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SERIES INVOLVING DIRICHLET ETA FUNCTION

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Dedicated to the Professor Dragoslav Mitrinović , famous Serbian professor which wrote many university books, one of which is dedicated to Special Functions "Specijalne Funkcije", to mark the year that marks 115 years since his birth.

Abstract. In this article, we obtain an integral representation for a remainder sum of the Dirichlet Eta function. We then obtain numerous generating functions and series concerning the usage of the obtained integral representation. Alternating Fibonacci sum of the remainder sum of the Dirichlet Eta function has been obtained, as well as the squared version of the Fibonacci series concerning the sum. A generalized representation of the product of polynomials concerning the remainder sum of the Dirichlet Eta function has been obtained. Numerous examples have been provided to showcase the derived results.

1. Introduction

Sums have been an ongoing topic of investigation since their introduction. The earliest record of people summing is the formula for the sum of the first n numbers. From then on people encountered more challenging sums as the science progressed, and with that the need to evaluate and analyze the obtained sums. The evaluation of those series was a focus of many great names we know today, such as Euler [7] and Malmsten [17]. Many sums concerning the use of special functions have been discovered since then, see the following books about sums [3, 5, 11, 14, 23]. Many papers have been written about them, see [9],[25, 26],[27, 28],[29]. Interested readers can find the related papers in the references given. The type of sums we will investigate in this paper can be found in the following book [20].

The first definition we will need in this sequel is as follows.
The polylogarithm, see [16] is defined by a power series in z, given by

$$\mathrm{Li}_s(z) = \sum_{k=1}^{+\infty} \frac{z^k}{k^s}. \tag{1.1}$$

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This definition is valid for arbitrary complex order s and for all complex numbers z with $|z| < 1$. We will also need the definition given by

$$\mathrm{Li}_s(z) = \int_0^z \frac{\mathrm{Li}_{s-1}(z)}{z} dz.$$

For $z = 1$ we get the Riemann zeta function ζ which is also a function of complex variable s. For more information see [6], [8], [13].

$$\mathrm{Li}_s(1) = \zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \Re(s) > 1.$$

The closely related function $\eta(s), s \in \mathbb{C}$, defined by

$$\eta(s) = \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k^s}, \Re(s) > 0, \tag{1.2}$$

is called an alternating zeta, or Dirichlet's eta, or Euler's eta function. The functions are connected by the following relation

$$\eta(s) = (1 - 2^{1-s})\zeta(s).$$

Integral representation of $\eta(s)$ is given by

$$\eta(s) = \frac{2^{s-1}}{\Gamma(s+1)} \int_0^{+\infty} \frac{t^s}{\cosh^2(t)} dt, \Re(s) > -1, s \neq 1.$$

A number of properties of the Eta function have been investigated recently by Sondow [21, 22], Milgram [18] and Alzer and Kwong , among others.

The second definition is given.

The gamma function is defined by a convergent improper integral, for $\Re(z) > 0$, see [1]

$$\Gamma(z) = \int_0^{+\infty} \tau^{z-1} e^{-\tau} d\tau.$$

The following relations hold

$$\Gamma(z+1) = z\Gamma(z),$$

$$\Gamma(z+1) = z!.$$

The skew harmonic numbers [19] are defined as follows,

$$\overline{H_k} = \sum_{l=1}^{k} \frac{(-1)^{l-1}}{l}. \tag{1.3}$$

Their integral representation is given by

$$\overline{H_k} = \ln(2) - \int_0^1 \frac{(-\tau)^k}{1+\tau} d\tau.$$

The generating function for the skew harmonic numbers holds

$$\sum_{n=1}^{+\infty} \overline{H_n}\tau^n = \frac{\ln(1+\tau)}{1-\tau}. \tag{1.4}$$

The generalized hypergeometric function ${}_qF_q(a;b;\tau)$ is defined as follows

$${}_pF_q(a;b;\tau) = \sum_{k=0}^{+\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{\tau^k}{k!} \tag{1.5}$$

where $(a)_k$ is the Pochhammer symbol defined as

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\dots(a+k-1).$$

Now we will define a sequence which represents the partial sum of the Dirichlet Eta function.

We define the following sequence

$$\eta_k(m) := \sum_{l=1}^{k} \frac{(-1)^{l-1}}{l^m}. \tag{1.6}$$

Throughout the years, many sums have been evaluated by various authors, we list some of them as the motivation behind evaluating the ones given in this paper. Choi et al.[4] evaluated various series in their paper, one of the interesting ones is

$$\sum_{k=1}^{+\infty} \frac{\zeta(2k)-1}{k+1} = \frac{3}{2} - \ln\pi.$$

Some of the series obtained by Kobayashi [15] are

$$\sum_{k=0}^{+\infty} \frac{\zeta(2k)}{2^{2k}} = 0,$$

$$\sum_{k=0}^{+\infty} \frac{\zeta(2k)}{4^{2k}} = -\frac{\pi}{8}.$$

Sums regarding the famous stieltjes constants have been investigated by Hu et al.[12]. For more information see the reference cited. The following sum was evaluated by Srivastava [24],

$$\sum_{k=2}^{+\infty} \frac{k-1}{2^k}(\zeta(k)-1) = \frac{\pi^2}{8} - 1.$$

Another wonderful sum evaluated in the same paper is

$$\sum_{k=1}^{+\infty} \frac{\zeta(2k+1)}{2k+1} 2^{-2k} = \ln 2 - \gamma.$$

We proceed with the main section of the paper.

2. Main results

We give our first crucial Lemma in this paper.

Lemma 2.1. *The following equalities hold for m such that both sides are defined*

$$\eta(2m)-\eta_k(2m)=\frac{(-1)^{k+1}}{\Gamma(2m)}\int_0^1\frac{\tau^k\ln^{2m-1}(\tau)}{1+\tau}d\tau, \tag{2.1}$$

$$\eta(2m+1)-\eta_k(2m+1)=\frac{(-1)^{k}}{\Gamma(2m+1)}\int_0^1\frac{\tau^k\ln^{2m}(\tau)}{1+\tau}d\tau, \tag{2.2}$$

$$\eta(2m)-\eta_{2k}(2m)=-\frac{1}{\Gamma(2m)}\int_0^1\frac{\tau^{2k}\ln^{2m-1}(\tau)}{1+\tau}d\tau, \tag{2.3}$$

$$\eta(2m+1)-\eta_{2k}(2m+1)=\frac{1}{\Gamma(2m+1)}\int_0^1\frac{\tau^{2k}\ln^{2m}(\tau)}{1+\tau}d\tau. \tag{2.4}$$

Proof. Let us consider the second equality, the first one is analogous and the third and fourth are the special cases of the first two.
Starting from the following integral equality

$$\int_0^1\tau^k\ln^m(\tau)d\tau=\frac{(-1)^m\Gamma(1+m)}{(1+k)^{1+m}}. \tag{2.5}$$

We see that if we set m to be even, then we don't have the alternating signs, and by considering the same integrand 2.5 but with $\frac{1}{1+\tau}$ in the denominator and expanding the $\frac{1}{1+\tau}$ in the denominator, we obtain the following

$$\int_0^1\frac{\tau^k\ln^{2m}(\tau)}{1+\tau}d\tau=\sum_{p=0}^{+\infty}\frac{\Gamma(2m+1)}{(1+k+p)^{2m+1}}(-1)^p.$$

This is a remainder of the Dirichlet Eta function, but since we have k in the denominator, and since we did not fix the k to be even or odd, we do not know whether the sign of the next term will be positive or negative. That is why we have $(-1)^k$ in front of the integral. Dividing by $\frac{1}{\Gamma(2m+1)}$ cancels the Gamma function, and we are left with the equality. Similar procedure can be applied to the first term, but since m is odd we will have a minus in front of the previously obtained equality. □

Theorem 2.2. *The following equalities hold for $|y|<1$ and for m such that the integral representation used in Lemma 2.1 is defined*

$$\sum_{k=0}^{+\infty}y^{2k}\left(\eta(2m+1)-\eta_{2k}(2m+1)\right)=\frac{1}{2}\left(\frac{\mathrm{Li}_{2m+1}(y)}{y+1}-\frac{\mathrm{Li}_{2m+1}(-y)}{y-1}\right) \tag{2.6}$$
$$-\frac{1}{y^2-1}(1-4^{-m})\zeta(2m+1),$$

$$\sum_{k=0}^{+\infty}y^{2k}\left(\eta(2m)-\eta_{2k}(2m)\right)=\frac{1}{2}\left(\frac{\mathrm{Li}_{2m}(y)}{1+y}-\frac{\mathrm{Li}_{2m}(-y)}{y-1}\right)-\frac{4^{-m}(4^m-2)\zeta(2m)}{y^2-1}, \tag{2.7}$$

$$\sum_{k=0}^{+\infty}(-1)^k y^k\left(\eta(2m)-\eta_k(2m)\right)=\frac{\Gamma(2m)\operatorname{Li}_{2m}(y)}{1+y}+\frac{1}{y+1}4^{-m}(4^m-2)\Gamma(2m)\zeta(2m). \tag{2.8}$$

Proof. We will prove the first equality, the second and the third one are analogous. The first case is where we have odd Dirichlet Eta function remainder, therefore we obtain

$$\sum_{k=0}^{+\infty} y^{2k}\left(\eta(2m+1)-\eta_{2k}(2m+1)\right)=\sum_{k=0}^{+\infty} y^{2k}\frac{1}{\Gamma(2m+1)}\int_0^1\frac{\tau^{2k}\ln^{2m}(\tau)}{1+\tau}d\tau.$$

Switching the order of integration and summation, we get the following

$$\sum_{k=0}^{+\infty} y^{2k}\left(\eta(2m+1)-\eta_{2k}(2m+1)\right)=\frac{1}{\Gamma(2m+1)}\int_0^1\frac{\ln^{2m}(\tau)}{(1+\tau)(1-\tau^2y^2)}d\tau.$$

Setting $\tau=e^{-z}$ we get the following

$$\frac{1}{\Gamma(2m+1)}\int_0^1\frac{\ln^{2m}(\tau)}{(1+\tau)(1-\tau^2y^2)}d\tau=\frac{1}{\Gamma(2m+1)}\int_0^{+\infty}\frac{z^{2m}e^{-z}}{(1+e^{-z})(1-e^{-2z}y^2)}dz.$$

Here we use following partial fraction $\frac{e^{-z}}{(1+e^{-z})(1-e^{-2z}y^2)}=\frac{y}{2(y+1)(e^z-y)}+\frac{y}{2(y-1)(y+e^z)}-\frac{1}{(y-1)(y+1)(e^z+1)}$. From which we get that our integral separates into the following

$$\frac{1}{\Gamma(2m+1)}\int_0^{+\infty}\frac{z^{2m}e^{-z}}{(1+e^{-z})(1-e^{-2z}y^2)}dz=\int_0^{+\infty}\frac{z^{2m}}{e^z-y}dz\frac{y}{2(y+1)}$$
$$+\frac{y}{2(y-1)}\int_0^{+\infty}\frac{z^{2m}}{y+e^z}dz-\frac{1}{(y-1)(y+1)}\int_0^1\frac{z^{2m}}{e^z+1}dz.$$

For the sake of completeness, we will solve the first integral and the third one, second one is analogous. Pulling e^z from the denominator and expanding the denominator into a geometric series, we obtain the following

$$\frac{y}{2(y+1)}\int_0^{+\infty}\frac{z^{2m}}{e^z-y}dz=\frac{y}{2(y+1)}\int_0^{+\infty}\sum_{n=0}^{+\infty}e^{-z}z^{2m}y^ne^{-zn}dz.$$

Switching the order of integration and summation, which is possible due to the terms being positive(Beppo-Levis) Theorem, we get

$$\frac{y}{2(y+1)}\int_0^{+\infty}\sum_{n=0}^{+\infty}e^{-z}z^{2m}y^ne^{-zn}dz=\frac{1}{2(y+1)}\sum_{n=0}^{+\infty}\frac{y^{n+1}}{(n+1)^{2m+1}}\Gamma(2m+1).$$

The third integral can be done in a similar way, using the formula

$$\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^s}=2^{-s}(2^s-2)\zeta(s).$$

Combining everything, we obtain the original formula. □

Corollary 2.3. *Setting $m = 2, y = \frac{1}{2}$ in the first equality of the Theorem, we obtain the following equality*

$$\sum_{k=0}^{+\infty} \frac{\eta(5) - \eta_{2k}(5)}{2^{2k}} = \mathrm{Li}_5\left(-\frac{1}{2}\right) + \frac{\mathrm{Li}_5\left(\frac{1}{2}\right)}{3} + \frac{5\zeta(5)}{4}. \tag{2.9}$$

Setting $m = 1, y = \frac{1}{2}$ in the second equality, we obtain

$$\sum_{k=0}^{+\infty} \frac{\eta(2) - \eta_{2k}(2)}{2^{2k}} = \frac{1}{18}\left(9\,\mathrm{Li}_2\left(\frac{1}{4}\right) + \pi^2 + 6\ln^2 2\right). \tag{2.10}$$

Setting $m = 1, y = \frac{1}{2}$ in the third equality, we obtain

$$\sum_{k=0}^{+\infty} \frac{(-1)^k(\eta(2) - \eta_k(2))}{2^k} = \frac{\pi^2}{9} - \frac{\ln^2(2)}{3}. \tag{2.11}$$

The following Theorem shows how generating functions can be paired up with the tails of the Dirichlet Eta function.

Theorem 2.4. *The following equality holds*

$$\sum_{k=1}^{+\infty} \frac{\eta(2) - \eta_k(2)}{k+1} = \frac{5\zeta(3)}{8} - \eta(2). \tag{2.12}$$

Proof. Utilizing the Lemma 1 while setting $m = 1$ we obtain

$$\sum_{k=1}^{+\infty} \frac{\eta(2) - \eta_k(2)}{k+1} = \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{(k+1)(1)!} \int_0^1 \frac{\tau^k \ln(\tau)}{1+\tau} d\tau.$$

Using the generating function

$$\sum_{k=1}^{+\infty} \frac{(-\tau)^k}{k+1} = \frac{\ln(1+\tau) - \tau}{\tau}$$

and switching the order of integration and summation, which is possible because the series is a constant with respect to the integration with respect to y, and the power series is convergent in $|\tau| < 1$ *and at* $\tau = 1$, and since $y \in (0,1)$, it is permitted, therefore we get

$$\sum_{k=1}^{+\infty} \frac{\eta(2) - \eta_k(2)}{k+1} = -\int_0^1 \sum_{k=1}^{+\infty} \frac{(-\tau)^k}{k+1} \frac{\ln(\tau)}{1+\tau} d\tau = -\int_0^1 \frac{-\tau + \ln(1+\tau)}{\tau} \frac{\ln(\tau)}{1+\tau} d\tau.$$

$$= -\int_0^1 (-\tau\ln(\tau) + \ln(\tau)\ln(1+\tau))\left(\frac{1}{\tau} - \frac{1}{1+\tau}\right) d\tau$$

$$= \int_0^1 \ln(\tau) d\tau - \int_0^1 \frac{\tau\ln(\tau)}{1+\tau} d\tau - \int_0^1 \frac{\ln(\tau)\ln(1+\tau)}{\tau} d\tau + \int_0^1 \frac{\ln(\tau)\ln(1+\tau)}{1+\tau} d\tau.$$

The first integral evaluates quite easily, the second one is solved by expanding the denominator and then realizing that it is indeed a Dirichlet eta function, just shifted by one, which at the end evaluates at the following

$$\int_0^1 \frac{\tau\ln(\tau)}{1+\tau}d\tau = \eta(2) - 1.$$

We focus now onto the fourth integral, the most challenging one

$$\int_0^1 \frac{\ln(1+\tau)\ln(\tau)}{1+\tau}d\tau.$$

We will solve it as an idefinite integral, performing partial integration with taking $u = \ln(1+\tau), du = \frac{1}{1+\tau}, dv = \frac{\ln(\tau)}{1+\tau}, v = \operatorname{Li}_2(-\tau) + \ln(\tau)\ln(1+\tau)$. We get the following

$$I = \ln(1+\tau)(\operatorname{Li}_2(-\tau)+\ln(\tau)\ln(1+\tau)) - \int \frac{1}{1+\tau}\left(\operatorname{Li}_2(-\tau)+\ln(\tau)\ln(1+\tau)\right)d\tau.$$

Simplifying and performing partial integration while taking $u = \operatorname{Li}_2(-\tau), du = -\frac{\ln(1+\tau)}{\tau}, dv = \frac{1}{1+\tau}, v = \ln(1+\tau)$

$$2I = \ln(1+\tau)(\operatorname{Li}_2(-\tau)+\ln(\tau)\ln(1+\tau))$$

$$-\left(\operatorname{Li}_2(-\tau)\ln(1+\tau) + \int \frac{\ln^2(1+\tau)}{\tau}d\tau\right).$$

Focusing on the integral inside, we perform partial integration taking $u = \ln^2(1+\tau), du = \frac{2\ln(1+\tau)}{1+\tau}, dv = \frac{1}{\tau}, v = \ln(-\tau)$ from which we get

$$\int \frac{\ln^2(1+\tau)}{\tau}d\tau = \ln(-\tau)\ln^2(1+\tau) - 2\int \frac{\ln(-\tau)\ln(1+\tau)}{1+\tau}d\tau.$$

Solving the leftover integral by taking $u = \ln(1+\tau), du = \frac{1}{1+\tau}, dv = \frac{\ln(-\tau)}{1+\tau}, v = -\operatorname{Li}_2(1+\tau)$,we obtain

$$\int \frac{\ln^2(1+\tau)}{\tau}d\tau = \ln(-\tau)\ln^2(1+\tau) + 2\ln(1+\tau)\operatorname{Li}_2(1+\tau) - 2\operatorname{Li}_3(1+\tau).$$

Which all together gives us the following

$$\int \frac{\ln(1+\tau)\ln(\tau)}{1+\tau}d\tau = \frac{1}{2}\Bigg(\ln(1+\tau)\operatorname{Li}_2(-\tau) + \ln(\tau)\ln^2(1+\tau)\Bigg)$$
$$-\frac{1}{2}\left(\operatorname{Li}_2(-\tau)\ln(1+\tau) + \ln(-\tau)\ln^2(1+\tau)\right)$$
$$-\frac{1}{2}\left(2\ln(1+\tau)\operatorname{Li}_2(1+\tau) - 2\operatorname{Li}_3(1+\tau)\right).$$

When putting in the limits $\tau \to 1$

$$\lim_{\tau\to 1}\frac{1}{2}\Bigg(\ln(1+\tau)\operatorname{Li}_2(-\tau) + \ln(\tau)\ln^2(1+\tau)\Bigg)$$
$$-\frac{1}{2}\left(\operatorname{Li}_2(-\tau)\ln(1+\tau) + \ln(-\tau)\ln^2(1+\tau)\right)$$

$$-\frac{1}{2}\left(2\ln(1+\tau)\operatorname{Li}_2(1+\tau)-2\operatorname{Li}_3(1+\tau)\right) = \frac{7\zeta(3)}{8}.$$

When putting in the limit as $\tau \to 0$

$$\lim_{\tau\to 0}\frac{1}{2}\bigg(\ln(1+\tau)\operatorname{Li}_2(-\tau)+\ln(\tau)\ln^2(1+\tau)\bigg)$$
$$-\frac{1}{2}\left(\operatorname{Li}_2(-\tau)\ln(1+\tau)+\ln(-\tau)\ln^2(1+\tau)\right)$$
$$-\frac{1}{2}\left(2\ln(1+\tau)\operatorname{Li}_2(1+\tau)-2\operatorname{Li}_3(1+\tau)\right) = \zeta(3).$$

All together, the integral evaluates at

$$\int_0^1 \frac{\ln(\tau)\ln(1+\tau)}{1+\tau}d\tau = -\frac{\zeta(3)}{8}.$$

The third integral

$$\int_0^1 \frac{\ln(\tau)\ln(1+\tau)}{\tau}d\tau$$

is easily solved by expanding the $\ln(1+\tau)$ into a series and then using the formula connecting the Dirichlet eta function and the Zeta function , namely

$$\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^s} = 2^{-s}(2^s-2)\zeta(s)$$

from which we get

$$\int_0^1 \frac{\ln(\tau)\ln(1+\tau)}{\tau}d\tau = -\frac{3\zeta(3)}{4}.$$

Combining everything, we get the following

$$\sum_{k=1}^{+\infty}\frac{\eta(2)-\eta_k(2)}{k+1} = -1-(\eta(2)-1)-\left(-\frac{3\zeta(3)}{4}\right)+\left(-\frac{\zeta(3)}{8}\right) = \frac{5\zeta(3)}{8}-\eta(2).$$

□

The following generalization of the sum of the remainder of the Dirichlet Eta function and the product of polynomials in the denominator holds.

Theorem 2.5. *The following equality holds for $l \geqslant 1$*

$$\sum_{n=1}^{+\infty}\frac{\eta(2m)-\eta_k(2m)}{(k+1)(k+2)(k+3)(k+4)...(k+l)} \tag{2.13}$$
$$= \frac{-1}{\Gamma(2m)(l-1)!}\int_0^1 \frac{y\ln^{2m-1}(y)}{(1+y)^2}\Bigg(\frac{{}_2F_1\left(1,1+l,;2+l;\frac{y}{1+y}\right)}{1+l}$$
$$-\frac{{}_2F_1\left(1,l;1+l;\frac{y}{1+y}\right)}{l}\Bigg)dy.$$

Proof. Rewriting the original sum in the following way, we get

$$\sum_{k=1}^{+\infty}\frac{\eta(2m)-\eta_k(2m)}{(k+1)(k+2)(k+3)...(k+l)}=\sum_{k=1}^{+\infty}(\eta(2m)-\eta_k(2m))\frac{k!}{(k+l)!}$$

$$=\frac{1}{l!}\sum_{k=1}^{+\infty}\frac{\eta(2m)-\eta_k(2m)}{\binom{l+k}{k}}.$$

Rewriting the difference of Dirichlet Eta functions using our Lemma and using the following representation for the denominator

$$\frac{1}{\binom{l+k}{k}}=l\int_0^1\tau^{l-1}(1-\tau)^k d\tau.$$

We obtain the following

$$-\frac{1}{\Gamma(2m)}\frac{1}{l!}\int_0^1\frac{\ln^{2m-1}(y)}{1+y}\left(\int_0^1 l\tau^{l-1}\sum_{k=1}^{+\infty}(1-\tau)^k(-y)^k d\tau\right)dy$$

$$=\frac{-1}{(2m-1)(l-1)!}\int_0^1\frac{\ln^{2m-1}(y)}{1+y}\left(\int_0^1\tau^{l-1}\frac{(1-\tau)y}{\tau y-y-1}d\tau\right)dy.$$

Computing the inside integral using the hypergeometric function 1.5 we obtain the following

$$=\frac{-1}{\Gamma(2m)(l-1)!}\int_0^1\frac{y\ln^{2m-1}(y)}{(1+y)^2}\Bigg(\frac{{}_2F_1\left(1,1+l,;2+l;\frac{y}{1+y}\right)}{1+l}$$
$$-\frac{{}_2F_1\left(1,l;1+l;\frac{y}{1+y}\right)}{l}\Bigg)dy.$$

□

Corollary 2.6. *The following equality holds*

$$\sum_{k=1}^{+\infty}\frac{\eta(2)-\eta_k(2)}{(k+1)(k+2)(k+3)(k+4)(k+5)}=\frac{1}{24}\left(-\frac{\pi^2}{60}-\frac{973}{576}+\frac{8\log(2)}{3}\right). \tag{2.14}$$

Proof. Using the similar procedure as in the previous Theorem, rewriting the original sum in the following way, we get

$$\sum_{k=1}^{+\infty}\frac{\eta(2)-\eta_k(2)}{(k+1)(k+2)(k+3)(k+4)(k+5)}=\sum_{k=1}^{+\infty}(\eta(2)-\eta_k(2))\frac{k!}{(k+5)!}$$

$$=\frac{1}{5!}\sum_{k=1}^{+\infty}\frac{\eta(2)-\eta_k(2)}{\binom{5+k}{k}}.$$

Now we use the same identity as in the main Theorem

$$\frac{1}{\binom{5+k}{k}}=5\int_0^1\tau^4(1-\tau)^k d\tau.$$

Using this identity together with Lemma 1 we get

$$\frac{1}{5!}\sum_{k=1}^{+\infty}\frac{\eta(2)-\eta_k(2)}{\binom{5+k}{k}} = -\frac{1}{5!}\int_0^1 \frac{\ln(y)}{1+y}\left(\int_0^1 5\tau^4\sum_{k=1}^{+\infty}(-y)^k(1-\tau)^k d\tau\right)dy$$

$$= -\frac{1}{4!}\int_0^1 \frac{\ln(y)}{1+y}\left(\int_0^1 \tau^4\frac{(1-\tau)y}{\tau y-y-1}d\tau\right)dy$$

$$= -\frac{1}{4!}\int_0^1 \frac{\ln(y)}{1+y}\left(\frac{(y+1)^4\log(y+1)}{y^5} - \frac{y(y(y(12y+125)+260)+210)+60}{60y^4}\right)dy.$$

Solving the resulting integral is fairly straightforward as expanding it and using the similar techniques to the previous Theorems yields the result. □

Corollary 2.7. *Setting $m, l = 1$ in Theorem 3, we get the following*

$$\sum_{k=1}^{+\infty}\frac{\eta(2)-\eta_k(2)}{k+1} = -\int_0^1 \frac{\ln(y)}{(1+y)^2}\left(\frac{{}_2F_1\left(1,2;3;\frac{y}{y+1}\right)}{2} - {}_2F_1\left(1,1;2;\frac{y}{y+1}\right)\right)dy.$$

Which we have evaluated in Theorem 2 2.12, therefore we get

$$\sum_{k=1}^{+\infty}\frac{\eta(2)-\eta_k(2)}{k+1} = -\int_0^1 \frac{\ln(y)}{(1+y)^2}\left(\frac{{}_2F_1\left(1,2;3;\frac{y}{y+1}\right)}{2} - {}_2F_1\left(1,1;2;\frac{y}{y+1}\right)\right)dy$$

$$= \frac{5\zeta(3)}{8} - \eta(2).$$

In the following Theorem we obtain a generating function for the alternating Fibonacci sequence multiplied with a difference of a Dirichlet Eta function and its partial sum.

Theorem 2.8. *The following equality holds for $|\tau| < \frac{2}{1+\sqrt{5}}, m \geqslant 1$*

$$\sum_{k=0}^{+\infty}(-\tau)^k F_k\left(\eta(2m)-\eta_k(2m)\right) = \frac{\left(\frac{1}{2}\right)^{2m+1}(4^m-2)\tau\zeta(2m)\Gamma(2m+1)}{m((\tau-1)\tau-1)\Gamma(2m)} \tag{2.15}$$

$$+\frac{\sqrt{5}-1}{\sqrt{5}(2\tau+\sqrt{5}-1)}\operatorname{Li}_{2m}\left(\frac{2\tau}{\sqrt{5}-1}\right) - \frac{1+\sqrt{5}}{\sqrt{5}(1+\sqrt{5}-2\tau)}\operatorname{Li}_{2m}\left(\frac{-2\tau}{1+\sqrt{5}}\right).$$

Proof. First rewriting the difference of the two Dirichlet Eta functions using the integral representation given in the Lemma, we get

$$\sum_{k=0}^{+\infty}(-\tau)^k F_k(\eta(2m)-\eta_k(2m)) = \sum_{k=0}^{+\infty}(-\tau)^k F_k\frac{(-1)^{k+1}}{\Gamma(2m)}\int_0^1 \frac{y^k\ln^{2m-1}(y)}{1+y}dy$$

$$= -\frac{1}{\Gamma(2m)} \int_0^1 \sum_{k=0}^{+\infty} F_k \frac{(\tau y)^k \ln^{2m-1}(y)}{y+1} dy.$$

Using the generating function for the Fibonacci sequence [10], namely

$$\sum_{k=0}^{+\infty} \tau^k F_k = \frac{\tau}{1-\tau-\tau^2}.$$

We obtain the following

$$-\frac{1}{\Gamma(2m)} \int_0^1 \sum_{k=0}^{+\infty} F_k \frac{(\tau y)^k \ln^{2m-1}(y)}{y+1} dy$$
$$= -\frac{1}{\Gamma(2m)} \int_0^1 \frac{\ln^{2m-1}(y)}{1+y} \frac{\tau y}{1-\tau y-\tau^2 y^2} dy.$$

In order to solve the given integral, we proceed with doing partial fraction decomposition on the denominator, from which we obtain the following

$$\frac{-y\tau}{\tau^2\left(y+\frac{1+\sqrt{5}}{2\tau}\right)\left(y-\frac{\sqrt{5}-1}{2\tau}\right)(1+y)} = -\frac{4\tau}{(1+\sqrt{5}-2\tau)(2\tau+\sqrt{5}-1)(y+1)} - \frac{2(\sqrt{5}-1)\tau}{\sqrt{5}(2\tau+\sqrt{5}-1)(2\tau y-\sqrt{5}+1)} + \frac{2\tau(1+\sqrt{5})}{\sqrt{5}(1+\sqrt{5}-2\tau)(2\tau y+\sqrt{5}+1)}.$$

Using it in our case, we obtain

$$\frac{1}{\Gamma(2m)} \int_0^1 \frac{4\ln^{2m-1}(y)\tau}{(1+\sqrt{5}-2\tau)(2\tau+\sqrt{5}-1)(y+1)} dy$$
$$+\frac{1}{\Gamma(2m)} \int_0^1 \frac{2\ln^{2m-1}(y)(\sqrt{5}-1)\tau}{\sqrt{5}(2\tau+\sqrt{5}-1)(2\tau y-\sqrt{5}+1)} dy$$
$$-\frac{1}{\Gamma(2m)} \int_0^1 \frac{2\ln^{2m-1}(y)\tau(1+\sqrt{5})}{\sqrt{5}(1+\sqrt{5}-2\tau)(2\tau y+\sqrt{5}+1)} dy.$$

Each of these integrals can be solved by expanding the denominator, namely the part with y, the first one reduces to an alternating Zeta function which can be summed using the known formula

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = 2^{-s}(2^s-2)\zeta(s).$$

The second and third integral are solved in a similar way. In the second integral, one should factor $\frac{1}{1-\sqrt{5}}$ and expand the denominator into a geometric series. We will solve the second integral for the educational purposes, the third one is analogous. Proceeding with the integral, we get

$$\int_0^1 \frac{\ln^{2m-1}(y)}{(2\tau y-\sqrt{5}+1)} dy = \int_0^1 \frac{\ln^{2m-1}(y)}{1-\sqrt{5}} \sum_{n=0}^{+\infty} \left(\frac{2\tau y}{1-\sqrt{5}}\right)^n (-1)^n dy$$
$$= \sum_{n=0}^{+\infty} \left(\frac{2\tau}{1-\sqrt{5}}\right)^n (-1)^n \frac{1}{1-\sqrt{5}} \int_0^1 \ln^{2m-1}(y) y^n dy$$
$$= -\Gamma(2m) \sum_{n=0}^{+\infty} \frac{\left(\frac{2\tau}{1-\sqrt{5}}\right)^n}{(n+1)^{2m}} (-1)^n = \frac{\Gamma(2m)(1-\sqrt{5})}{2\tau} \operatorname{Li}_{2m}\left(\frac{2\tau}{\sqrt{5}-1}\right).$$

Summing it all up gives us the desired result. □

Corollary 2.9. *Setting $\tau = \frac{1}{2}, m = 1$ in the previously derived Theorem, we obtain the equality*

$$\sum_{k=0}^{+\infty} \frac{(-1)^k}{2^k} F_k \left(\eta(2) - \eta_k(2)\right) \tag{2.16}$$
$$= \frac{1}{5}\left(\sqrt{5}-1\right) \operatorname{Li}_2\left(\frac{1}{\sqrt{5}-1}\right) - \frac{1}{5}\left(\sqrt{5}+1\right) \operatorname{Li}_2\left(-\frac{1}{\sqrt{5}+1}\right) - \frac{\pi^2}{30}.$$

Corollary 2.10. *Setting $\tau = \frac{1+\sqrt{5}}{2^{p+1}}$ and $\tau = \frac{1-\sqrt{5}}{2^{p+1}}, p \geqslant 2$ respectfully in the derived Theorem, we obtain*

$$\sum_{k=0}^{+\infty}(-1)^k F_k \left(\frac{1+\sqrt{5}}{2^{p+1}}\right)^k \left(\eta(2m) - \eta_k(2m)\right) = \frac{2^p \operatorname{Li}_{2m}\left(-2^{-p}\right)}{\sqrt{5}\left(1-2^p\right)} \tag{2.17}$$
$$-\frac{\left(\sqrt{5}-1\right) 2^p \operatorname{Li}_{2m}\left(2^{-p-1}\left(\sqrt{5}+3\right)\right)}{\left(\sqrt{5}-5\right) 2^p - \sqrt{5} - 5} - \frac{\left(\sqrt{5}+1\right)\left(4^m-2\right) 2^{p-2m}\zeta(2m)}{\left(2^p-1\right)\left(2^{p+1}+\sqrt{5}+3\right)}$$

$$\sum_{k=0}^{+\infty}(-1)^k F_k \left(\frac{1-\sqrt{5}}{2^{p+1}}\right)^k \left(\eta(2m) - \eta_k(2m)\right) = \frac{2^p \operatorname{Li}_{2m}\left(-2^{-p}\right)}{\sqrt{5}\left(2^p-1\right)} \tag{2.18}$$
$$-\frac{\left(\sqrt{5}+1\right) 2^p \operatorname{Li}_{2m}\left(-2^{-p-1}\left(\sqrt{5}-3\right)\right)}{\left(\sqrt{5}+5\right) 2^p - \sqrt{5} + 5} - \frac{\left(\sqrt{5}-1\right)\left(4^m-2\right) 2^{p-2m}\zeta(2m)}{\left(2^p-1\right)\left(-2^{p+1}+\sqrt{5}-3\right)}.$$

Subtracting the second from the first series and multiplying by $\frac{1}{\sqrt{5}}$ we get the following

$$\sum_{k=0}^{+\infty}(-1)^k F_k \frac{\left(\left(\frac{1+\sqrt{5}}{2^{p+1}}\right)^k - \left(\frac{1-\sqrt{5}}{2^{p+1}}\right)^k\right)}{\sqrt{5}} \left(\eta(2m) - \eta_k(2m)\right).$$

Taking $\frac{1}{2^p}$ in the front, we are left with

$$\sum_{k=0}^{+\infty}(-1)^k F_k \frac{\left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right)}{\sqrt{5}} \frac{\left(\eta(2m) - \eta_k(2m)\right)}{2^{kp}}.$$

Observing that the expression inside the brackets is a Fibonacci sequence, $F_k = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k}{\sqrt{5}}$, we obtain the following equality

$$\sum_{k=0}^{+\infty}(-1)^k F_k^2 \frac{\eta(2m) - \eta_k(2m)}{2^{kp}} \tag{2.19}$$
$$= \frac{1}{\sqrt{5}}\Bigg(\frac{2^p \operatorname{Li}_{2m}\left(-2^{-p}\right)}{\sqrt{5}\left(1-2^p\right)} - \frac{2^p \operatorname{Li}_{2m}\left(-2^{-p}\right)}{\sqrt{5}\left(2^p-1\right)}$$
$$+\frac{1}{\sqrt{5}} \frac{\left(\sqrt{5}+1\right) 2^p \operatorname{Li}_{2m}\left(-2^{-p-1}\left(\sqrt{5}-3\right)\right)}{\left(\sqrt{5}+5\right) 2^p - \sqrt{5} + 5}\Bigg)$$

$$+\frac{1}{\sqrt{5}}\left(-\frac{\left(\sqrt{5}-1\right)2^p\operatorname{Li}_{2m}\left(2^{-p-1}\left(\sqrt{5}+3\right)\right)}{\left(\sqrt{5}-5\right)2^p-\sqrt{5}-5}+\frac{\left(\sqrt{5}-1\right)\left(4^m-2\right)2^{p-2m}\zeta(2m)}{\left(2^p-1\right)\left(-2^{p+1}+\sqrt{5}-3\right)}\right)$$
$$-\frac{1}{\sqrt{5}}\left(\frac{\left(\sqrt{5}+1\right)\left(4^m-2\right)2^{p-2m}\zeta(2m)}{\left(2^p-1\right)\left(2^{p+1}+\sqrt{5}+3\right)}\right).$$

Setting $p=2, m=1$ in the expression we got, we obtain

$$\sum_{k=0}^{+\infty}(-1)^kF_k^2\frac{\eta(2)-\eta_k(2)}{2^{2k}} \tag{2.20}$$
$$=-\frac{1}{15}8\operatorname{Li}_2\left(-\frac{1}{4}\right)$$
$$+\frac{2\left(\left(11\sqrt{5}-5\right)\operatorname{Li}_2\left(\frac{1}{8}\left(\sqrt{5}+3\right)\right)+\left(11\sqrt{5}+5\right)\operatorname{Li}_2\left(\frac{1}{8}\left(3-\sqrt{5}\right)\right)\right)}{145\sqrt{5}}-\frac{5\pi^2}{261}.$$

As a consequence, we can obtain many series of the Fibonacci squared-zeta type using the formula above. The similar procedure of obtaining the generating function for $F_k^2\frac{\eta(2m)-\eta_k(2m)}{2^{kp}}$ can be done by using Theorem 6, introducing a substitution $\tau=-\tau$ and then doing the same procedure as above with introducing the substitutions $\tau=\frac{1+\sqrt{5}}{2^{p+1}}$ and $\tau=\frac{1-\sqrt{5}}{2^{p+1}}$.

The following Theorem is beautiful in particular, because it connects $\zeta(3),\pi,\ln(2)$ which we all know are irrational numbers.

Theorem 2.11. *The following equality holds*

$$\sum_{k=1}^{+\infty}\left(\eta(2)-\eta_{k-1}(2)\right)\left(\overline{H_k}(-1)^{k+1}+\frac{1}{2}\left(H_{\frac{k-1}{2}}-H_{\frac{k-2}{2}}\right)\right)=\frac{5\zeta(3)}{8}+\frac{\pi^2}{8}\ln(2). \tag{2.21}$$

Proof. We will separate this proof in two problems of evaluating sums. Therefore, we focus on to the first one. Let us first consider

$$\sum_{k=1}^{+\infty}(\eta(2)-\eta_{k-1})\overline{H_k}(-1)^{k+1}.$$

We will be using the Lemma concerning the difference of the Dirichlet Eta function and its partial sum, which gives us the following

$$\eta(2)-\eta_{k-1}(2)=(-1)^k\int_0^1\frac{\tau^{k-1}\ln(\tau)}{1+\tau}d\tau.$$

We will also need the generating function of skew harmonic numbers 1.4

$$\sum_{k=1}^{+\infty}(\eta(2)-\eta_{k-1})\overline{H_k}(-1)^{k+1}=\sum_{k=1}^{+\infty}(-1)^{k+1}(-1)^k\overline{H_k}\int_0^1\frac{\tau^{k-1}\ln(\tau)}{1+\tau}d\tau$$
$$=-\int_0^1\frac{\ln(\tau)}{1+\tau}\sum_{k=1}^{+\infty}\overline{H_k}\tau^{k-1}d\tau=-\int_0^1\frac{\ln(\tau)\ln(1+\tau)}{\tau(1+\tau)(1-\tau)}d\tau.$$

Here we employ the partial fractions, $\frac{1}{\tau(1+\tau)(1-\tau)} = -\frac{1}{\tau} + \frac{1}{2(1+\tau)} + \frac{1}{2(\tau-1)}$. Using it in our case, we proceed

$$-\int_0^1 \frac{\ln(\tau)\ln(1+\tau)}{\tau(1+\tau)(1-\tau)}d\tau = \int_0^1 \ln(1+\tau)\ln(\tau)\left(-\frac{1}{\tau} + \frac{1}{2(\tau+1)} + \frac{1}{2(\tau-1)}\right)d\tau$$

$$= -\int_0^1 \frac{\ln(1+\tau)\ln(\tau)}{\tau}d\tau + \frac{1}{2}\int_0^1 \frac{\ln(\tau)\ln(1+\tau)}{(1+\tau)}d\tau - \frac{1}{2}\int_0^1 \frac{\ln(1+\tau)\ln(\tau)}{1-\tau}d\tau.$$

The second integral has already been solved in Theorem 2 2.12 while the first one can be easily shown to be equal to $\frac{3\zeta(3)}{4}$. The third integral can be solved as an indefinite integral, but it is not easy to evaluate the limits of it, even the `Mathematica` takes some time to evaluate it. Therefore, we will make the substitution $y = \int_0^1 \frac{\ln(1+\tau)\ln(\tau)}{1-\tau}d\tau$. All together, we get

$$\sum_{k=1}^{+\infty}(\eta(2) - \eta_{k-1})\overline{H_k}(-1)^{k+1} = \frac{3\zeta(3)}{4} - \frac{\zeta(3)}{16} - \frac{1}{2}y = \frac{11\zeta(3)}{16} - \frac{y}{2}.$$

Now we put our focus on to the second sum, namely

$$\sum_{k=1}^{+\infty}(\eta(2) - \eta_{k-1}(2))\frac{1}{2}\left(H_{\frac{k-1}{2}} - H_{\frac{H_{k-2}}{2}}\right).$$

We will use the same integral representation for the difference of the Dirichlet Eta functions and the following integral representation

$$\int_0^1 \frac{\tau^n}{1+\tau}d\tau = \frac{1}{2}\left(H_{\frac{n}{2}} - H_{\frac{n-1}{2}}\right).$$

Proceeding, we get

$$\sum_{k=1}^{+\infty}(\eta(2) - \eta_{k-1}(2))\frac{1}{2}\left(H_{\frac{k-1}{2}} - H_{\frac{H_{k-2}}{2}}\right) = \sum_{k=1}^{+\infty}(-1)^k \int_0^1 \frac{\ln(\tau)\tau^{k-1}}{1+\tau}d\tau \int_0^1 \frac{y^{k-1}}{1+y}dy$$

$$= -\int_0^1 \frac{\ln(\tau)}{1+\tau}\left(\int_0^1 \sum_{k=1}^{+\infty}\frac{(-\tau y)^{k-1}}{1+y}dy\right)d\tau$$

$$= -\int_0^1 \frac{\ln(\tau)}{1+\tau}\left(\int_0^1 \frac{1}{(1+y)(1+\tau y)}dy\right)d\tau$$

$$= \int_0^1 \frac{\ln(\tau)}{(1-\tau)(1+\tau)}(\ln(1+\tau) - \ln(2))d\tau$$

$$= \int_0^1 \frac{\ln(\tau)\ln(1+\tau)}{(1-\tau)(1+\tau)}d\tau - \ln(2)\int_0^1 \frac{\ln(\tau)}{(1-\tau)(1+\tau)}d\tau.$$

Using partial fractions on both of these integrals, we get

$$= \frac{1}{2}\int_0^1 \frac{\ln(\tau)\ln(1+\tau)}{1+\tau}d\tau + \frac{1}{2}\int_0^1 \frac{\ln(\tau)\ln(1+\tau)}{1-\tau}d\tau - \ln(2)\int_0^1 \frac{\ln(\tau)}{(1-\tau)(1+\tau)}d\tau$$

$$= -\frac{\zeta(3)}{16} + \frac{y}{2} + \frac{\pi^2\ln(2)}{8}.$$

The first integral has been solved in Theorem 2 2.12 while the third integral can be solved by using the partial fractions and by solving each integral as an indefinite integral. Adding then the two integrals and taking the limit we get the result given.

Adding the two obtained results, we obtain the stated equality.

□

Theorem 2.12. *The following equality holds for $m \in \mathbb{N}$*

$$\sum_{k=0}^{+\infty}(\eta(2m)-\eta_k(2m))\left((\overline{H_k}-\ln(2))(-1)^k+\frac{1}{2}(H_{\frac{k}{2}}-H_{\frac{k-1}{2}})\right)=0, \tag{2.22}$$

$$\sum_{k=0}^{+\infty}(\eta(2m+1)-\eta_k(2m+1))\left((\overline{H_k}-\ln(2))(-1)^k+\frac{1}{2}(H_{\frac{k}{2}}-H_{\frac{k-1}{2}})\right)=0. \tag{2.23}$$

Proof. The proof is similar to the one given in Theorem 5, with addition to using the integral representation $\overline{H_k}-\ln(2)=-\int_0^1\frac{(-\tau)^k}{1+\tau}d\tau$. Proceeding with using the difference of Dirichlet Eta functions and the given integral representation, we obtain

$$\begin{aligned}&\sum_{k=0}^{+\infty}(\eta(2m)-\eta_k(2m))\left(\overline{H_k}-\ln 2\right)(-1)^k\\&=\int_0^1\frac{\ln^{2m-1}(\tau)}{1+\tau}\left(\int_0^1\frac{1}{(1+\tau y)(1+y)}dy\right)d\tau\\&=-\int_0^1\frac{\ln^{2m-1}(\tau)}{1+\tau}\left(\frac{\ln(1+\tau)-\ln(2)}{1-\tau}\right)d\tau.\end{aligned}$$

Which we have already solved in Theorem 5. Using the same approach using the integral representation given in Theorem 5, we obtain that

$$\begin{aligned}&\sum_{k=0}^{+\infty}(\eta(2m)-\eta_k(2m))\frac{1}{2}(H_{\frac{k}{2}}-H_{\frac{k-1}{2}})\\&=-\int_0^1\frac{\ln^{2m-1}(\tau)}{1+\tau}\left(\int_0^1\frac{1}{(1+\tau y)(1+y)}dy\right)d\tau.\end{aligned}$$

Which can be seen to be of the opposite sign, therefore adding them gives us the desired equality.
The proof of the second equality is equivalent.

□

3. Conclusion

In this paper various series concerning the remainder of the Dirichlet Eta function have been obtained. All the analytical evaluations have been verified using `Mathematica`. In this paper we evaluated various Dirichlet Eta series of the form found in the new book [20]. We also proved a useful integral representation of the tail of the Dirichlet Eta function. Moreover, it was shown how generating functions of the known sums can be paired up with the integral representation of the remainder of the Dirichlet Eta function to produce further results. Questions arise whether tails of other special functions can be found in terms of the integral representations.

References

1. Abramowitz, M., Stegun, I.A. (1964). *Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables*, Dover Books on Mathematics, https://doi.org/10.1119/1.15378
2. Alzer, H., Kwong, M. K. (2015), On the concavity of Dirichlet's eta function and related functional inequalities, J. Number Theory 151 , 172–196. https://doi.org/10.1016/j.jnt.2014.12.009
3. Bonar D.D, Kourym, M.J. (2006). *Real Infinite Series*, MAA, Washington DC, 2006.
4. Choi, J., Srivastava, H.M., Quine, J.R. (1995). Some series involving the zeta function. Bulletin of the Australian Mathematical Society , Volume 51(3), 383-393.https://doi.org/10.1017/S0004972700014210
5. Davis, H.T. (2015). *The Summation of Series (Dover Books on Mathematics)*, Mineola, New York: Dover Publications.
6. Edwards, M.H. (2001). *Riemann's Zeta Function*, Dover Publications; Dover Ed edition.
7. Euler, L. (1743). De Inventione Integralium, si Post Integrationem Variabili Quantitati Determinatus Valor Tribuatur; Miscellanea Berolinensia; University of the Pacific: Stockton, CA, USA, Volume 7, pp. 129–171.
8. Fabiano, N. (2020). Zeta function and some of its properties, *Vojnotehnički glasnik/Military Technical Courier*, 68(4),pp.895-906.https://doi.org/10.5937/vojtehg68-28535
9. Furdui, O. (2005). Two surprising series with harmonic numbers and the tail of zeta(2), Gazeta Matematica, Seria A, 33(112)(1-2),1-8.
10. Grimaldi, R. (2012). *Fibonacci and Catalan Numbers: An Introduction*, Wiley.DOI:10.1002/9781118159743
11. Hirschman, I.I. (2014), *Infinite Series*, Dover Publications, Reprint edition.
12. Hu, S., Kim, M. S. (2022). On the Stieltjes constants and gamma functions with respect to alternating Hurwitz zeta functions, Journal of Mathematical Analysis and Applications,Volume 509(1).https://doi.org/10.1016/j.jmaa.2021.125930
13. Ivić ,A. (1983). *Topics in recent zeta function theory*, Publications Mathéma- tiques d'Orsay, Orsay.
14. Knopp, K. (1990). *Theory and Applications of Infinite Series*, Dover Publications, New York.
15. Kobayashi, M. (2021).New relations on zeta and L functions.
16. Lewin L. (1981). *Polylogarithms and associated functions*, Elsevier, North Holland.https://doi.org/10.1137/1025037
17. Malmsten, S.J. (1865). Om Definita Integraler Mellan Imaginära Gränsor; Norstedt and Söner: Stockholm, Sweden.
18. Milgram, M. S. (2013). Integral and series representations of Riemann's zeta function and Dirichlet's eta function and a medley of related results, J. Math.https://doi.org/10.1155/2013/181724
19. Olaikan, A. S. (2021). An Introduction to the Harmonic Series and Logarithmic Integrals; Ali Shadhar Olaikhan, Amazon Digital Services LLC KDP Print US: Seattle, WA, USA.
20. Sîntămărian,A.,Furdui,O. (2021).*Sharpening Mathematical Analysis Skills*, Springer.https://doi.org/10.1007/978-3-030-77139-3
21. Sondow, J. (2003). Zeros of the alternating zeta function on the line R(s) = 1, Amer. Math. Monthly 110 , 435–437.https://doi.org/10.2307/3647831
22. Sondow, J. (2005). Double integrals for Euler's constant and $\ln \frac{4}{\pi}$ and an analog of Hadjicostas's formula, Amer. Math. Monthly 112 , 61–65.https://doi.org/10.2307/30037385

23. Srivastava, H. M., Choi, J. (2001). Series associated with the Zeta and related functions, Springer Science+Business Media.
24. Srivastava, H. M. (1989). A unified presentation of certain classes of series of the Riemann zeta function. Riv. Mat. Univ. Parma (4)14 , 1–23.
25. Stojiljković ,V. (2021). Some Series Associated with Central Binomial Coefficients and Harmonic Numbers, *Octogon Mathematical Magazine* , 29(2), 749-763.https://doi.org/10.6084/m9.figshare.19337258.v1
26. Stojiljković, V., Fabiano, N., Šešum Čavić, V. (2022). Harmonic series with polylogarithmic functions. Vojnoteh. Glas. Tech. Cour., 70(1), 3–61.DOI:10.5937/vojtehg70-35148
27. Stojiljković, V.; Fabiano, N.; Pantović, M.; Radojević, S.; Radenović, S.; Šešum Čavić, V. (2022). Various Series Related to the Polylogarithmic Function. Axioms, 11, 174.https://doi.org/10.3390/axioms11040174
28. Stojiljkovic, V. (2022). Generalized order n fractional integrals. Innovative Journal of Mathematics (IJM), 1(2), 71–81. https://doi.org/10.55059/ijm.2022.1.2/27
29. Stojiljkovic, V. (2022). Various Series Concerning the Zeta Function. International Journal of Emerging Multidisciplinaries: Mathematics, 1(2), 23–34.https://doi.org/10.54938/ijemdm.2022.01.2.65

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