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ON sfp-INJECTIVE AND sfp-FLAT MODULES

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ABSTRACT. Let R be a ring. A left R-module M is said to be sfp-injective if, for every exact sequence $0 \to K \to L$ with K and L super finitely presented left R-modules, the induced sequence $Hom(L, M) \to Hom(K, M) \to 0$ is exact. A right R-module N is called sfp-flat if, for every exact sequence $0 \to K \to L$ with K and L super finitely presented left R-modules, the induced sequence $0 \to N \otimes K \to N \otimes L$ is exact. We study precovers and preenvelopes by sfp-injective and sfp-flat modules, including their properties under (almost) excellent extensions of rings.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, R denotes an associative ring with identity and all modules are unitary. As usual, $pd_R(M)$, $id_R(M)$, and $fd_R(M)$ will denote the projective, injective and flat dimensions of an R-module M, respectively. The character module $Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+ and for a class of R-modules $\mathcal{C}$, we denote by $\mathcal{C}^+ = \{C^+ \mid C \in \mathcal{C}\}$. Let M, N be R-modules, then we use $Hom(M, N)$ (resp. $M \otimes N$) to denote $Hom_R(M, N)$ (resp. $M \otimes_R N$). For unexplained concepts and notations, we refer the reader to [3, 11, 16].

We first recall some known notions and facts needed in the sequel.

Let $\mathcal{C}$ be a class of right R-modules and M a right R-module. Following [3], we say that a map $f \in Hom_R(C, M)$ with $C \in \mathcal{C}$ is a *$\mathcal{C}$-precover* of M, if the group homomorphism $Hom_R(C', f) : Hom_R(C', C) \to Hom_R(C', M)$ is surjective for each $C' \in \mathcal{C}$. A $\mathcal{C}$-precover $f \in Hom_R(C, M)$ of M is called a *$\mathcal{C}$-cover* of M if f is right minimal, that is, if $fg = f$ implies that g is an automorphism for each $g \in End_R(C)$. Dually, we have the definition of *$\mathcal{C}$-preenvelope* (*$\mathcal{C}$-envelope*). In general, $\mathcal{C}$-covers ($\mathcal{C}$-envelopes) may not exist, if exists, they are unique up to isomorphism.

A right R-module M is called *FP-injective*(or absolutely pure) [12] if $Ext^1_R(F, M) = 0$ for all finitely presented right R-modules F.

A left R-module A is said to be *n-presented* if there exists an exact sequence of left R-modules: $F_n \to F_{n-1} \to \cdots \to F_0 \to A \to 0$, in which every F_i is a finitely generated free or projective left R-module. Clearly, a module is 0-presented (resp.

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1-presented) if and only if it is finitely generated (resp. finitely presented), and every m-presented module is n-presented for $m \geq n$. For any two non-negative integers n, d, a left R-module M is called (n, d)-*injective* [17] if $Ext_R^{d+1}(P, M) = 0$ for all n-presented right R-modules P; a right R-module M is called (n, d)-*flat* [17] if $Tor_{d+1}^R(M, P) = 0$ for all n-presented left R-modules P.

A left R-module M is called *super finitely presented* [5] if there exists an exact sequence of left R-modules: $\cdots \to F_1 \to F_0 \to M \to 0$, where each F_i is finitely generated and projective. Recently, Gao and Wang introduced the notion of weak injective and weak flat modules [6]. A left R-module M is called *weak injective* if $Ext_R^1(F, M) = 0$ for any super finitely presented left R-module F. A right R-module N is called *weak flat* if $Tor_1^R(N, F) = 0$ for any super finitely presented left R-module F. The *left super finitely presented dimension*, denoted by $l.sp.gldim(R)$, of a ring R is defined as

$$l.sp.gldim(R) = sup\,\{pd_R(M) | M \text{ is a super finitely presented left } R\text{-module}\}.$$

Garkusha and Generalov [7] introduced the notions of being fp-injective and fp-flat modules as generalization of FP-injective and flat modules respectively. Mao [9] studied further properties of fp-injective (resp. fp-flat) modules through the theory of covers and envelopes. He also investigated fp-injective and fp-flat modules under (almost) excellent extensions of rings. Recall that a left R-module M is said to be *fp-injective* if for every exact sequence $0 \to K \to L$ with K and L finitely presented left R-modules, the induced sequence $Hom(L, M) \to Hom(K, M) \to 0$ is exact. A right R-module N is called *fp-flat* if, for every exact sequence $0 \to K \to L$ with K and L finitely presented left R-modules, the induced sequence $0 \to N \otimes K \to N \otimes L$ is exact.

In [14], Wei and Zhang introduced and studied the notions of fp_n-injective and fp_n-flat modules as generalization of $(n, 0)$-injective and $(n, 0)$-flat modules respectively. A left R-module M is said to be fp_n-*injective* if for every exact sequence $0 \to K \to L$ with K and L n-presented left R-modules, the induced sequence $Hom(L, M) \to Hom(K, M) \to 0$ is exact. A right R-module N is called fp_n-*flat* if, for every exact sequence $0 \to K \to L$ with K and L n-presented left R-modules, the induced sequence $0 \to N \otimes K \to N \otimes L$ is exact.

Using the definition of super finitely presented modules, in this paper, we give a generalization of weak injective modules (resp. weak flat modules) and fp_n-injective modules (resp. fp_n-flat modules), by introducing the notion of sfp-injective modules (resp. sfp-flat modules).

In section 2, sfp-injective modules and sfp-flat modules are defined and studied. We prove that every left R-module has a sfp-injective cover (resp. preenvelope) and every right R-module has a sfp-flat cover (resp. preenvelope). Also we introduced the notion of sfp-projective modules and give some equivalent characterization to $l.sp.gldim(R) = 0$ in terms of sfp-projective modules.

In Section 3, we further study the exchange properties of sfp-injective and sfp-flat modules, as well as sfp-injective precovers (resp. preenvelopes) and sfp-flat precovers (resp. preenvelopes) under (almost) excellent extensions of rings.

2. sfp-injective and sfp-flat Modules

We begin the section with the following lemma.

Lemma 2.1. [13] *Let $0 \to A \to B \to C \to 0$ be an exact sequence of left R-modules. If any two of A, B, C are super finitely presented, then so is the third one.*

Now we give the concepts of sfp-injective modules and sfp-flat modules.

Definition 2.2.

(1) A left R-module M is said to be *sfp-injective* if for every exact sequence $0 \to K \to L$ with K and L super finitely presented left R-modules, the induced sequence $Hom(L, M) \to Hom(K, M) \to 0$ is exact.

(2) A right R-module N is called *sfp-flat* if, for every exact sequence $0 \to K \to L$ with K and L super finitely presented left R-modules, the induced sequence $0 \to N \otimes K \to N \otimes L$ is exact.

Remark 2.3.

(1) Note that every injective (resp. flat) module is sfp-injective (resp. sfp-flat) module. Also we have the following implications:

$$\begin{array}{ccccccc} \text{FP-injective} & \Rightarrow & (2,0)\text{-injective} & \Rightarrow & \cdots & \Rightarrow & \text{weak injective} \\ \Downarrow & & \Downarrow & & & & \Downarrow \\ \text{fp-injective} & \Rightarrow & fp_2\text{-injective} & \Rightarrow & \cdots & \Rightarrow & \text{sfp-injective} \end{array}$$

and

$$\begin{array}{ccccccc} \text{Flat} & \Rightarrow & (2,0)\text{-flat} & \Rightarrow & \cdots & \Rightarrow & \text{weak flat} \\ \Downarrow & & \Downarrow & & & & \Downarrow \\ \text{fp-flat} & \Rightarrow & fp_2\text{-flat} & \Rightarrow & \cdots & \Rightarrow & \text{sfp-flat} \end{array}$$

(2) If R is a coherent ring then all the modules in the first diagram coincide with FP-injective modules and all the modules in the second diagram coincide with flat modules by [6, Remark 2.2(1)] and [7, Theorem 2.4].

Proposition 2.4. *Let $\{M_i\}_I$ be a class of left R-modules, and $\{N_i\}_I$ a class of right R-modules. Then,*

(1) *$\prod_I M_i$ (resp. $\bigoplus_I M_i$) is sfp-injective if and only if each M_i is a sfp-injective module.*

(2) *$\prod_I N_i$ (resp. $\bigoplus_I N_i$) is sfp-flat if and only if each N_i is a sfp-flat module.*

Proof. (1). We know that $Hom(A, \prod_I M_i) \cong \prod_I Hom(A, M_i)$ for any left R-module A and in addition that $Hom(A, \bigoplus_I M_i) \cong \bigoplus_I Hom(A, M_i)$ for any finitely presented left R-module A. Since every super finitely presented module is finitely presented, the assertion holds.

(2). Since $(\bigoplus_I N_i) \otimes A \cong \bigoplus_I (N_i \otimes A)$ for any left R-module A and $(\prod_I N_i) \otimes A \cong \prod_I (N_i \otimes A)$ for any finitely presented left R-module A, the assertion holds. □

In what follows let us denote the class of all sfp-injective left R-modules by $\mathcal{SI}$ and the class of all sfp-flat right R-modules by $\mathcal{SF}$.

Lemma 2.5. *The following are true for any ring R:*

1. *A left R-module M is sfp-injective if and only if M^+ is sfp-flat.*
2. *A right R-module N is sfp-flat if and only if N^+ is sfp-injective.*
3. *The class $\mathcal{SI}$ (resp., $\mathcal{SF}$) is closed under pure submodules, pure quotients and direct limits.*
4. *A super finitely presented left R-module is sfp-injective if and only if it is weak injective.*

Proof. (1). Let $0 \to K \to L$ be any exact sequence with K and L are super finitely presented left R-modules. Consider the following commutative diagram:

$$\begin{tikzcd} M^+ \otimes K \arrow[r] \arrow[d, "\sigma_K"] & M^+ \otimes L \arrow[d, "\sigma_L"] \\ Hom(K, M)^+ \arrow[r] & Hom(L, M)^+. \end{tikzcd}$$

Since any super finitely presented left R-module is finitely presented, σ_K and σ_L are isomorphisms by [11, Lemma 3.60]. Then, M is sfp-injective if and only if $Hom(L, M) \to Hom(K, M) \to 0$ is exact if and only if $0 \to Hom(K, M)^+ \to Hom(L, M)^+$ is exact if and only if $0 \to M^+ \otimes K \to M^+ \otimes L$ is exact if and only if M^+ is sfp-flat by the defnition.

(2). Let $0 \to A \to B$ be any exact sequence with A and B super finitely presented left R-modules. Then N is sfp-flat if and only if $0 \to N \otimes A \to N \otimes B$ is exact if and only if $(N \otimes B)^+ \to (N \otimes A)^+ \to 0$ is exact if and only if $Hom(B, N^+) \to Hom(A, N^+) \to 0$ is exact if and only if N^+ is sfp-injective by the defnition.

(3). Let $0 \to A \to B \to C \to 0$ be a pure exact sequence with B sfp-injective. Then, the sequence $0 \to C^+ \to B^+ \to A^+ \to 0$ is split. Since B^+ is sfp-flat by (1), it follows that A^+ and C^+ are sfp-flat by Proposition 2.4. So A and C are sfp-injective by (1) again. Thus, the class $\mathcal{SI}$ is closed under pure submodules and pure quotients. Similarly, we get that class $\mathcal{SF}$ is closed under pure submodules and pure quotients. Since for every finitely presented left R-module A, we have

$$Hom(A, \lim_{\to} M_i) \cong \lim_{\to} Hom(A, M_i)$$

and for any left R-module A, we have

$$(\lim_{\to} N_i) \otimes A \cong \lim_{\to} (N_i \otimes A)$$

the class $\mathcal{SI}$(resp., $\mathcal{SF}$) is closed under direct limits.

(4). It is enough to show that every super finitely presented sfp-injective module is weak injective. Let A be a super finitely presented sfp-injective module and let $0 \to A \to B \to C \to 0$ be any exact sequence of left R-modules with C super finitely presented. Then, B is super finitely presented by Lemma 2.1, and $0 \to Hom(C, A) \to Hom(B, A) \to Hom(A, A) \to 0$ is exact by the definition of sfp-injective modules. Thus, $0 \to A \to B \to C \to 0$ is split, that is, $Ext^1(C, A) = 0$. So A is weak injective. □

Theorem 2.6. *The following hold for any ring R:*

(1) *Every left (resp. right) R-module has a sfp-injective (resp. sfp-flat) cover.*
(2) *Every left (resp. right) R-module has a sfp-injective (resp. sfp-flat) preenvelope.*

Proof. (1). From Proposition 2.4 and Lemma 2.5, we get $\mathcal{SI}$ and $\mathcal{SF}$ are closed under direct sums and pure quotients. Then, every left (resp. right) R-module has a sfp-injective (resp. sfp-flat) cover by [8, Theorem 2.5].

(2). Since $\mathcal{SI}$ and $\mathcal{SF}$ are closed under direct products and pure submodules by Proposition 2.4 and Lemma 2.5, every left (resp. right) R-module has a sfp-injective (resp. sfp-flat) preenvelope by [10, Corollary 3.5]. □

Proposition 2.7. *The following hold for any ring R:*

(1) *If $M \to N$ is a sfp-injective preenvelope of a left R-module M, then $N^+ \to M^+$ is a sfp-flat precover of M^+.*
(2) *If $M \to N$ is a sfp-flat preenvelope of a right R-module M, then $N^+ \to M^+$ is a sfp-injective precover of M^+.*

Proof. By Lemma 2.5, we have $\mathcal{SF}^+ \subseteq \mathcal{SI}$ and $\mathcal{SI}^+ \subseteq \mathcal{SF}$. Now both the assertions follows immediately from [4, Theorem 3.1]. □

Proposition 2.8. *For any ring R, the following are equivalent:*

(1) *R is weak injective as a left R-module;*
(2) *Every left R-module has an epic sfp-injective cover;*
(3) *Every right R-module has a monic sfp-flat preenvelope.*

Proof. (1) $\Rightarrow$ (2). Let M be a left R-module. Then there exists an exact sequence $F \to M \to 0$ with F free. Note that R is sfp-injective as a left R-module. From Proposition 2.4, we get that F is sfp-injective. Since M has a sfp-injective cover $f : I \to M$ by Theorem 2.6, then f is epic.

(2) $\Rightarrow$ (1). Assume that $f : M \to {}_RR$ is an epic sfp-injective cover of ${}_RR$. Then, ${}_RR$ is isomorphic to a direct summand of M, and so it is sfp-injective by Proposition 2.4. Since ${}_RR$ is super finitely presented, ${}_RR$ is weak injective by Lemma 2.5.

(1) $\Rightarrow$ (3). Let M be a right R-module. By Theorem 2.6 M has a sfp-flat preenvelope $g : M \to F$. Since $({}_RR)^+$ is an injective cogenerator in the category of right R-modules, there exists an exact sequence $0 \to M \to \prod({}_RR)^+$. By assumption, ${}_RR$ is weak injective module, and thus sfp-injective, and so $({}_RR)^+$ is sfp-flat by Lemma 2.5. Thus, $\prod({}_RR)^+$ is sfp-flat by Proposition 2.4. Therefore, g is monic.

(3) $\Rightarrow$ (1). Since the injective right R-module $({}_RR)^+$ has a monic sfp-flat preenvelope $h : ({}_RR)^+ \to F$ by assumption, then $({}_RR)^+$ is a direct summand of F and so it is sfp-flat by Proposition 2.4, which implies that ${}_RR$ is sfp-injective from Lemma 2.5. Since ${}_RR$ is super finitely presented, ${}_RR$ is weak injective by Lemma 2.5. It follows that R is weak injective as a left R-module. □

Theorem 2.9. *The following are equivalent for a ring R:*

(1) *Every quotient of any sfp-injective left R-module is sfp-injective;*

(2) *Every left R-module has a monic sfp-injective cover;*
(3) *Every submodule of any sfp-flat right R-module is sfp-flat;*
(4) *Every right R-module has an epic sfp-flat envelope.*

Proof. (1) $\Rightarrow$ (2). For any left R-module M, there is a sfp-injective cover $f : E \to M$ by Theorem 2.6(1). Note that $im(f)$ is sfp-injective by (1). It is easy to verify that $im(f) \to M$ is a monic sfp-injective cover.

(2) $\Rightarrow$ (3). Let N be a submodule of a sfp-flat right R-module M. Then the exact sequence $0 \to N \to M$ induces the exact sequence $M^+ \to N^+ \to 0$. Note that M^+ is sfp-injective and N^+ has a monic sfp-injective cover by (2). Thus N^+ is sfp-injective, and so N is sfp-flat.

(3) $\Rightarrow$ (4). For any right R-module M, there is a sfp-flat preenvelope $f : M \to F$ by Theorem 2.6(2). Since $im(f)$ is sfp-flat by (3), we have $M \to im(f)$ is an epic sfp-flat preenvelope, equivalently, an epic sfp-flat envelope.

(4) $\Rightarrow$ (1). Let M be any sfp-injective left R-module and N any submodule of M. Then we get a short exact sequence $0 \to N \to M \to M/N \to 0$ which induces the exact sequence $0 \to (M/N)^+ \to M^+$. Since $(M/N)^+$ has an epic sfp-flat envelope and M^+ is sfp-flat, we have $(M/N)^+$ is sfp-flat, and so M/N is sfp-injective. □

Theorem 2.10. *The following are equivalent for a ring R:*

(1) *Every left R-module is sfp-injective;*
(2) *Every right R-module is sfp-flat;*
(3) *Every super finitely presented left R-module is sfp-injective;*
(4) *Every super finitely presented submodule of any super finitely presented left R-module is its direct summands;*
(5) *R is weak injective as a left R-module and every submodule of any sfp-flat right R-module is sfp-flat.*

Proof. (1) $\Rightarrow$ (2). Let N be any right R-module. Since N^+ is sfp-injective, N is sfp-flat by Lemma 2.5.

(2) $\Rightarrow$ (1). For any left R-module M, since M^+ is sfp-flat, M is sfp-injective by Lemma 2.5.

(1) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (4). Let A be a super finitely presented submodule of any super finitely presented left R-module B. Then we get an exact sequence $0 \to A \to B \to C \to 0$. Note that C is super finitely presented by Lemma 2.1. Since A is weak injective by (3) and Lemma 2.5(4), $Ext^1(C, A) = 0$. So the exact sequence $0 \to A \to B \to C \to 0$ is split. Thus A is a direct summand of B.

(4) $\Rightarrow$ (5). Let $0 \to A \to B \to C \to 0$ be any exact sequence with A and B super finitely presented left R-modules. Then, this sequence is split. So every left R-module is sfp-injective and every right R-module is sfp-flat. Hence the assertions holds.

(5) $\Rightarrow$ (1). Let M be a left R-module. Then there is an exact sequence $F \to M \to 0$ with F free. Since ${}_RR$ is weak injective as a left R-module, F is weak injective, and so is sfp-injective. Thus M is sfp-injective by Theorem 2.9. □

Theorem 2.11. *The following are equivalent for a ring R:*

(1) *Every sfp-injective left R-module is weak injective;*
(2) *Every sfp-flat right R-module is weak flat;*
(3) *The character module of a sfp-injective left R-module is weak flat;*
(4) *A left R-module sfp-injective if and only if its character module is weak flat.*

Proof. By using [6, Theorem 2.10, Remark 2.2(2)] and Lemma 2.5 it is easy to verify the equivalence. □

Next we introduce the concept of sfp-projective modules, which may be viewed as a dual of sfp-injective modules.

Definition 2.12. A left R-module M is said to be *sfp-projective* if for every exact sequence $A \to B \to 0$ with A and B super finitely presented left R-modules, the induced sequence $Hom(M, A) \to Hom(M, B) \to 0$ is exact.

Theorem 2.13. *The following are equivalent for a ring R:*

(1) $l.sp.gldim(R) = 0$*;*
(2) *Every left R-module is sfp-projective;*
(3) *Every super finitely presented left R-module is sfp-projective.*

Proof. (1) $\Rightarrow$ (2). Since every epimorphism $K \to L$ with K and L super finitely presented left R-modules is split by (1), every left R-module is sfp-projective.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). Let M be any super finitely presented left R-module. There is an epimorphism $P \to M$ with P finitely generated projective. Since M is sfp-projective, $P \to M$ is a split epimorphism. Thus M is projective. So $l.sp.gldim(R) = 0$. □

Remark 2.14.

(1) Note that every fp-projective module (see [9, Definition 4.1]) is spf-projective. If R is a coherent ring then sfp-projective modules are coincide with fp-projective modules.
(2) As Gao mentioned in [6, Remark 3.11] there exists a ring with $l.sp.gldim(R) = 0$ which is not von Neumann regular. By Theorem 2.13 and [9, Theorem 4.2] there exists sfp-projective module which is not fp-projective.

3. sfp-injective and sfp-flat modules under (almost) excellent extensions of rings

Recall that a ring S is said to be an *almost excellent extension* of a subring R if the following conditions are satisfied:

(1) S is a finite normalizing extension of R, that is, R and S have the same identity and there are elements $s_1, \cdots, s_n \in S$ such that $S = Rs_1 + \cdots + Rs_n$ and $Rs_i = s_iR$ for all $i = 1, \cdots, n$,
(2) S_R is flat and ${}_RS$ is projective,
(3) S is left R-projective, that is, if ${}_SM$ is a submodule of ${}_SN$ and ${}_RM$ is a direct summand of ${}_RN$, then ${}_SM$ is a direct summand of ${}_SN$.

Further, S is an excellent extension of R if S is an almost excellent extension of R and S is free with basis $s_1, \cdots, s_n$ as both a right and a left R-module with $s_1 = 1_R$.

Lemma 3.1. *Let S be a ring extension of a ring R.*

1. *If S_R (resp. ${}_RS$) is flat, then an R-module ${}_RA$ (resp. A_R) is super finitely presented implies that $S \otimes_R A$ (resp. $A \otimes_R S$) is super finitely presented S-module.*
2. *If S_R (resp. ${}_RS$) is finitely generated projective, then an S-module A_S (resp. ${}_SA$) is super finitely presented implies that A_R (resp. ${}_RA$) is super finitely presented R-module.*

Proof. (1). Since S_R is flat, the assertion holds by the standard isomorphisms $S \otimes_R R^n \cong {}_SS^n$ for all $n \geq 1$.

(2). If ${}_RS$ is a finitely generated projective left R-module, then we have a isomorphism $Hom_S(S, S^n) \cong {}_RS^n$ for all $n \geq 1$. The assertion easily follows from the definition of super finitely presented modules. □

Proposition 3.2. *Let S be an almost excellent extension of a subring R and ${}_SM$ a left S-module. Then the following are equivalent:*

1. *${}_SM$ is sfp-injective;*
2. *${}_RM$ is sfp-injective;*
3. *$Hom_R(S, M)$ is a sfp-injective left S-module.*

Proof. (1) ⇒ (2). Let $0 \to {}_RA \to {}_RB$ be an exact sequence of left R-modules with ${}_RA$ and ${}_RB$ super finitely presented. Since S_R is a flat, from the definition of almost excellent extensions we get an exact sequence of left S-modules

$$0 \to S_R \otimes_R A \to S_R \otimes_R B.$$

Then, $S_R \otimes_R A$ and $S_R \otimes_R B$ is super finitely presented left S-modules by Lemma 3.1 since S is an almost excellent extension of a subring R. By (1), the sequence

$$Hom_S(S_R \otimes_R B, M) \to Hom_S(S_R \otimes_R A, M) \to 0$$

is exact, which implies that the sequence

$$Hom_R(B, Hom_S(S, M)) \to Hom_R(A, Hom_S(S, M)) \to 0$$

is also exact. Thus, we get an exact sequence

$$Hom_R(B, M) \to Hom_R(A, M) \to 0,$$

that is, ${}_RM$ is a sfp-injective left R-module.

(2) ⇒ (3). Assume that $0 \to {}_SA \to {}_SB$ is an exact sequence of left S-modules with ${}_SA$ and ${}_SB$ super finitely presented. Since ${}_RS$ is finitely generated projective, ${}_RA$ and ${}_RB$ are super finitely presented left R-modules by Lemma 3.1. By (2), we get an exact sequence

$$Hom_R(B, M) \to Hom_R(A, M) \to 0,$$

which yields the exactness of the sequence

$$Hom_R(S \otimes_S B, M) \to Hom_R(S \otimes_S A, M) \to 0.$$

Then, we get the exact sequence

$$Hom_S(B, Hom_R(S, M)) \to Hom_S(A, Hom_R(S, M)) \to 0.$$

Thus, $Hom_R(S, M)$ is a sfp-injective left S-module.

(3) $\Rightarrow$ (1). By [15, Lemma 1.1], ${}_SM$ is isomorphic to a direct summand of $Hom_R(S, M)$. Thus, ${}_SM$ is an sfp-injective left S-module. $\square$

Corollary 3.3. *Let S be an almost excellent extension of a subring R and M_S a right S-module. Then, the following are equivalent:*

1. *M_S is sfp-flat;*
2. *M_R is sfp-flat;*
3. *$M \otimes_R S$ is sfp-flat right S-module.*

Proof. (1) $\Leftrightarrow$ (2). By Lemma 2.5 and Proposition 3.2, M_S is a sfp-flat right S-module if and only if ${}_S(M^+)$ is a sfp-injective left S-module if and only if ${}_R(M^+)$ is a sfp-injective left R-module if and only if M_R is a sfp-flat right R-module.

(1) $\Leftrightarrow$ (3). By Lemma 2.5 and Proposition 3.2, M_S is a sfp-flat right S-module if and only if ${}_S(M^+)$ is a sfp-injective left S-module, if and only if $(M \otimes_R S)^+ \cong Hom_R(S, M^+)$ is a sfp-injective left S-module if and only if $M \otimes_R S$ is a sfp-flat right S-module. $\square$

Let S be an almost excellent extension of a subring R. From [15, Lemma 1.1], we see that a left S-module ${}_SM$ is isomorphic to a direct summand of $Hom_R(S, M)$ and $S \otimes_R M$. In the sequel, we denote $\lambda_M : {}_SM \to Hom_R(S, M)$ (resp. $\tau_M : {}_SM \to S \otimes_R M$) as the inclusion and $\pi_M : Hom_R(S, M) \to {}_SM$ (resp. $\rho_M : S \otimes_R M \to {}_SM$) as the canonical projection. Let $\theta : M \to N$ be a homomorphism of left R-modules, we use θ_* to denote the induced homomorphism: $\theta_* = Hom_R(S, \theta) : Hom_R(S, M) \to Hom_R(S, N)$ for any left R-module S.

Theorem 3.4. *Suppose that S is an almost excellent extension of a subring R and $\theta : {}_SM \to {}_SN$ is an S-homomorphism.*

1. *If the R-homomorphism $\theta : {}_RM \to {}_RN$ is a sfp-injective precover of ${}_RN$, then $\pi_N\theta_* : Hom_R(S, M) \to {}_SN$ is a sfp-injective precover of ${}_SN$.*
2. *If the R-homomorphism $\theta : {}_RM \to {}_RN$ is a sfp-injective preenvelope of ${}_RM$, then $\theta_*\lambda_M : {}_SM \to Hom_R(S, N)$ is a sfp-injective preenvelope of ${}_SM$.*

Proof. (1). Since ${}_RM$ is sfp-injective, $Hom_R(S, M)$ is a sfp-injective left S-module by Proposition 3.2. For any sfp-injective left S-module ${}_SG$ and any S-homomorphism $\alpha : {}_SG \to {}_SN$, ${}_RG$ is a sfp-injective left R-module by Proposition 3.2. So there exists $\beta : {}_RG \to {}_RM$ such that $\theta\beta = \alpha$. Thus we obtain the following commutative diagram:

$$\begin{tikzcd}
Hom_R(S, M) \arrow[d, equal] & Hom_R(S, G) \arrow[l, "\beta_*"'] \arrow[r, shift right, "\lambda_G"'] \arrow[d, "\alpha_*"'] & {}_SG \arrow[l, shift right, "\pi_G"'] \arrow[d, "\alpha"] \\
Hom_R(S, M) \arrow[r, "\theta_*"'] & Hom_R(S, N) \arrow[r, shift right, "\lambda_N"'] & {}_SN \arrow[l, shift right, "\pi_N"']
\end{tikzcd}$$

So we have,

$$(\pi_N\theta_*)(\beta_*\lambda_G) = \pi_N(\theta\beta)_*\lambda_G = \pi_N\alpha_*\lambda_G = \pi_N\lambda_N\alpha = \alpha.$$

Hence $\pi_N\theta_*$ is a sfp-injective precover of ${}_SN$.

(2) can be proved dually. $\square$

Lemma 3.5. *Let S be an excellent extension of a subring R and ${}_SM$ a left S-module. Then the following are equivalent:*

(1) *${}_SM$ is sfp-flat;*
(2) *${}_RM$ is sfp-flat;*
(3) *$S \otimes_R M$ is a sfp-flat left S-module.*

Proof. (1) $\Rightarrow$ (2). Let $0 \to K_R \to L_R$ be an exact sequence with K_R and L_R super finitely presented right R-modules. Since ${}_RS$ is flat, we get the S-module exact sequence

$$0 \to K \otimes_R S \to L \otimes_R S.$$

Note that $K \otimes_R S$ and $L \otimes_R S$ are super finitely presented right S-modules. Thus by (1), we get the exact sequence

$$0 \to (K \otimes_R S) \otimes_S M \to (L \otimes_R S) \otimes_S M,$$

which gives the exactness of the sequence

$$0 \to K \otimes_R (S \otimes_S M) \to L \otimes_R (S \otimes_S M),$$

So we have the exact sequence $0 \to K \otimes_R M \to L \otimes_R M$. Therefore ${}_RM$ is sfp-flat.

(2) $\Rightarrow$ (3). Let $0 \to A_S \to B_S$ be an exact sequence with A_S and B_S super finitely presented right S-modules. Since S_R is finitely generated free, A_R and B_R are super finitely presented right R-modules. So by (2), we obtain the exact sequence

$$0 \to A \otimes_R M \to B \otimes_R M,$$

which gives rise to the exact sequence

$$0 \to (A \otimes_S S) \otimes_R M \to (B \otimes_S S) \otimes_R M.$$

Consequently, we have the exact sequence

$$0 \to A \otimes_S (S \otimes_R M) \to B \otimes_S (S \otimes_R M).$$

Thus $(S \otimes_R M)$ is a sfp-flat left S-module.

(3) $\Rightarrow$ (1). By [15, Lemma 1.1] ${}_SM$ is isomorphic to a direct summand of $S \otimes_R M$, the assertion holds. $\square$

Theorem 3.6. *Let S be an excellent extension of a subring R and $\theta : {}_SM \to {}_SN$ is an S-homomorphism.*

(1) *If the R-homomorphism $\theta : {}_RM \to {}_RN$ is a sfp-flat preenvelope of ${}_RM$, then $(1 \otimes \theta)\tau_M : {}_SM \to S \otimes_R N$ is a sfp-flat preenvelope of ${}_SM$.*

(2) *If the R-homomorphism* $\theta : {}_RM \to {}_RN$ *is a sfp-flat precover of* ${}_RN$*, then* $\rho_N(1 \otimes \theta) : S \otimes_R M \to {}_SN$ *is a sfp-flat precover of* ${}_SN$*.*

Proof. (1) By Lemma 3.5, $S \otimes_R N$ is a sfp-flat left S-module. For any sfp-flat left S-module ${}_SQ$ and any S-homomorphism $\alpha : {}_SM \to {}_SQ$, since ${}_SQ$ is a sfp-flat left R-module by Lemma 3.5, there exists $\beta : {}_RN \to {}_RQ$ such that $\theta\beta = \alpha$. Consider the following commutative diagram:

$$\begin{tikzcd}
{}_SM \arrow[r, shift right, "\rho_M"'] \arrow[d, "\alpha"'] & S \otimes_R M \arrow[l, shift right, "\tau_M"'] \arrow[r, "1\otimes\theta"] \arrow[d, "1\otimes\alpha"] & S \otimes_R N \arrow[d, equal] \\
{}_SQ \arrow[r, shift left, "\tau_Q"] & S \otimes_R Q \arrow[l, shift left, "\rho_Q"] & S \otimes_R N \arrow[l, "1\otimes\beta"']
\end{tikzcd}$$

Thus we obtain

$$\rho_Q(1 \otimes \beta)(1 \otimes \theta)\tau_M = \rho_Q(1 \otimes (\beta\theta))\tau_M = \rho_Q(1 \otimes \alpha)\tau_M = \rho_Q\tau_Q\alpha = \alpha.$$

So $(1 \otimes \theta)\tau_M$ is a sfp-flat preenvelope of ${}_SM$.

(2) can be proved dually. □

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