modules. □

**Proposition 3.2.** Let $S$ be an almost excellent extension of a subring $R$ and $SM$ a left $S$-module. Then the following are equivalent:

1. $SM$ is sfp-injective;
2. $RM$ is sfp-injective;
3. $\text{Hom}_R(S,M)$ is a sfp-injective left $S$-module.

**Proof.** (1) $\Rightarrow$ (2). Let $0 \rightarrow RA \rightarrow RB$ be an exact sequence of left $R$-modules with $RA$ and $RB$ super finitely presented. Since $S_R$ is a flat, from the definition of almost excellent extensions we get an exact sequence of left $S$-modules

$$0 \rightarrow S_R \otimes_R A \rightarrow S_R \otimes_R B.$$ 

Then, $S_R \otimes_R A$ and $S_R \otimes_R B$ is super finitely presented left $S$-modules by Lemma 3.1 since $S$ is an almost excellent extension of a subring $R$. By (1), the sequence

$$\text{Hom}_S(S_R \otimes_R B,M) \rightarrow \text{Hom}_S(S_R \otimes_R A,M) \rightarrow 0$$

is exact, which implies that the sequence

$$\text{Hom}_R(B,\text{Hom}_S(S,M)) \rightarrow \text{Hom}_R(A,\text{Hom}_S(S,M)) \rightarrow 0$$

is also exact. Thus, we get an exact sequence

$$\text{Hom}_R(B,M) \rightarrow \text{Hom}_R(A,M) \rightarrow 0,$$

that is, $RM$ is a sfp-injective left $R$-module.

(2) $\Rightarrow$ (3). Assume that $0 \rightarrow SA \rightarrow SB$ is an exact sequence of left $S$-modules with $SA$ and $SB$ super finitely presented. Since $rS$ is finitely generated projective, $RA$ and $RB$ are super finitely presented left $R$-modules by Lemma 3.1. By (2), we get an exact sequence

$$\text{Hom}_R(B,M) \rightarrow \text{Hom}_R(A,M) \rightarrow 0,$$

which yields the exactness of the sequence

$$\text{Hom}_R(S \otimes_S B,M) \rightarrow \text{Hom}_R(S \otimes_S A,M) \rightarrow 0.$$
Then, we get the exact sequence

\[ \text{Hom}_S(B, \text{Hom}_R(S, M)) \to \text{Hom}_S(A, \text{Hom}_R(S, M)) \to 0. \]

Thus, \( \text{Hom}_R(S, M) \) is a sfp-injective left \( S \)-module.

(3) \( \Rightarrow \) (1). By [15, Lemma 1.1], \( sM \) is isomorphic to a direct summand of \( \text{Hom}_R(S, M) \). Thus, \( sM \) is an sfp-injective left \( S \)-module.

\[ \square \]

**Corollary 3.3.** Let \( S \) be an almost excellent extension of a subring \( R \) and \( M_S \) a right \( S \)-module. Then, the following are equivalent:

1. \( M_S \) is sfp-flat;
2. \( M_R \) is sfp-flat;
3. \( M \otimes_R S \) is sfp-flat right \( S \)-module.

**Proof.** (1) \( \Leftrightarrow \) (2). By Lemma 2.5 and Proposition 3.2, \( M_S \) is a sfp-flat right \( S \)-module if and only if \( s(M^+) \) is a sfp-injective left \( S \)-module if and only if \( R(M^+) \) is a sfp-injective left \( R \)-module if and only if \( M_R \) is a sfp-flat right \( R \)-module.

(1) \( \Leftrightarrow \) (3). By Lemma 2.5 and Proposition 3.2, \( M_S \) is a sfp-flat right \( S \)-module if and only if \( s(M^+) \) is a sfp-injective left \( S \)-module, if and only if \( (M \otimes_R S)^+ \cong \text{Hom}_R(S, M^+) \) is a sfp-injective left \( S \)-module if and only if \( M \otimes_R S \) is a sfp-flat right \( S \)-module.

Let \( S \) be an almost excellent extension of a subring \( R \). From [15, Lemma 1.1], we see that a left \( S \)-module \( sM \) is isomorphic to a direct summand of \( \text{Hom}_R(S, M) \) and \( s \otimes_R M \). In the sequel, we denote \( \lambda_M : sM \to \text{Hom}_R(S, M) \) (resp. \( \tau_M : sM \to s \otimes_R M \)) as the inclusion and \( \pi_M : \text{Hom}_R(S, M) \to sM \) (resp. \( \rho_M : s \otimes_R M \to sM \)) as the canonical projection. Let \( \theta : M \to N \) be a homomorphism of left \( R \)-modules, we use \( \theta_* \) to denote the induced homomorphism: \( \theta_* = \text{Hom}_R(S, \theta) : \text{Hom}_R(S, M) \to \text{Hom}_R(S, N) \) for any left \( R \)-module \( S \).

**Theorem 3.4.** Suppose that \( S \) is an almost excellent extension of a subring \( R \) and \( \theta : sM \to sN \) is an \( S \)-homomorphism.

1. If the \( R \)-homomorphism \( \theta : R_M \to R_N \) is a sfp-injective precover of \( R_N \), then \( \pi_N \theta_* : \text{Hom}_R(S, M) \to sN \) is a sfp-injective precover of \( sN \).
2. If the \( R \)-homomorphism \( \theta : R_M \to R_N \) is a sfp-injective preenvelope of \( R_M \), then \( \theta_* \lambda_M : sM \to \text{Hom}_R(S, N) \) is a sfp-injective preenvelope of \( sM \).

**Proof.** (1). Since \( R_M \) is sfp-injective, \( \text{Hom}_R(S, M) \) is a sfp-injective left \( S \)-module by Proposition 3.2. For any sfp-injective left \( S \)-module \( sG \) and any \( S \)-homomorphism \( \alpha : sG \to sN \), \( R_G \) is a sfp-injective left \( R \)-module by Proposition 3.2. So there exists \( \beta : R_G \to R_M \) such that \( \theta \beta = \alpha \). Thus we obtain the following commutative diagram:

\[
\begin{array}{ccc}
\text{Hom}_R(S, M) & \xleftarrow{\beta_*} & \text{Hom}_R(S, G) & \xleftarrow{\pi_G} & sG \\
\downarrow{\alpha_*} & & \downarrow{\lambda_G} & & \downarrow{\alpha} \\
\text{Hom}_R(S, M) & \xrightarrow{\theta_*} & \text{Hom}_R(S, N) & \xrightarrow{\pi_N} & sN
\end{array}
\]
So we have,

\[(\pi_N\theta_*)(\beta_\lambda_G) = \pi_N(\theta\beta)_\lambda G = \pi_N\alpha_\lambda G = \pi_N\lambda_G = \alpha.\]

Hence \(\pi_N\theta_*\) is a sfp-injective precover of \(S_N\).

(2) can be proved dually. \(\square\)

**Lemma 3.5.** Let \(S\) be an excellent extension of a subring \(R\) and \(S_M\) a left \(S\)-module. Then the following are equivalent:

1. \(S_M\) is sfp-flat;
2. \(R_M\) is sfp-flat;
3. \(S \otimes_R M\) is a sfp-flat left \(S\)-module.

**Proof.** (1) \(\Rightarrow\) (2). Let \(0 \to K_R \to L_R\) be an exact sequence with \(K_R\) and \(L_R\) super finitely presented right \(R\)-modules. Since \(R_S\) is flat, we get the \(S\)-module exact sequence

\[0 \to K \otimes_R S \to L \otimes_R S.\]

Note that \(K \otimes_R S\) and \(L \otimes_R S\) are super finitely presented right \(S\)-modules. Thus by (1), we get the exact sequence

\[0 \to (K \otimes_R S) \otimes_S M \to (L \otimes_R S) \otimes_S M,\]

which gives the exactness of the sequence

\[0 \to K \otimes_R (S \otimes_S M) \to L \otimes_R (S \otimes_S M),\]

So we have the exact sequence \(0 \to K \otimes_R M \to L \otimes_R M\). Therefore \(R_M\) is sfp-flat.

(2) \(\Rightarrow\) (3). Let \(0 \to A_S \to B_S\) be an exact sequence with \(A_S\) and \(B_S\) super finitely presented right \(S\)-modules. Since \(S_R\) is finitely generated free, \(A_R\) and \(B_R\) are super finitely presented right \(R\)-modules. So by (2), we obtain the exact sequence

\[0 \to A \otimes_R M \to B \otimes_R M,\]

which gives rise to the exact sequence

\[0 \to (A \otimes_S S) \otimes_R M \to (B \otimes_S S) \otimes_R M.\]

Consequently, we have the exact sequence

\[0 \to A \otimes_S (S \otimes_R M) \to B \otimes_S (S \otimes_R M).\]

Thus \((S \otimes_R M)\) is a sfp-flat left \(S\)-module.

(3) \(\Rightarrow\) (1). By [15, Lemma 1.1] \(S_M\) is isomorphic to a direct summand of \(S \otimes_R M\), the assertion holds. \(\square\)

**Theorem 3.6.** Let \(S\) be an excellent extension of a subring \(R\) and \(\theta : S_M \to S_N\) is an \(S\)-homomorphism.

1. If the \(R\)-homomorphism \(\theta : R_M \to R_N\) is a sfp-flat preenvelope of \(R_M\), then \((1 \otimes \theta)\tau_M : S_M \to S \otimes_R N\) is a sfp-flat preenvelope of \(S_M\).
(2) If the $R$-homomorphism $\theta : R^M \to R^N$ is a sfp-flat precover of $R^N$, then $\rho_N(1 \otimes \theta) : S \otimes_R M \to S^N$ is a sfp-flat precover of $S^N$.

Proof. (1) By Lemma 3.5, $S \otimes_R N$ is a sfp-flat left $S$-module. For any sfp-flat left $S$-module $S^Q$ and any $S$-homomorphism $\alpha : S^M \to S^Q$, since $S^Q$ is a sfp-flat left $R$-module by Lemma 3.5, there exists $\beta : R^N \to R^Q$ such that $\theta \beta = \alpha$. Consider the following commutative diagram:

$$
\begin{array}{ccc}
S^M & \xrightarrow{\tau_M} & S \otimes_R M \\
\downarrow{\rho_M} & & \downarrow{1 \otimes \theta} \\
S^Q & \xrightarrow{\tau_Q} & S \otimes_R Q \\
\end{array}
$$

Thus we obtain

$$
\rho_Q(1 \otimes \beta)(1 \otimes \theta)\tau_M = \rho_Q(1 \otimes (\beta \theta))\tau_M = \rho_Q(1 \otimes \alpha)\tau_M = \rho_Q\tau_Q\alpha = \alpha.
$$

So $(1 \otimes \theta)\tau_M$ is a sfp-flat preenvelope of $S^M$.

(2) can be proved dually. \qed

References


---

1 Department of Mathematics, Periyar University, Salem - 636 011, TN, India. 
_E-mail address_: selvavlr@yahoo.com

2 Department of Mathematics, Periyar University, Salem - 636 011, TN, India. 
_E-mail address_: prabakaranpvkr@gmail.com