

CONVERGENCE RESULTS FOR NEARLY ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN HYPERBOLIC SPACES

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ABSTRACT. The aim of this article is to study modified three-step iterations for nearly asymptotically quasi-nonexpansive mappings and establish strong and Δ -convergence theorems for above mentioned iteration scheme and mappings in the setting of hyperbolic spaces. Our results extend and generalize the corresponding results of [2, 3, 6, 16, 26, 30, 34, 35, 39, 40, 47] and many others.

1. INTRODUCTION

The class of asymptotically nonexpansive mapping, introduced by Goebel and Kirk [9] in 1972, is an important generalization of the class of nonexpansive mapping. They proved that if C is a nonempty bounded, closed and convex subset of a uniformly convex Banach space, then every asymptotically nonexpansive self mapping of C has a fixed point.

There are number of papers dealing with the approximation of fixed points of asymptotically nonexpansive and asymptotically quasi-nonexpansive mappings in uniformly convex Banach spaces using modified Mann, Ishikawa and three-step iteration processes and have been studied by many authors (see, e.g., [3, 22, 23, 33, 37, 38, 42, 45, 46]).

The concept of Δ -convergence in a general metric space was introduced by Lim [21]. In 2008, Kirk and Panyanak [19] used the notion of Δ -convergence introduced by Lim [21] to prove in the $CAT(0)$ space and analogous of some Banach space results which involve weak convergence. Further, Dhompongsa and Panyanak [6] obtained Δ -convergence theorems for the Picard, Mann and Ishikawa iterations in a $CAT(0)$ space. Since then, the existence problem and the Δ -convergence problem of iterative sequences to a fixed point for nonexpansive mapping, asymptotically nonexpansive mapping, nearly asymptotically nonexpansive mapping, asymptotically quasi-nonexpansive mapping in the intermediate sense, total asymptotically nonexpansive mapping and asymptotically quasi-nonexpansive mapping through Picard, Mann [24], Ishikawa [13], modified

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Agarwal et al. [2] have been rapidly developed in the framework of CAT(0) space and many papers have appeared in this direction (see, e.g., [1, 5, 6, 16, 25, 34]).

The aim of this paper is to establish convergence results via modified three-step iteration scheme which contains modified S -iteration process for a wider class of asymptotically nonexpansive mappings in uniformly convex hyperbolic spaces which include both uniformly convex Banach spaces and CAT(0) spaces. Our results extend and improve the corresponding results of Abbas et al. [1], Dhompongsa and Panyanak [6], Khan and Abbas [16] and many other results in this direction.

2. PRELIMINARIES

Let $F(T) = \{x \in K : Tx = x\}$ denotes the set of fixed points of the mapping T . We begin with the following definitions.

Definition 2.1. Let (X, d) be a metric space and K be its nonempty subset. Then $T: K \rightarrow K$ said to be

- (1) nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$;
- (2) asymptotically nonexpansive if there exists a sequence $\{u_n\} \subset [0, \infty)$ with $\lim_{n \rightarrow \infty} u_n = 0$ such that $d(T^n x, T^n y) \leq (1 + u_n)d(x, y)$ for all $x, y \in K$ and $n \geq 1$;
- (3) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{u_n\} \subset [0, \infty)$ with $\lim_{n \rightarrow \infty} u_n = 0$ such that $d(T^n x, p) \leq (1 + u_n)d(x, p)$ for all $x \in K, p \in F(T)$ and $n \geq 1$;
- (4) uniformly L -Lipschitzian if there exists a constant $L > 0$ such that $d(T^n x, T^n y) \leq L d(x, y)$ for all $x, y \in K$ and $n \geq 1$;
- (5) quasi L -Lipschitzian if $F(T) \neq \emptyset$ and there exists a constant $L > 0$ such that $d(Tx, y) \leq L d(x, y)$ for all $x \in K$ and $y \in F(T)$;
- (6) semi-compact if for a sequence $\{x_n\}$ in K with $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow p \in K$ as $k \rightarrow \infty$.
- (7) a sequence $\{x_n\}$ in K is called approximate fixed point sequence for T (AFPS, in short) if $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$.

The class of nearly Lipschitzian mappings is an important generalization of the class of Lipschitzian mappings and was introduced by Sahu [36].

Definition 2.2. Let K be a nonempty subset of a metric space (X, d) and fix a sequence $\{a_n\} \subset [0, \infty)$ with $\lim_{n \rightarrow \infty} a_n = 0$. A mapping $T: K \rightarrow K$ said to be nearly Lipschitzian with respect to $\{a_n\}$ if for all $n \geq 1$, there exists a constant $k_n \geq 0$ such that $d(T^n x, T^n y) \leq k_n[d(x, y) + a_n]$ for all $x, y \in K$.

The infimum of the constants k_n for which the above inequality holds is denoted by $\eta(T^n)$ and is called nearly Lipschitz constant of T^n .

A nearly Lipschitzian mapping T with sequence $\{a_n, \eta(T^n)\}$ is said to be:

- (i) nearly nonexpansive if $\eta(T^n) = 1$ for all $n \geq 1$;
- (ii) nearly asymptotically nonexpansive if $\eta(T^n) \geq 1$ for all $n \geq 1$ and $\lim_{n \rightarrow \infty} \eta(T^n) = 1$;
- (iii) nearly uniformly k -Lipschitzian if $\eta(T^n) \leq k$ for all $n \geq 1$.

Definition 2.3. ([15]) Let K be a nonempty subset of a metric space (X, d) and fix a sequence $\{a_n\} \subset [0, \infty)$ with $\lim_{n \rightarrow \infty} a_n = 0$. A mapping $T: K \rightarrow K$ said to be nearly asymptotically quasi-nonexpansive with respect to $\{a_n\}$ if $F(T) \neq \emptyset$ and there exists a sequence $\{u_n\}$ in $[0, \infty)$ with $u_n \rightarrow 0$ as $n \rightarrow \infty$ such that $d(T^n x, p) \leq (1 + u_n)d(x, p) + a_n$ for all $x \in K$, $p \in F(T)$ and $n \geq 1$.

In fact if T is a nearly asymptotically nonexpansive mapping and $F(T) \neq \emptyset$, then T is nearly asymptotically quasi-nonexpansive. The following is an example of a nearly asymptotically quasi-nonexpansive mapping with $F(T) \neq \emptyset$.

Example 2.4. ([1]) Let $X = \mathbb{R}$, $K = (-\infty, 3]$ and $T: K \rightarrow K$ be a mapping

$$T(x) = \begin{cases} \frac{1}{2}x, & \text{if } x \in (\infty, 2], \\ x - 1, & \text{if } x \in (2, 3]. \end{cases}$$

Clearly, $F(T) = \{0\}$ and also T is asymptotically quasi-nonexpansive mapping with sequences $\{u_n\} = \{1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots\}$ and $\{a_n\} = \{1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots\}$.

A nearly asymptotically quasi-nonexpansive is called a nearly quasi-nonexpansive (asymptotically quasi-nonexpansive mapping) if $u_n = 0$ for all $n \geq 1$ ($a_n = 0$ for all $n \geq 1$).

The following example shows that T is a nearly quasi-nonexpansive mapping but not Lipschitzian and quasi-nonexpansive.

Example 2.5. Let $X = \mathbb{R}$, $K = [-\frac{1}{\pi}, \frac{1}{\pi}]$ and $k \in (0, 1)$. For each $x \in K$, define

$$T(x) = \begin{cases} kx \sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Since $T: K \rightarrow K$ is continuous, it easily follows that it is uniformly continuous. Note that $F(T) = \{0\}$ and $\{T^n x\} \rightarrow 0$ uniformly on K as $n \rightarrow \infty$ but T is not Lipschitzian. In fact, suppose that there exists $k > 0$ such that $|Tx - Ty| \leq k|x - y|$ for all $x, y \in K$. If we take $x = \frac{2}{5\pi}$ and $y = \frac{2}{3\pi}$, then

$$|Tx - Ty| = \left| k \cdot \frac{2}{5\pi} \cdot \sin\left(\frac{5\pi}{2}\right) - k \cdot \frac{2}{3\pi} \cdot \sin\left(\frac{3\pi}{2}\right) \right| = \frac{16k}{15\pi},$$

whereas

$$k|x - y| = k \left| \frac{2}{5\pi} - \frac{2}{3\pi} \right| = \frac{4k}{15\pi},$$

and hence it is not Lipschitzian mapping.

Throughout this paper, we work in the setting of hyperbolic space introduced by Kohlenbach [20]. It is worth noting that they are different from Gromov hyperbolic space [4] or from other notions of hyperbolic space that can be found in the literature (see for example [10, 18, 32]).

A hyperbolic space [20] is a triple (X, d, W) , where (X, d) is a metric space and $W: X^2 \times [0, 1] \rightarrow X$ is such that

$$(W1) \quad d(u, W(x, y, \alpha)) \leq \alpha d(u, x) + (1 - \alpha)d(u, y)$$

$$(W2) \quad d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$$

$$(W3) \quad W(x, y, \alpha) = W(x, y, (1 - \alpha))$$

$$(W4) \quad d(W(x, z, \alpha), W(y, w, \beta)) \leq \alpha d(x, y) + (1 - \alpha) d(z, w)$$

for all $x, y, z, w \in X$ and $\alpha, \beta \in [0, 1]$.

The class of hyperbolic spaces in the sense of Kohlenbach [20] contains all normed linear spaces and convex subsets thereof as well as Hadamard manifolds and CAT(0) spaces in the sense of Gromov [11]. An important example of a hyperbolic space is the open unit ball B_H in a real Hilbert space H is as follows.

Let B_H be the open unit ball in H . Then

$$k_{B_H}(x, y) = \arg \tanh(1 - \sigma(x, y))^{1/2},$$

where

$$\sigma(x, y) = \frac{(1 - \|x\|^2)(1 - \|y\|^2)}{|1 - \langle x, y \rangle|^2}$$

for all $x, y \in B_H$, defines a metric on B_H (also known as Kobayashi distance).

The convexity mapping W was first introduced by Takahashi in [44], where a triple (X, d, W) satisfying (W1) is called a convex metric space (see, more details [44]). If (X, d, W) satisfies (W1)-(W3), then we obtain the notion of space of hyperbolic type in the sense of Goebel and Kirk [10]. (W4) was already considered by Itoh [14] under the name “condition III” and it is used by Reich and Shafrir [32] and Kirk [18] to define their notions of hyperbolic space. Hyperbolic spaces are a natural generalization of both uniformly convex normed space and CAT(0) spaces.

A hyperbolic space (X, d, W) is said to be uniformly convex [43] if for all $u, x, y \in X$, $r > 0$ and $\varepsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that $d(W(x, y, \frac{1}{2}), u) \leq (1 - \delta)r$ whenever $d(x, u) \leq r$, $d(y, u) \leq r$ and $d(x, y) \geq \varepsilon r$.

A mapping $\eta: (0, \infty) \times (0, 2] \rightarrow (0, 1]$ which provides such a $\delta = \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$, is known as modulus of uniform convexity. We call η monotone if it decreases with r (for a fixed ε).

Let K be a nonempty subset of hyperbolic space X . Let $\{x_n\}$ be a bounded sequence in a hyperbolic space X . For $x \in X$, define a continuous functional $r(\cdot, \{x_n\}): X \rightarrow [0, \infty)$ by $r(x, \{x_n\}) = \limsup_{n \rightarrow \infty} d(x, x_n)$. The asymptotic radius $\rho = r(\{x_n\})$ of $\{x_n\}$ is given by $\rho = \inf\{r(x, \{x_n\}) : x \in X\}$. The asymptotic center $A_K(\{x_n\})$ of a bounded sequence $\{x_n\}$ with respect to a subset K of X is defined as follows:

$$A_K(\{x_n\}) = \{x \in X : r(x, \{x_n\}) \leq r(y, \{x_n\})\} \quad \text{for any } y \in K.$$

The set of all asymptotic center of $\{x_n\}$ is denoted by $A(\{x_n\})$.

It has been shown in [43] that bounded sequences have unique asymptotic center with respect to closed convex subsets in a complete and uniformly hyperbolic space with monotone modulus of uniform convexity.

A sequence $\{x_n\}$ in X is said to Δ -converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$ [19]. In this case, we write $\Delta\text{-}\lim_n x_n = x$ and call x is the Δ -limit of $\{x_n\}$.

Recall that Δ -convergence coincides with weak convergence in Banach space with Opial's property [29].

In the sequel we need the following lemmas.

Lemma 2.6. ([17]) *Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η . Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n \rightarrow \infty} d(x_n, x) \leq r$, $\limsup_{n \rightarrow \infty} d(y_n, x) \leq r$ and $\lim_{n \rightarrow \infty} d(W(x_n, y_n, \alpha_n), x) = r$ for some $r \geq 0$, then $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$.*

Lemma 2.7. ([17]) *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space X and $\{x_n\}$ a bounded sequence in K such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in K such that $\lim_{m \rightarrow \infty} r(y_m, \{x_n\}) = \rho$, then $\lim_{m \rightarrow \infty} y_m = y$.*

Lemma 2.8. ([45]) *Let $\{p_n\}_{n=1}^\infty$, $\{q_n\}_{n=1}^\infty$ and $\{r_n\}_{n=1}^\infty$ be sequences of nonnegative numbers satisfying the inequality*

$$p_{n+1} \leq (1 + q_n)p_n + r_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^\infty q_n < \infty$ and $\sum_{n=1}^\infty r_n < \infty$, then $\lim_{n \rightarrow \infty} p_n$ exists.

First, we define the modified three-step iteration scheme in hyperbolic space as follows.

Let K be a nonempty closed convex subset of a hyperbolic space X and $T: K \rightarrow K$ be a nearly asymptotically quasi-nonexpansive mapping. Then, for an arbitrary chosen $x_1 \in K$, we construct the sequence $\{x_n\}$ in K such that

$$\begin{cases} x_{n+1} = W(T^n x_n, T^n y_n, \alpha_n), \\ y_n = W(x_n, T^n z_n, \beta_n), \\ z_n = W(x_n, T^n x_n, \gamma_n), \quad n \geq 1, \end{cases} \quad (2.1)$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are appropriate sequences in $(0,1)$ is called modified three-step iteration scheme. Iteration scheme (2.1) is independent of Noor iteration, modified Ishikawa iteration and modified Mann iteration schemes.

If $\gamma_n = 0$ for all $n \geq 1$, then iteration scheme (2.1) reduces to the following.

$$\begin{cases} x_{n+1} = W(T^n x_n, T^n y_n, \alpha_n), \\ y_n = W(x_n, T^n x_n, \beta_n), \quad n \geq 1, \end{cases} \quad (2.2)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate sequences in $(0,1)$ is called modified S -iteration scheme in hyperbolic space.

The three-step iterative approximation problems were studied extensively by Noor [27, 28], Glowinsky and Le Tallec [8], and Haubruge et al [12]. It has been shown [8] that three-step iterative scheme gives better numerical results than the two step and one step approximate iterations. Thus we conclude that three step scheme plays an important and significant role in solving various problems, which arise in pure and applied sciences.

3. MAIN RESULTS

Lemma 3.1. *Let K be a nonempty convex subset of a hyperbolic space X and let $T: K \rightarrow K$ be a nearly asymptotically quasi-nonexpansive mapping with sequence $\{(a_n, u_n)\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} u_n < \infty$. Let $\{x_n\}$ be a sequence in K defined by (2.1). Then $\lim_{n \rightarrow \infty} d(x_n, p)$ and $\lim_{n \rightarrow \infty} d(x_n, F(T))$ both exist for each $p \in F(T)$.*

Proof. Let $p \in F(T)$. From (2.1) and by definition 2.3, we have

$$\begin{aligned} d(z_n, p) &= d(W(x_n, T^n x_n, \gamma_n), p) \\ &\leq (1 - \gamma_n)d(x_n, p) + \gamma_n d(T^n x_n, p) \\ &\leq (1 - \gamma_n)d(x_n, p) + \gamma_n[(1 + u_n)d(x_n, p) + a_n] \\ &= (1 - \gamma_n)d(x_n, p) + \gamma_n(1 + u_n)d(x_n, p) + \gamma_n a_n \\ &\leq (1 + \gamma_n u_n)d(x_n, p) + a_n. \end{aligned} \quad (3.1)$$

Again using (2.1), (3.1) and definition 2.3, we have

$$\begin{aligned}
d(y_n, p) &= d(W(x_n, T^n z_n, \beta_n), p) \\
&\leq (1 - \beta_n)d(x_n, p) + \beta_n d(T^n z_n, p) \\
&\leq (1 - \beta_n)d(x_n, p) + \beta_n [(1 + u_n)d(z_n, p) + a_n] \\
&= (1 - \beta_n)d(x_n, p) + \beta_n (1 + u_n)d(z_n, p) + \beta_n a_n \\
&\leq (1 - \beta_n)d(x_n, p) + \beta_n (1 + u_n)[(1 + \gamma_n u_n)d(x_n, p) \\
&\quad + a_n] + \beta_n a_n \\
&= (1 - \beta_n)d(x_n, p) + \beta_n (1 + u_n)(1 + \gamma_n u_n)d(x_n, p) \\
&\quad + \beta_n (1 + u_n)a_n + \beta_n a_n \\
&\leq [1 + \beta_n u_n(1 + \gamma_n + \gamma_n u_n)]d(x_n, p) + (2 + u_n)a_n. \tag{3.2}
\end{aligned}$$

Finally, using (2.1), (3.2) and definition 2.3, we get

$$\begin{aligned}
d(x_{n+1}, p) &= d(W(T^n x_n, T^n y_n, \alpha_n), p) \\
&\leq (1 - \alpha_n)d(T^n x_n, p) + \alpha_n d(T^n y_n, p) \\
&\leq (1 - \alpha_n)[(1 + u_n)d(x_n, p) + a_n] + \alpha_n [(1 + u_n)d(y_n, p) + a_n] \\
&= (1 - \alpha_n)(1 + u_n)d(x_n, p) + \alpha_n (1 + u_n)d(y_n, p) + a_n \\
&\leq (1 - \alpha_n)(1 + u_n)d(x_n, p) + \alpha_n (1 + u_n) \times \\
&\quad [1 + \beta_n u_n(1 + \gamma_n + \gamma_n u_n)]d(x_n, p) + (2 + u_n)a_n + a_n \\
&\leq [1 + u_n\{1 + \alpha_n \beta_n (1 + u_n + \gamma_n + 2\gamma_n u_n + \gamma_n u_n^2)\}]d(x_n, p) \\
&\quad + (3 + 3u_n + u_n^2)a_n \\
&\leq [1 + Qu_n]d(x_n, p) + Q'a_n, \tag{3.3}
\end{aligned}$$

for some $Q, Q' > 0$.

Taking infimum over all $p \in F(T)$, we have

$$d(x_{n+1}, F(T)) \leq [1 + Qu_n]d(x_n, F(T)) + Q'a_n, \tag{3.4}$$

Since by hypothesis $\sum_{n=1}^{\infty} u_n < \infty$ and $\sum_{n=1}^{\infty} a_n < \infty$, it follows from Lemma 2.8, (3.3) and (3.4) that $\lim_{n \rightarrow \infty} d(x_n, p)$ and $\lim_{n \rightarrow \infty} d(x_n, F(T))$ both exist. This completes the proof. $\square$

Lemma 3.2. *Let K be a nonempty convex subset of a hyperbolic space X and let $T: K \rightarrow K$ be a quasi- L -Lipschitzian mapping. If $\{x_n\}$ be a sequence in K and $\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0$ and $\lim_{n \rightarrow \infty} x_n = z \in K$. Then z is a fixed point of T .*

Proof. Since $\lim_{n \rightarrow \infty} x_n = z$, then for each $\varepsilon > 0$, there exists a natural number n_1 such that

$$d(x_n, z) < \frac{\varepsilon}{2(1+L)} \text{ for all } n \geq n_1.$$

By $\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0$, there exists a natural number $n_2 \geq n_1$ such that

$$d(x_{n+1}, F(T)) < \frac{\varepsilon}{3(1+3L)} \text{ for all } n \geq n_2$$

and hence

$$d(x_{n_2}, F(T)) < \frac{\varepsilon}{3(1+3L)}.$$

It follows that there exists a point $u \in F(T)$ such that

$$d(x_{n_2}, u) < \frac{\varepsilon}{2(1+3L)}.$$

Thus, we have

$$\begin{aligned} d(Tz, z) &\leq d(Tz, u) + d(u, Tx_{n_2}) + d(Tx_{n_2}, u) + d(u, x_{n_2}) + d(x_{n_2}, z) \\ &\leq Ld(z, u) + (1+2L)d(u, x_{n_2}) + d(x_{n_2}, z) \\ &\leq L[d(z, x_{n_2}) + d(x_{n_2}, u)] + (1+2L)d(u, x_{n_2}) + d(x_{n_2}, z) \\ &= (1+L)d(z, x_{n_2}) + (1+3L)d(u, x_{n_2}) \\ &< (1+L) \cdot \frac{\varepsilon}{2(1+L)} + (1+3L) \cdot \frac{\varepsilon}{2(1+3L)} = \varepsilon. \end{aligned}$$

Therefore z is a fixed point of T , since ε is arbitrary. This completes the proof. $\square$

Lemma 3.3. *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $T: K \rightarrow K$ be a nearly asymptotically quasi-nonexpansive mapping with sequence $\{(a_n, u_n)\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} u_n < \infty$. Let $\{x_n\}$ be a sequence in K defined by (2.1). Assume that $F(T) \neq \emptyset$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequence in $[l, m]$ for some $l, m \in (0, 1)$. If $d(x, T^n y) \leq d(T^n x, T^n y)$ for all $x, y \in K$, then $\lim_{n \rightarrow \infty} d(x_n, T^n x_n) = 0$, $\lim_{n \rightarrow \infty} d(x_n, T^n y_n) = 0$ and $\lim_{n \rightarrow \infty} d(x_n, T^n z_n) = 0$.*

Proof. From Lemma 3.1, we obtain $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for each $p \in F(T)$. Suppose that $\lim_{n \rightarrow \infty} d(x_n, p) = r \geq 0$. Since

$$d(T^n x_n, p) \leq (1 + u_n)d(x_n, p) + a_n \quad \text{for all } n \geq 1,$$

we have

$$\limsup_{n \rightarrow \infty} d(T^n x_n, p) \leq r.$$

Also (3.2) yields that

$$d(y_n, p) \leq r. \tag{3.5}$$

Hence

$$\limsup_{n \rightarrow \infty} d(T^n y_n, p) \leq \limsup_{n \rightarrow \infty} [(1 + u_n)d(y_n, p) + a_n] \leq r. \tag{3.6}$$

Again (3.1) yields that

$$d(z_n, p) \leq r. \tag{3.7}$$

Hence

$$\limsup_{n \rightarrow \infty} d(T^n z_n, p) \leq \limsup_{n \rightarrow \infty} [(1 + u_n)d(z_n, p) + a_n] \leq r. \tag{3.8}$$

Since

$$r = \lim_{n \rightarrow \infty} d(x_{n+1}, p) = \lim_{n \rightarrow \infty} d(W(T^n x_n, T^n y_n, \alpha_n), p),$$

it follows from Lemma 2.6 that

$$\lim_{n \rightarrow \infty} d(T^n x_n, T^n y_n) = 0. \quad (3.9)$$

From (2.1) and (3.9), we have

$$\begin{aligned} d(x_{n+1}, T^n x_n) &= d(W(T^n x_n, T^n y_n, \alpha_n), T^n x_n) \\ &\leq \alpha_n d(T^n x_n, T^n y_n) \\ &\leq m d(T^n x_n, T^n y_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (3.10)$$

Hence from (3.9) and (3.10), we have

$$\begin{aligned} d(x_{n+1}, T^n y_n) &\leq d(x_{n+1}, T^n x_n) + d(T^n x_n, T^n y_n) \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (3.11)$$

Now using (3.11), we have

$$\begin{aligned} d(x_{n+1}, p) &\leq d(x_{n+1}, T^n y_n) + d(T^n y_n, p) \\ &\leq d(x_{n+1}, T^n y_n) + (1 + u_n)(d(y_n, p) + a_n). \end{aligned} \quad (3.12)$$

The inequality (3.12) gives

$$r \leq \liminf_{n \rightarrow \infty} d(y_n, p). \quad (3.13)$$

From (3.5) and (3.13), we get

$$r = \lim_{n \rightarrow \infty} d(y_n, p) = \lim_{n \rightarrow \infty} d(W(x_n, T^n z_n, \beta_n), p). \quad (3.14)$$

Applying Lemma 2.6 in (3.14), we obtain

$$\lim_{n \rightarrow \infty} d(x_n, T^n z_n) = 0. \quad (3.15)$$

Now using (3.15) and hypothesis of the theorem $d(x, T^n y) \leq d(T^n x, T^n y)$ for all $x, y \in K$, we get

$$\begin{aligned} d(x_n, T^n x_n) &\leq d(x_n, T^n y_n) + d(T^n y_n, T^n x_n) \\ &\leq d(T^n x_n, T^n y_n) + d(T^n y_n, T^n x_n) \\ &= 2 d(T^n x_n, T^n y_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (3.16)$$

Again note that

$$d(x_n, T^n y_n) \leq d(x_n, T^n x_n) + d(T^n x_n, T^n y_n). \quad (3.17)$$

From (3.9) and (3.17), we obtain

$$\lim_{n \rightarrow \infty} d(x_n, T^n y_n) = 0. \quad (3.18)$$

This completes the proof. $\square$

Theorem 3.4. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $T: K \rightarrow K$ be uniformly continuous nearly asymptotically nonexpansive mapping with $F(T) \neq \emptyset$ and sequence $\{(a_n, \eta(T^n))\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} (\eta(T^n) - 1) < \infty$. Let $\{x_n\}$ be a sequence in K defined by (2.1). Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequence in $[l, m]$ for some $l, m \in (0, 1)$. Then $\{x_n\}$ is Δ -convergence to a point in $F(T)$.*

Proof. By Lemma 3.3, $\lim_{n \rightarrow \infty} d(x_n, T^n x_n) = 0$. By uniform continuity of T , $\lim_{n \rightarrow \infty} d(x_n, T^n x_n) = 0$ implies $\lim_{n \rightarrow \infty} d(Tx_n, T^{n+1}x_n) = 0$. From (3.16) and (3.18), we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(W(T^n x_n, T^n y_n, \alpha_n), x_n) \\ &\leq (1 - \alpha_n)d(x_n, T^n x_n) + \alpha_n d(T^n y_n, x_n) \\ &\leq (1 - l)d(x_n, T^n x_n) + m d(T^n y_n, x_n) \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (3.19)$$

Also

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) \\ &\quad + d(T^{n+1}x_{n+1}, T^{n+1}x_n) + d(T^{n+1}x_n, Tx_n) \\ &\leq (1 + \eta(T^{n+1}))d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) \\ &\quad + d(T^{n+1}x_n, Tx_n) + a_{n+1}. \end{aligned} \quad (3.20)$$

The above inequality gives

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0. \quad (3.21)$$

Now, we have to show that $\{x_n\}$ is Δ -convergence to a point in $F(T)$.

Since $\{x_n\}$ is bounded by Lemma 3.1, therefore the sequence $\{x_n\}$ has a unique asymptotic center (see, [43]), that is, $A(\{x_n\}) = \{x\}$ (say). Let $A(\{y_n\}) = \{v\}$. Then by equation (3.21) of Lemma 3.3, $\lim_{n \rightarrow \infty} d(y_n, Ty_n) = 0$. Since T is nearly asymptotically nonexpansive mappings with sequence $\{(a_n, \eta(T^n))\}$. By uniform continuity of T , we have

$$\lim_{n \rightarrow \infty} d(T^i y_n, T^{i+1} y_n) = 0 \quad \text{for } i = 1, 2, \dots \quad (3.22)$$

Now we claim that v is a fixed point of T . For this, we define a sequence $\{z_n\}$ in K by $z_m = T^m v$ for all $m \geq 1$. For integers $m, n \geq 1$, we have

$$\begin{aligned} d(z_m, y_n) &\leq d(T^m v, T^m y_n) + d(T^m y_n, T^{m-1} y_n) + \dots + d(T y_n, y_n) \\ &\leq \eta(T^n)(d(v, y_n) + a_m) + \sum_{i=0}^{m-1} d(T^i y_n, T^{i+1} y_n). \end{aligned} \quad (3.23)$$

Then from (3.22) and (3.23), we have

$$\begin{aligned} r(z_m, \{y_n\}) &= \limsup_{m \rightarrow \infty} d(z_m, y_n) \\ &\leq \eta(T^m)[r(v, \{y_n\}) + a_m]. \end{aligned}$$

Hence

$$\limsup_{m \rightarrow \infty} r(z_m, \{y_n\}) \leq r(v, \{y_n\}). \quad (3.24)$$

Since $A_K(\{y_n\}) = \{v\}$, by definition of asymptotic center $A_K(\{y_n\})$ of a bounded sequence $\{y_n\}$ with respect to $K \subset X$, we have

$$r(v, \{y_n\}) \leq r(y, \{y_n\}), \quad \forall y \in K.$$

This implies that

$$\liminf_{m \rightarrow \infty} r(z_m, \{y_n\}) \geq r(v, \{y_n\}), \quad (3.25)$$

therefore, from (3.24) and (3.25), we have

$$\lim_{m \rightarrow \infty} r(z_m, \{y_n\}) = r(v, \{y_n\}).$$

It follows from Lemma 2.7 that $T^m v \rightarrow v$. By uniform continuity of T , we have

$$Tv = S(\lim_{m \rightarrow \infty} T^m v) = T^{m+1} v = v,$$

which implies that v is a fixed point of T , that is, $v \in F(T)$.

Next, we claim that v is the unique asymptotic center for each subsequence $\{y_n\}$ of $\{x_n\}$.

Assume contrarily, that is, $x \neq v$. Since $\lim_{n \rightarrow \infty} d(x_n, v)$ exists by Lemma 3.1, therefore, by the uniqueness of asymptotic centers, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} d(y_n, v) &< \limsup_{n \rightarrow \infty} d(y_n, x) \\ &\leq \limsup_{n \rightarrow \infty} d(x_n, x) \\ &< \limsup_{n \rightarrow \infty} d(x_n, v) \\ &= \limsup_{n \rightarrow \infty} d(y_n, v), \end{aligned}$$

a contradiction and hence $x = v$. Since $\{y_n\}$ is an arbitrary subsequence of $\{x_n\}$, therefore, $A_K(\{y_n\}) = \{v\}$ for all subsequence $\{y_n\}$ of $\{x_n\}$. This shows that $\{x_n\}$ Δ -converges to a point in $F(T)$. This completes the proof. $\square$

We now establish some strong convergence theorems of newly defined modified three-step iteration scheme for non-Lipschitzian mappings in the framework of uniformly convex hyperbolic space.

Theorem 3.5. *Let K be a nonempty convex subset of a hyperbolic space X and let $T: K \rightarrow K$ be a nearly asymptotically quasi-nonexpansive mapping with sequence $\{(a_n, u_n)\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} u_n < \infty$. Let $\{x_n\}$ be a sequence in K defined by (2.1). Assume that $F(T) \neq \emptyset$ closed set. Then $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf_{n \rightarrow \infty} d(x_n, F(T)) = 0$.*

Proof. The necessity is obvious. Conversely, suppose that $\liminf_{n \rightarrow \infty} d(x_n, F(T)) = 0$. As proved in Lemma 3.1, for all $p \in F(T)$, $\lim_{n \rightarrow \infty} d(x_n, F(T))$ exists. Thus by hypothesis, $\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence in K . With the help of inequality $1 + x \leq e^x$, $x \geq 0$. For any integer $m \geq 1$, we have from (3.3)

$$\begin{aligned}
d(x_{n+m}, p) &\leq (1 + Qu_{n+m-1})d(x_{n+m-1}, p) + Q'a_{n+m-1} \\
&\leq e^{Qu_{n+m-1}}d(x_{n+m-1}, p) + Q'a_{n+m-1} \\
&\leq e^{Qu_{n+m-1}}[e^{Qu_{n+m-2}}d(x_{n+m-2}, p) + Q'a_{n+m-2}] \\
&\quad + Q'a_{n+m-1} \\
&\leq e^{Q(u_{n+m-1}+u_{n+m-2})}d(x_{n+m-2}, p) + e^{Q(u_{n+m-1}+u_{n+m-2})} \times \\
&\quad Q'[a_{n+m-1} + a_{n+m-2}] \\
&\leq \dots \\
&\leq \left(e^{Q\sum_{k=n}^{n+m-1} u_k}\right)d(x_n, p) + Q'\left(e^{Q\sum_{k=n}^{n+m-1} u_k}\right) \sum_{k=n}^{n+m-1} a_k \\
&= M\left(d(x_n, p) + \sum_{k=n}^{n+m-1} c_k\right) \\
&\leq M\left(d(x_n, p) + \sum_{k=n}^{\infty} c_k\right), \tag{3.26}
\end{aligned}$$

for every $p \in F(T)$, where $M = e^{Q(\sum_{k=n}^{n+m-1} u_k)} > 0$ and $c_k = Q'a_k$. As $\sum_{n=1}^{\infty} u_n < \infty$, so $M^* = \left(e^{Q(\sum_{n=1}^{\infty} u_n)}\right) \geq M = \left(e^{Q(\sum_{k=n}^{n+m-1} u_k)}\right) > 0$.

Since $\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0$, without loss of generality, we may assume that a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{p_{n_k}\} \subset F(T)$ such that $d(x_{n_k}, p_{n_k}) \rightarrow 0$ as $k \rightarrow \infty$. Then for any $\varepsilon > 0$, there exists $k_\varepsilon > 0$ such that

$$d(x_{n_k}, p_{n_k}) < \frac{\varepsilon}{3M^*} \quad \text{and} \quad \sum_{k=n_{k_\varepsilon}}^{\infty} c_k < \frac{\varepsilon}{6M^*}, \tag{3.27}$$

for all $k \geq k_\varepsilon$.

For any $m \geq 1$ and for all $n \geq n_{k_\varepsilon}$, by (3.26) and (3.27), we have

$$\begin{aligned}
d(x_{n+m}, x_n) &\leq d(x_{n+m}, p_{n_k}) + d(x_n, p_{n_k}) \\
&\leq 2M^*[d(x_n, p_{n_k}) + \sum_{k=n_{k_\varepsilon}}^{\infty} c_k] \\
&< 2M^*\left(\frac{\varepsilon}{3M^*} + \frac{\varepsilon}{6M^*}\right) = \varepsilon. \tag{3.28}
\end{aligned}$$

This proves that $\{x_n\}$ is a Cauchy sequence in closed subset K of a complete hyperbolic space X and so it must converge to a point z in K , that is, $\lim_{n \rightarrow \infty} x_n = z$. Now, $\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0$ gives $d(z, F(T)) = 0$. Since $F(T)$ is closed, we have $z \in F(T)$. Thus $\{x_n\}$ converges strongly to a point in $F(T)$. This completes the proof. $\square$

Remark 3.6. Theorem 3.5 extends and improves Theorem 3.1 of Ghosh and Deb-nath [7], Theorem 1.1 and 1.1' of Petryshyn and Williamson [31] and Theorem 1 of Liu [22] in the more general space setting.

Corollary 3.7. *Let K be a nonempty convex subset of a hyperbolic space X and let $T: K \rightarrow K$ be a nearly asymptotically nonexpansive mapping with sequence $\{(a_n, u_n)\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} u_n < \infty$. Let $\{x_n\}$ be a sequence in K defined by (2.1). Assume that $F(T) \neq \emptyset$ closed set. Then $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf_{n \rightarrow \infty} d(x_n, F(T)) = 0$.*

Theorem 3.8. *Let K be a nonempty convex subset of a hyperbolic space X and let $T: K \rightarrow K$ be an asymptotically quasi-nonexpansive mapping with sequence $\{u_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} u_n < \infty$. Let $\{x_n\}$ be a sequence in K defined by (2.1). Then $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf_{n \rightarrow \infty} d(x_n, F(T)) = 0$.*

Proof. As in the proof of Theorem 3.5, it can be easily shown that $\{x_n\}$ is a Cauchy sequence in K . Let $\lim_{n \rightarrow \infty} x_n = z$. Since every asymptotically quasi-nonexpansive mapping is quasi L -Lipschitzian, it follows from Lemma 3.2 that z is a fixed point of T . This completes the proof. $\square$

Theorem 3.9. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $T: K \rightarrow K$ be uniformly continuous nearly asymptotically nonexpansive mapping with $F(T) \neq \emptyset$ and sequence $\{(a_n, \eta(T^n))\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} (\eta(T^n) - 1) < \infty$. Let $\{x_n\}$ be a sequence in K defined by (2.1). Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequence in $[l, m]$ for some $l, m \in (0, 1)$. If T^m for some $m \geq 1$ is demicompact, then $\{x_n\}$ converges strongly to a fixed point of T .*

Proof. By equation (3.21) of Lemma 3.3, we have $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$. By the uniform continuity of T , we have

$$\begin{aligned} d(x_n, Tx_n) \rightarrow 0 &\Rightarrow d(Tx_n, T^2x_n) \rightarrow 0 \Rightarrow \\ &\dots \Rightarrow d(T^i x_n, T^{i+1} x_n) \rightarrow 0 \end{aligned}$$

for all $i = 1, 2, 3, \dots$, it follows that

$$d(x_n, T^m x_n) \leq \sum_{i=0}^{m-1} d(T^i x_n, T^{i+1} x_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Since $d(x_n, T^m x_n) \rightarrow 0$ and T^m is semi-compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j \rightarrow \infty} T^m x_{n_j} = x \in K$.

Note that

$$d(x_{n_j}, x) \leq d(x_{n_j}, T^m x_{n_j}) + d(T^m x_{n_j}, x) \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

Since $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$, we get $x \in F(T)$. Since $\lim_{n \rightarrow \infty} d(x_n, x)$ exists by Lemma 3.1 and $\lim_{j \rightarrow \infty} d(x_{n_j}, x) = 0$, we conclude that $x_n \rightarrow x \in F(T)$. This

shows that the sequence $\{x_n\}$ converges strongly to a fixed point of T . This completes the proof. $\square$

Senter and Dotson [41] introduced the concept of Condition (A) as follows.

Definition 3.10. (See [41]) A mapping $T: K \rightarrow K$ is said to satisfy condition (A) if there exists a non-decreasing function $f: [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $d(x, Tx) \geq f(d(x, F(T)))$, for all $x \in K$.

As an application of Theorem 3.5, we establish another strong convergence result employing condition (A).

Theorem 3.11. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $T: K \rightarrow K$ be uniformly continuous nearly asymptotically nonexpansive mapping with $F(T) \neq \emptyset$ and sequence $\{(a_n, \eta(T^n))\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} (\eta(T^n) - 1) < \infty$. Let $\{x_n\}$ be a sequence in K defined by (2.1). Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequence in $[l, m]$ for some $l, m \in (0, 1)$. Suppose that T satisfies condition (A). Then $\{x_n\}$ converges strongly to a fixed point of T .*

Proof. By Lemma 3.2, for any $p \in F(T)$, we see that

$$\lim_{n \rightarrow \infty} d(x_n, p) \quad \text{and} \quad \lim_{n \rightarrow \infty} d(x_n, F(T))$$

both exist. Let $\lim_{n \rightarrow \infty} d(x_n, F(T)) = r$ for some $r > 0$. Now in view of Theorem 3.5, to complete the proof we must show that $r = 0$. Since T satisfies Condition (A), so we have

$$d(x_n, Tx_n) \geq f(d(x_n, F(T))) \geq f(r).$$

Since by equation (3.21) of Lemma 3.3, $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$, we get that $f(r) = 0$ and so $r = 0$. This completes the proof. $\square$

4. CONCLUSION

1. We prove strong and Δ -convergence theorems of modified three-step iteration process which contains modified S -iteration process in the framework of uniformly convex hyperbolic spaces.

2. Theorem 3.4 extends Theorem 3.8 of Agarwal et al. [2] to the case of modified three-step iteration scheme and from uniformly convex Banach space to a uniformly convex hyperbolic space considered in this paper.

3. Theorem 3.4 also extends Theorem 3.3 of Dhompongsa and Panyanak [6] to the case of more general class of nonexpansive mappings which are not necessarily Lipschitzian, modified three-step iteration scheme and from CAT(0) space to a uniformly convex hyperbolic space considered in this paper.

4. Theorem 3.4 also extends Theorem 3.5 of Niwongsa and Panyanak [26] to the case of more general class of asymptotically nonexpansive mappings which

are not necessarily Lipschitzian, modified three-step iteration scheme and from $CAT(0)$ space to a uniformly convex hyperbolic space considered in this paper.

5. Our results also extend and generalize the corresponding results of Xu and Noor [47] to the case of more general class of asymptotically nonexpansive mappings, modified three-step iteration scheme and from a Banach space to a uniformly convex hyperbolic space considered in this paper.

6. Our results also extend and generalize the corresponding results of [3, 16, 30, 34, 35, 39, 40] for a more general class of non-Lipschitzian mappings, modified three-step iteration scheme and from uniformly convex metric space, $CAT(0)$ space to a uniformly convex hyperbolic space considered in this paper.

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