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ON OPEN NEIGHBORHOOD ENERGY OF GRAPHS

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Abstract. The open neighborhood of a vertex u, represented by $N(u)$ in a simple connected graph G is the collection of all nodes that are adjacent to u, other than u. In this paper, we introduce a square matrix of order n, called the open neighborhood matrix $ONM(G)$ of G whose $(i,\ j)^{th}$ entry is $\frac{|N(v_i)\cap N(v_j)|}{d_i+d_j}$ whenever $v_i \sim v_j,\ i \neq j$; and zero otherwise, where d_i and d_j are the degrees of v_i and v_j, respectively. We then establish the relationship between the connectedness of the graph G and multiplicity of the eigenvalue zero of the matrix $ONM(G)$, if it exists. Furthermore, we give some bounds for the largest open neighborhood eigenvalue and open neighborhood energy of graphs.

1. Introduction

Throughout this paper, let $G = (V,\ E)$ be a simple, connected, undirected graph of order n with vertex set $V(G) = \{v_1,\ v_2, \cdots,\ v_n\}$ and size m with an edge set $E(G) = \{e_1,\ e_2, \cdots,\ e_m\}$. The neighborhood $N(v_i)$ of every node v_i in G is the collection of nodes that are connected to v_i in G. The neighborhood graph $N(G)$ is defined to be the intersection graph of the neighborhoods of the nodes in G. That is G and $N(G)$ have same vertex set and two vertices of $N(G)$ are connected by an edge if and only if they have at least one common neighbor. Exoo and Harary [7] and subsequently Greenburg, Lundgren, and Maybee [9] investigated these graphs as 2-step graphs. Acharya and Vartak initially presented the concept of Neighborhood graphs in [1] and identified some of their characteristics. The neighborhood complex $\mathcal{N}(G)$ of a graph G has been introduced in [5], which is defined as

$$\mathcal{N}(G) = \{W|\ W \subseteq V(G),\ \exists v \in V(G):\ W \subseteq N(v)\}$$

The neighborhood polynomial of a graph G, represented by $N(G,\ x)$, is defined to be the generating function for the neighborhood complex of G. The definition of the neighborhood polynomial was first introduced in [5], and it can be expressed

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as

$$N(G,\ x) = \sum_{v \in \mathcal{N}(G)} x^{|v|}.$$

Generally graphs are connected to several matrices. We may state in general that, if $\mathbf{M}(G)$ is a matrix associated with a graph G, whose elements are given by

$$\mathbf{M}_{ij} = \psi(d_i,\ d_j)\ for\ \mathbf{R}(i,\ j),$$

where $\psi(d_i,\ d_j)$ is a function on the degrees of vertices v_i and v_j, while $\mathbf{R}(i,\ j)$ is the relation between the vertices v_i and v_j. The adjacency matrix $A(G)$ of G is the most extensively studied such matrix [17], also the concepts of degree sum matrix [15], degree exponent matrix [16], degree exponent sum matrix [2], Harary matrix [14] etc., have also been proposed. Inspired by these previous works, we aim to present a new type of matrix known as the open neighborhood matrix $ONM(G)$ of G, whose $(i,\ j)^{th}$ entry is $\dfrac{|N(v_i) \cap N(v_j)|}{(d_i + d_j)}$ whenever $i \neq j,\ v_i \sim v_j$ in G, and zero otherwise. There is a pleasant relation between the open neighborhood matrix $ONM(G)$ of a graph G, and the adjacency matrix $A(G)$ of G. More precisely, if $i \neq j$, then the $(i,\ j)^{th}$ entry of the $ONM(G)$ is same as the $(i,\ j)^{th}$ entry of $\dfrac{1}{d_i + d_j} A^2(G)$ and which is the ratio of the number of two-walks between the vertices v_i and v_j to the sum of its degrees d_i and d_j. In some cases, it is possible to calculate the adjacency energy of specific families of graphs if the open neighborhood energy is already known. For instance, if G is a strongly regular graph with parameters,$(n,\ k,\ \lambda,\ \mu)$, then $\mathcal{E}(G) = \dfrac{2k}{\lambda} ONE(G)$. More specifically, if $G = K_n$, the complete graph on $n \geq 2$ vertices, then $\mathcal{E}(K_n) = 2(ONE(K_n)+1)$.

Since $ONM(G)$ is a real symmetric matrix, all of its eigenvalues are real. Consequently, we may arrange these eigenvalues(called the open neighborhood eigenvalues) as, $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$. We define the open neighborhood energy $ONE(G)$ of G to be

$$ONE(G) = \sum_{i=1}^{n} |\rho_i|.$$

The structure of the paper is organized in the following manner: Section 2 includes essential concepts and results that will serve as a basis for the subsequent sections. The key findings of this study are presented in Section 3, which encompasses the bounds for the spectral radius ρ_1 of the open neighborhood matrix $ONM(G)$ and the open neighborhood energy $ONE(G)$. Furthermore, we try to examine the relationship between the connectivity of a given graph G with the multiplicity of the open neighborhood eigenvalue zero.

2. Preliminaries

First we recall some known results that will be used in the following sections.

Lemma 2.1. [6] *For real numbers* $x_1 \leq x_2 \leq \cdots \leq x_n$ *and* $y_1 \leq y_2 \leq \cdots \leq y_n$,

$$\left(\sum_{j=1}^{n} x_j\right)\left(\sum_{j=1}^{n} y_j\right) \leq n\left(\sum_{j=1}^{n} x_j y_j\right)$$

with equality holds if and only if $x_1 = x_2 = \cdots = x_n$ *or* $y_1 = y_2 = \cdots = y_n$.

Lemma 2.2. [13] *For non-negative numbers* $y_1,\ y_2,\ \cdots\ y_n$ *and* $\alpha \geq 2$,

$$\sum_{j=1}^{n} (y_j)^{\alpha} \leq \left(\sum_{j=1}^{n} y_j^2\right)^{\frac{\alpha}{2}}$$

Lemma 2.3. [11] *Suppose that* $B = (b_{r,s})$ *and* $C = (c_{r,s})$ *are two nonnegative symmetric matrices of order* n. *If* $B \geq C$, *i.e.,* $b_{r,s} \geq c_{r,s}$ *for all* $r,\ s$, *then* $\rho_1(B) \geq \rho_1(C)$.

Lemma 2.4. [4] *Let* G *be a graph of order* n *with degree sequence* $d_1,\ d_2,\ ...,\ d_n$ *and firse Zagreb index* M_1. *Then*

$$\lambda_1 \geq \sqrt{\frac{\sum_{j=1}^{n} d_j^2}{n}} = \sqrt{\frac{M_1}{n}},$$

with equality holds if and only if G *is regular or semiregular.*

Lemma 2.5. [10] *If* G *is a connected graph of order* n *with size* m, *then*

$$\lambda_1 \leq \sqrt{2m - n + 1},$$

with equality holds if and only if G *is isomorphic to* K_n *or* $K_{1,\ n-1}$.

Lemma 2.6. [8] *Let* G *be a graph with* n *vertices,* m *edges, and the first Zagreb index* M_1. *The maximum degree of* G, Δ, *satisfies the inequality,*

$$\Delta \leq \frac{2m}{n} + \frac{n-1}{n}\sqrt{\frac{nM_1 - 4m^2}{n-1}}.$$

Lemma 2.7. [3] *Let* x_j *and* y_j *be nonnegative real numbers. Then*

$$\left| n\sum_{j=1}^{n} x_j y_j - \sum_{j=1}^{n} x_j \sum_{j=1}^{n} y_j \right| \leq \alpha(n)(A-a)(B-b),$$

where $a,\ b,\ A$ *and* B *are real constants such that* $a \leq x_i \leq A$ *and* $b \leq y_i \leq B$. *Further,* $\alpha(n) = n\lfloor \frac{n}{2} \rfloor \left(1 - \frac{1}{n}\lfloor \frac{n}{2} \rfloor\right)$.

3. Main results

The key findings of this study are presented in this section.

Definition 3.1. Let G be a graph of order n. The open neighborhood matrix of G, denoted by $ONM(G) = [o_{ij}]$, is a matrix of order n whose $(i,\ j)^{th}$ entry is,

$$o_{ij} = \begin{cases} \dfrac{|N(v_i) \cap N(v_j)|}{d_i + d_j} & if\ v_i \sim v_j,\ i \neq j, \\ 0 & otherwise. \end{cases}$$

Since $|N(v_i) \cup N(v_j)| = |N(v_i)| + |N(v_j)| - |N(v_i) \cap N(v_j)| > 0$, for $i \neq j$ and $v_iv_j \in E(G)$ we have, $0 \leq o_{ij} < 1$. Also, note that $ONM(G)$ is a real symmetric matrix.

Example 3.2. Suppose G is the complete graph K_n with $n \geq 2$ vertices, then we have for any edge $e = v_iv_j$ in G, $|N(v_i) \cap N(v_j)| = n - 2$ and $d_i = d_j = n - 1$. Thus $ONM(G) = \dfrac{n-2}{2(n-1)}A(G)$, where $A(G)$ is the adjacency matrix of G.

Example 3.3. Consider a tree T of order n, then $|N(v_i) \cap N(v_j)| = 0$ for any edge $e = v_iv_j$ in T. Otherwise if $|N(v_i) \cap N(v_j)| \geq 1$, there exists at least one common neighbour of v_i and v_j, say x in T. Thus $v_ixv_jv_i$ together contribute a cycle in T. Since a tree does not contain any cycles, we have reached a contradiction and hence $ONM(T) = 0$.

Example 3.4. Let G be the complete bipartite graph $K_{m,n}$, with the partite sets X and X' such that $|X| = m$ and $|X'| = n$, respectively. Since G is a complete bipartite graph, we have $X \cap X' = \phi$, thus $ONM(G) = 0$.

Proposition 3.5. *Let $ONM(G)$ be the open neighborhood matrix of a graph G with eigenvalues $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$. Then,*

(1) $\sum_{j=1}^{n} \rho_j = 0.$

(2) $\sum_{j=1}^{n} \rho_j^2 = 2 \sum_{v_i \sim v_j} \left(\dfrac{|N(v_i) \cap N(v_j)|}{d_i + d_j} \right)^2 \leq 2m.$

Proof. (1) Since $Trace(ONM(G)) = 0$, we have $\sum_{j=1}^{n} \rho_j = 0$.

(2) We have

$$\begin{aligned} \sum_{j=1}^{n} \rho_j^2 &= Trace(ONM(G)^2) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} o_{ij}o_{ji} \qquad (3.1) \\ &= 2 \sum_{v_i \sim v_j} \left(\frac{|N(v_i) \cap N(v_j)|}{d_i + d_j} \right)^2. \end{aligned}$$

Since $\dfrac{|N(v_i)\cap N(v_j)|}{d_i+d_j} < 1$, and considering equation 3.1, the following holds

$$\sum_{j=1}^{n}\rho_i^2 = 2\sum_{v_i\sim v_j}\left(\frac{|N(v_i)\cap N(v_j)|}{d_i+d_j}\right)^2 \leq 2\sum_{v_i\sim v_j}(1)^2 = 2m.$$

□

Theorem 3.6. *Let G be a graph of order n with maximum degree Δ, minimum degree δ and first Zegrab index M_1. Then*

$$\frac{1}{2\Delta}\sqrt{\frac{M_1}{n}} < \rho_1 < \frac{2m}{n\delta} + \frac{(n-1)}{n\delta}\sqrt{\frac{nM_1^2-4m^2}{(n-1)}}.$$

Proof. Let G be a graph of order n with maximum degree Δ, minimum degree δ first Zegrab index M_1 and open neighborhood matrix $ONM(G)=[o_{ij}]$. Let J be the matrix of all one's, then observe that

$$\frac{1}{2\Delta}A(G) < ONM(G) < \frac{\Delta}{\delta}J. \tag{3.2}$$

This is because, if v_iv_j is an edge in G, then

$$0 \leq o_{ij} = \frac{|N(v_i)\cap N(v_j)|}{d_i+d_j} < \frac{\Delta}{\delta} \text{ and } o_{ij} > \frac{1}{2\Delta}.$$

Now apply Lemma 2.3 to equation 3.2 then again apply Lemmas 2.4 and 2.6 to obtain

$$\frac{1}{2\Delta}\sqrt{\frac{M_1}{n}} < \rho_1 < \frac{2m}{n\delta} + \frac{n-1}{n\delta}\sqrt{\frac{nM_1^2-4m^2}{(n-1)}}.$$

□

Theorem 3.7. *Let G be a graph of order n. Then*

$$\rho_1 < \left(n-1+\frac{1}{n-1}\right)\sqrt{2m-n+1}.$$

Proof. Let the open neighborhood matrix of G be $ONM(G)=[o_{ij}]$ and take $p=n-1$. Since $f(t)=t+\dfrac{1}{t}$ is an increasing function on $[1,\ \infty)$. Now for any edge v_iv_j in $E(G)$, it follows that

$$o_{ij} < \frac{\Delta}{\delta} + \frac{\delta}{\Delta} \leq p + \frac{1}{p}. \tag{3.3}$$

Let θ_1 denote the magnitude of the largest eigenvalue of the matrix $\left(p+\dfrac{1}{p}\right)A(G)$. Also, note that $ONM(G) < \left(p+\dfrac{1}{p}\right)A(G)$, hence by Lemmas 2.3, 2.5 and equation 3.3 we have the inequality,

$$\rho_1 < \theta_1 \leq \left(p+\frac{1}{p}\right)\sqrt{2m-n+1} = \left(n-1+\frac{1}{n-1}\right)\sqrt{2m-n+1}.$$

Hence the theorem. □

Proposition 3.8. *Let G be a graph with open neighborhood matrix $ONM(G)$. Then number of zero rows or columns corresponds to the number of pendant vertices of G.*

Proof. Suppose the i^{th} row(or column) of $ONM(G)$ be zero, then the vertex v_i will either an isolated vertex or $N(v_i) \cap N(w) = \phi$ for each w in $V(G)$. Since G is connected for any edge $v_i w$ in G, $|N(v_i) \cap N(w)| = 0$, implies v_i must be a pendant vertex. Thus the total number of zero rows(or columns) is same as the number of pendant vertices of G. □

From hereafter we use $\mathcal{C}(G,\ n,\ m)$ to denote the collection of all connected graphs of order n and size m with at least one cycle and at least one pendant vertex.

Theorem 3.9. *Let G be a graph in $\mathcal{C}(G,\ n,\ m)$ with open neighborhood matrix $ONM(G)$. Then the multiplicity of the open neighborhood eigenvalue zero is same as the number of pendant vertices of G.*

Proof. Suppose G be a graph with open neighborhood matrix $ONM(G)$. Let k be the multiplicity of the open neighborhood eigenvalue zero, we have to prove that G contains k pendant vertices. Since G contains at least one cycle and one pendant vertex, clearly G is not isomorphic to a tree or a complete bipartiate graph. Let X denotes the collection of all pendant vertices in G and Y be $V(G) - X$. Let t be a positive integer with $k < t < n$, and if possible suppose that $|X| = t$. Since G is connected and using Proposition 3.8, the rank of the matrix $ONM(G)$ is $n - t$. Also, since $ONM(G)$ is a real symmetric matrix, the characterestic polynomial can be expressed as

$$\mathbf{P}(\rho) = |ONM(G) - \rho I| = \rho^t g(\rho), \tag{3.4}$$

where $g(\rho)$ is a polynomial with degree $n - t$ such that $g(0) \neq 0$. Thus we have the multiplicity of open neighborhood eigen value zero becomes t, which is a contradiction to our assumption. In the same way if we set $|X| = t$, were $t < k < n$, again it results a contradiction. Hence $|X| = k$ as required. □

Note that an edge $e = uv$ in a connected graph G is called a bridge if and only if e is a cut edge, that is the graph $G - e$ is disconnected. The following corollaries are immidiate consequences of the Theorem 3.9 and Proposition 3.8.

Corollary 3.10. *Let G be a graph with open neighborhood eigenvalue zero having multiplicity t. Then G has at least t components, each of which is isomorphic to K_1.*

Corollary 3.11. *Let G be a graph of order n and let z denotes the number of zero rows of $ONM(G)$. Then all clique's in G have a cardinality upto $n - z$.*

Remark 3.12. Let G be a graph of order n and $\mathcal{D}_j$ represent the collection of all open neighborhood matrices of size $j \times j$ constructed by selecting any set of j vertices from the vertex set $V(G)$. Then we have the following;

(1) The number of triangles in G is equal to the total number of $D \in \mathcal{D}_3$ such that $|D| = \frac{1}{32}$.
(2) The number of four-cycles in G corresponds to the total number of non-zero $D \in \mathcal{D}_4$ such that any two rows or columns of D are identical.

Theorem 3.13. *For any integer $n \geq 2$ there is a graph G with more than n vertices such that $ONE(G) = n$.*

Proof. Suppose $n \geq 2$ is the given integer. Now observe from Example 3.2 that, $ONM(K_n) = \frac{n-2}{2(n-1)} A(K_n)$. Hence we have

$$ONE(K_n) = \frac{n-2}{2(n-1)} \mathcal{E}(K_n) = n - 2. \tag{3.5}$$

If we set $G = K_{n+2}$, the complete graph on $n+2$ vertices. Then from equation 3.5, we can conclude that G is a graph with more than n vertices, and $ONE(G) = n$ as desired. Hence the theorem. □

Theorem 3.14. *Let G be a graph of order n and size m. Then*

$$\rho_1 \leq \sqrt{2m\left(1 - \frac{1}{n}\right)}$$

Proof. Using Cauchy-Schwartz inequality on $\rho_2 \geq \rho_3 \geq \cdots \geq \rho_n$, we have

$$\left(\sum_{j=2}^{n} |\rho_j|\right)^2 \leq (n-1) \sum_{j=2}^{n} \rho_j^2. \tag{3.6}$$

Using Proposition 3.5, we can simplify the above inequality to obtain

$$\rho_1^2 \leq (n-1)(2m - \rho_1^2).$$

This can be further simplified to

$$\rho_1 \leq \sqrt{2m\left(1 - \frac{1}{n}\right)}.$$

□

Theorem 3.15. *Let G be a graph of order n and size m. Then*

$$0 \leq ONE(G) \leq \sqrt{8m\left(1 - \frac{1}{n}\right)}.$$

Proof. Considering equation 3.6, we have the following inequality

$$0 \leq \sum_{j=2}^{n} |\rho_j| \leq \sqrt{(n-1)(2m - \rho_1^2)},$$

which again reduces to

$$0 \leq ONE(G) \leq \rho_1 + \sqrt{(n-1)(2m - \rho_1^2)}. \tag{3.7}$$

Now consider the function $f(t) = t + \sqrt{(n-1)(2m-t^2)}$, which is a decearsing function on $t \in \left(\sqrt{\frac{2m}{n}},\ \sqrt{2m}\right)$ and $f(t)$ is maximum when $t = \sqrt{\frac{2m(n-1)}{n}}$. Therefore, from equation 3.7, we have the inequality

$$0 \leq ONE(G) \leq \sqrt{8m\left(1-\frac{1}{n}\right)}.$$

□

Theorem 3.16. *Let G be a graph of order n. Then* $ONE(G) \geq 2\sqrt{\sum_{v_i \sim v_j}\left(\frac{|N(v_i)\cap N(v_j)|}{d_i+d_j}\right)^2}$.

Proof. Let G be a graph with open neighborhood matrix $ONM(G)$. Then by Proposition 3.5, we have

$$\sum_{i=1}^{n}\rho_i^2 = -2\sum_{1\leq i<j\leq n}\rho_i\rho_j \tag{3.8}$$

Again we have

$$\begin{aligned}(ONE(G))^2 = \left(\sum_{i=1}^{n}|\rho_i|\right)^2 &= \sum_{i=1}^{n}|\rho_i|^2 + 2\sum_{1\leq i<j\leq n}|\rho_i\rho_j| \\ &\geq \sum_{i=1}^{n}\rho_i^2 + 2\left|\sum_{1\leq i<j\leq n}\rho_i\rho_j\right| \qquad (3.9)\\ &\geq 2\sum_{i=1}^{n}\rho_i^2 = 4\sum_{i\sim j}\left(\frac{|N(v_i)\cap N(v_j)|}{d_i+d_j}\right)^2.\end{aligned}$$

Taking the square root on both sides we obtain the required inequality. Hence the proof. □

Corollary 3.17. *Let G be a graph in $\mathcal{C}(G,\ n,\ m)$ with open neighbourhood matrix $ONM(G)$. Then $ONE(G) \geq 2\sqrt{\chi_{-4}(G)}$, where χ_n is the generalized sum connectivity index of G.*

Proof. Suppose X and X' represents two subsets of the vertex set $V(G)$ as described in Theorem 3.9. Consider an edge e that connects any two vertices v_i and v_j in G. If both the ends of e are in X or if one end is in X and other is in X', in either cases it follows that $|N(v_i)\cap N(v_j)| = 0$. Therefore, we must consider those edges $e = v_iv_j$ in G with $|N(v_i)\cap N(v_j)| \neq 0$, for those edges we have the inequality

$$|N(v_i)\cap N(v_j)| \geq \frac{1}{d_i+d_j}. \tag{3.10}$$

Using the above equation the following holds,

$$\sum_{v_i\sim v_j}\left(\frac{|N(v_i)\cap N(v_j)|}{d_i+d_j}\right)^2 \geq \sum_{v_i\sim v_j}\frac{1}{(d_i+d_j)^4} = \chi_{-4}(G). \tag{3.11}$$

Now the corollary follows from equation 3.11 and Theorem 3.16. □

Corollary 3.18. *Let G be a graph in $\mathcal{C}(G,\ n,\ m)$ with open neighbourhood matrix $ONM(G)$. Then $ONE(G) \geq \frac{1}{\Delta^2}\sqrt{\frac{m}{2}}$, where Δ represents the maximum degree of G.*

Proof. Starting with the same proof of Corollary 3.17, and taking equation 3.10, it can again simplified to

$$|N(v_i) \cap N(v_j)| \geq \frac{1}{2\Delta}.$$

Therefore, equation 3.11 can be simplified to

$$\sum_{v_i \sim v_j} \left(\frac{|N(v_i) \cap N(v_j)|}{d_i + d_j} \right)^2 \geq \sum_{v_i \sim v_j} \frac{1}{(2\Delta)^4} = \frac{m}{(2\Delta)^4}$$

Now by Theorem 3.16, we have

$$(ONE(G))^2 \geq \frac{m}{2(\Delta)^4}. \tag{3.12}$$

Taking the square root on both sides of equation 3.12, we obtain

$$ONE(G) \geq \frac{1}{\Delta^2}\sqrt{\frac{m}{2}}.$$

Thus the proof. □

Theorem 3.19. *Let G be a graph of order n and size m and let $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ be the eigenvalues of open neighborhood matrix $ONM(G)$. Then*

$$ONE(G) \leq \sqrt{2mn + \alpha(n)\left(|\rho_1| - |\rho_n|\right)^2},$$

where $\alpha(n) = n\lfloor \frac{n}{2} \rfloor \left(1 - \frac{1}{n}\lfloor \frac{n}{2} \rfloor\right)$.

Proof. Let G be a graph with open neighborhood matrix $ONM(G)$ with eigenvalues $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$. Letting $x_i = y_i = |\rho_i|,\ a = b = |\rho_n|$ and $A = B = |\rho_1|$ in Lemma 2.7, we get

$$\left| n \sum_{j=1}^{n} |\rho_j|^2 - \left(\sum_{j=1}^{n} |\rho_j| \right)^2 \right| \leq \alpha(n)\left(|\rho_1| - |\rho_n|\right)^2. \tag{3.13}$$

A rearrangement of above equation along with Proposition 3.5, we obtain

$$\begin{aligned} (ONE(G))^2 &\leq \alpha(n)\left(|\rho_1| - |\rho_n|\right)^2 + n \sum_{i=1}^{n} |\rho_i|^2 \\ &\leq 2mn + \alpha(n)\left(|\rho_1| - |\rho_n|\right)^2. \end{aligned} \tag{3.14}$$

Taking the square root on both sides we have the inequality

$$ONE(G) \leq \sqrt{2mn + \alpha(n)\left(|\rho_1| - |\rho_n|\right)^2},$$

where $\alpha(n) = n\lfloor \frac{n}{2} \rfloor \left(1 - \frac{1}{n}\lfloor \frac{n}{2} \rfloor\right)$. Hence the theorem. □

Theorem 3.20. *For any $\epsilon > 0$, there exist infinitely many n such that for each value of n there exists a k-regular graph G of order n, with $k+1 < n$ and*

$$\frac{ONE(G)}{k+\sqrt{k(n-1)(n-k)}} < \epsilon.$$

Proof. In order to prove the theorem, it is necessary to show that there exists an infinite sequence of graphs that satisfies the requirements. To do this we take Paley graphs. Consider the prime p such that $p \equiv 1(mod\ 4)$ and $p \geq 11$. The Paley graph G_p with p vertices has its vertex set consisting of elements from the finite field $GF(p)$. Two vertices in G_p are adjacent if and only if their difference is a non-zero square in $GF(p)$. Also, note that Paley graphs are $k = \frac{p-1}{2}$ regular and any adjacent vertices of G_p has exactly two common neighboures. Thus for any v_iv_j in $E(G_p)$ we have $|N(v_i) \cap N(v_j)| = 2$. Hence the open neighborhood matrix of G_p is given by,

$$ONM(G_p) = [o_{ij}] = \begin{cases} \frac{2}{(p-1)} & i \neq j,\ v_iv_j\ in\ E(G_p), \\ 0 & otherwise, \end{cases}$$

or equivalently, $ONM(G_p) = \frac{2}{p-1}A(G_p)$, where $A(G_p)$ is the adjacency matrix of G_p. Since $A(G_p)$ has eigenvalues $\frac{p-1}{2}$ with multiplicity one and $\frac{-1\pm\sqrt{p}}{2}$ with both having multiplicity $\frac{p-1}{2}$. Hence the open neighborhood eigenvalues are one with multiplicity one and $\frac{1\pm\sqrt{p}}{(p-1)}$ with both having multiplicity $\frac{p-1}{2}$. Hence the open neighborhood energy of G_p is,

$$\begin{aligned} ONE(G_p) &= 1 + \left(\frac{p-1}{2}\right)\frac{1+\sqrt{p}}{p-1} + \left(\frac{p-1}{2}\right)\frac{\sqrt{p}-1}{p-1} \\ &= 1+\sqrt{p} < 2p \end{aligned} \tag{3.15}$$

Now by letting $k = \frac{p-1}{2}$ along with equation 3.15, we arrive

$$\frac{ONE(G_p)}{k+\sqrt{k(n-1)(n-k)}} = \frac{1+\sqrt{p}}{\frac{p-1}{2}(1+\sqrt{p+1})}$$

$$< \frac{2p}{p^{\frac{3}{2}}} = \frac{2}{\sqrt{p}} \to 0,\ whenever\ p \to \infty.$$

Hence the theorem. □

Theorem 3.21. *Let G be a graph of order n and size m and let $\mathcal{E}(G)$ be the adjacency energy of G. Then*

$$\mathcal{E}(G) < \frac{2mn}{ONE(G)}. \tag{3.16}$$

Proof. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\rho_1 \geq \rho_2 \cdots \geq \rho_n$ be the eigenvalues of the adjacency matrix $A(G)$ and open neighborhood matrix $ONM(G)$, respectively. Now using Lemma 2.3, we have $ONM(G) \leq A(G)$ and thus $\rho_1 \leq \lambda_1$. Also, note that $\sum_{j=1}^{n} |\lambda_j|^2 = 2m$. Now consider,

$$\begin{aligned} \Gamma &= \sum_{i=1}^{n}\sum_{j=1}^{n} (|\rho_i| - |\lambda_j|)^2 \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \left(|\rho_i|^2 + |\lambda_j|^2 - 2|\rho_i||\lambda_j|\right) \\ &= n\sum_{i=1}^{n} |\rho_i|^2 + n\sum_{j=1}^{n} |\lambda_j|^2 - 2\sum_{i=1}^{n} |\rho_i| \sum_{j=1}^{n} |\lambda_j| \\ &= n\sum_{i=1}^{n} |\rho_i|^2 + 2mn - 2ONE(G)\mathcal{E}(G) \end{aligned} \tag{3.17}$$

Using Proposition 3.5 and observing the fact that $\Gamma > 0$, equation 3.17 again simplified to

$$2ONE(G)\mathcal{E}(G) < 4mn.$$

Hence we get

$$\mathcal{E}(G) < \frac{2mn}{ONE(G)}.$$

□

4. Conclusion

In this paper, we discussed the open neighborhood matrix $ONM(G)$ and open neighborhood energy $ONE(G)$ of a graph G. We deduced that for every integer n, there exists a graph G with $|G| > n$ and $ONE(G) = n$, as well as some bounds for the spectral radius ρ_1 of the matrix $ONM(G)$. We then obtained the relationship between the multiplicity of the zero eigenvalue (if it exists) with the connectivity of the graph G. Also, we concluded that for any graph G of order n and size m the inequality $\mathcal{E}(G) < \frac{2mn}{ONE(G)}$ holds, where $\mathcal{E}(G)$ is the adjacency energy of G.

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