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CHARACTERISTIC SUBMULTIGROUPS OF A MULTIGROUP

A.M. IBRAHIM[1] AND P.A. EJEGWA[2*]

Abstract. In this paper, the notion of characteristic submultigroups is introduced and some of its homomorphic properties are presented. It is shown that every characteristic submultigroup of a multigroup is normal. Some related results are established with respect to factor multigroups and normal submultigroups.

1. Introduction

The idea of multigroups was proposed by Nazmul *et al.* [10] as algebraic structures of multisets that generalized the concept of groups. The idea is consistent with other non-standard groups in [1] [4], [9], [11], [13], [15], [16], etc. Although other works in [3], [5], [8], [12], [14], [19], [20] earlier used the term multigroup as an extension of group theory (with divergent views), the notion of multigroup in [10] is quite accepted because it agrees with other non-classical groups and defined over multiset (see [7], [17], [18], [21] for details of multisets). Further studies on the concept of multigroups via multisets can be found in [2], [6].

In this paper, we introduce the notion of characteristic submultigroups, present some of its homomorphic properties and obtain some related results. We show that every characteristic submultigroup of a multigroup is normal.

2. preliminaries

Definition 2.1 ([17]). Let $X = \{x_1, x_2, ..., x_n, ...\}$ be a set. A multiset A over X is a cardinal-valued function, that is, $C_A : X \to \mathbb{N}$ such that for $x \in Dom(A)$ implies $A(x)$ is a cardinal and $A(x) = C_A(x) > 0$, where $C_A(x)$ denoted the number of times an object x occur in A. Whenever $C_A(x) = 0$, implies $x \notin Dom(A)$. The set X is called the ground or generic or root set of the class of all multisets (for short, msets) containing objects from X.
The set of all multisets over X is denoted by $MS(X)$.

Definition 2.2 ([18]). Let $A, B \in MS(X)$, A is called a submultiset of B written as $A \subseteq B$ if $C_A(x) \leq C_B(x) \forall x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper submultiset of B and denoted as $A \subset B$. A multiset is called the parent in relation to its submultiset.

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[2] Corresponding author.
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Definition 2.3. Let X be a group. A multiset G is called a multigroup of X if the count function of G, that is, $C_G : X \to \mathbb{N}$ satisfies the following conditions:

(i) $C_G(xy) \geq C_G(x) \wedge C_G(y) \forall x, y \in X$,
(ii) $C_G(x^{-1}) = C_G(x) \forall x \in X$.

By implication, a multiset G is called a multigroup of a group X if

$$C_G(xy^{-1}) \geq C_G(x) \wedge C_G(y), \forall x, y \in X.$$

It follows immediately from the definition that,

$$C_G(e) \geq C_G(x) \forall x \in X,$$

where e is the identity element of X. We denote the set of all multigroups of X by $MG(X)$.

Remark 2.4. Every multigroup is a multiset but the converse is not necessarily true.

Definition 2.5. Let $A \in MG(X)$. A nonempty submultiset B of A is called a submultigroup of A denoted by $B \sqsubseteq A$ if B form a multigroup. A submultigroup B of A is a proper submultigroup denoted by $B \sqsubset A$, if $B \sqsubseteq A$ and $A \neq B$.

Definition 2.6. Let $A, B \in MG(X)$ such that $A \subseteq B$. A multigroup A is called a normal submultigroup of B if

$$C_A(xyx^{-1}) \geq C_A(y) \forall x, y \in X.$$

Definition 2.7 ([10])**.** Let $A \in MG(X)$. Then A is said to be abelian if for all $x, y \in X$, $C_A(xy) = C_A(yx)$.

Definition 2.8 ([10])**.** Let $A \in MG(X)$. Then the sets A_* and A^* are defined as

$$A_* = \{x \in X \mid C_A(x) > 0\}$$

and

$$A^* = \{x \in X \mid C_A(x) = C_A(e)\}$$

Proposition 2.9 ([10])**.** *Let $A \in MG(X)$. Then A_* and A^* are subgroups of X.*

Definition 2.10 ([10])**.** Let $A, B \in MG(X)$. Then the product of A and B denoted by $A \circ B$ is defined as

$$C_{A \circ B}(x) = \bigvee \{C_A(y) \wedge C_B(z) \mid x = yz, \forall y, z \in X\}.$$

Definition 2.11. Let X be a group. For any submultigroup A of a multigroup G of X, the submultiset yA of G for $y \in G$ defined by

$$C_{yA}(x) = C_A(y^{-1}x) \forall x \in A$$

is called the left comultiset of A. Similarly, the submultiset Ay of G for $y \in G$ defined by

$$C_{Ay}(x) = C_A(xy^{-1}) \forall x \in A$$

is called the right comultiset of A.
It is notice that, $xA = yA$, $Ay \circ Az = Ayz$ and $yA \circ zA = yzA \, \forall x, y, z \in X$.

Definition 2.12. Let A be a multigroup of a group X and B a normal submultigroup of A. Then the set of right/left comultisets of B with the property $xB \circ yB = xyB \, \forall x, y \in X$ form a multigroup called factor or quotient multigroup of A determined by B, denoted as A/B.

Definition 2.13. Let X and Y be groups and let $f : X \to Y$ be an homomorphism. Let A and B be multisets over X and Y respectively. Then

(i) the image of A under f, denoted by $f(A)$, is a multiset of Y defined by

$$C_{f(A)}(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} C_A(x), & f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

for each $y \in Y$.

(ii) the inverse image of B under f, denoted by $f^{-1}(B)$, is a multiset of X defined by

$$C_{f^{-1}(B)}(x) = C_B(f(x)) \forall x \in X.$$

Definition 2.14. Let X and Y be groups and let $A \in MG(X)$ and $B \in MG(Y)$, respectively.

(i) A homomorphism f from X to Y is called a weak homomorphism from A to B if $f(A) \subseteq B$. If f is a weak homomorphism from A to B, then we say that, A is weakly homomorphic to B denoted by $A \sim B$.

(ii) An isomorphism f from X to Y is called a weak isomorphism from A to B if $f(A) \subseteq B$. If f is a weak isomorphism from A to B, then we say that, A is weakly isomorphic to B denoted by $A \simeq B$.

(iii) A homomorphism f from X to Y is called a homomorphism from A to B if $f(A) = B$. If f is a homomorphism from A to B, then A is homomorphic to B denoted by $A \approx B$.

(iv) An isomorphism f from X to Y is called an isomorphism from A to B if $f(A) = B$. If f is an isomorphism from A to B, then A is isomorphic to B denoted by $A \cong B$.

3. Main results

A subgroup Y of a group X is a characteristic subgroup if $Y^\theta = Y$ for every automorphism θ of X, where Y^θ denotes $\theta(Y)$. Now, we define the analogue of this notion in multigroup setting.

Definition 3.1. Let $A, B \in MG(X)$ such that $A \subseteq B$. Then A is called a characteristic (fully invariant) submultigroup of B if

$$C_{A^\theta}(x) = C_A(x) \, \forall x \in X$$

for every automorphism, θ of X. That is, $\theta(A) \subseteq A$ for every $\theta \in Aut(X)$.

Definition 3.2. Let A be a multigroup of a group X and θ a function from X into itself. Define the multiset A^θ of X by

$$C_{A^\theta}(x) = C_A(x^\theta), \text{ where } x^\theta = \theta(x) = x \, \forall x \in X.$$

Proposition 3.3. *Let $A, B \in MG(X)$ such that $A \subseteq B$. If θ is a homomorphism of X into itself, then A^θ is a multigroup of X.*

Proof. Let $x, y \in X$. Then $C_{A^\theta}(xy) = C_A((xy)^\theta) = C_A(x^\theta y^\theta)$ since θ is a homomorphism. Since A is a multigroup of X, we have

$$C_A(x^\theta y^\theta) \geq C_A(x^\theta) \wedge C_A(y^\theta) = C_{A^\theta}(x) \wedge C_{A^\theta}(y).$$

Thus,

$$C_{A^\theta}(xy) \geq C_{A^\theta}(x) \wedge C_{A^\theta}(y).$$

Also,

$$C_{A^\theta}(x^{-1}) = C_A((x^{-1})^\theta) = C_A((x^\theta)^{-1}) = C_A(x^\theta) = C_{A^\theta}(x).$$

Hence, A^θ is a multigroup of X. □

Proposition 3.4. *Let X be a group. Every characteristic submultigroup of a multigroup B of X is normal.*

Proof. Let $x, y \in X$. Suppose that A is a characteristic submultigroup of B. To prove that A is a normal submultigroup of B, we have to show that

$$C_A(xyx^{-1}) \geq C_A(y) \forall x, y \in X.$$

Now, since A is a characteristic submultigroup of B, we have

$$\begin{aligned} C_{A^\theta}(xyx^{-1}) &= C_A((xyx^{-1})^\theta) = C_A(x^\theta(yx^{-1})^\theta) \\ &= C_{(x^\theta)^{-1}A}(y^\theta(x^{-1})^\theta) \\ &= C_{A(x^\theta)^{-1}}(y^\theta(x^\theta)^{-1}) \\ &= C_A(y^\theta(x^\theta)^{-1}(x^\theta)) \\ &\geq C_A(y^\theta) = C_{A^\theta}(y) \end{aligned}$$

$\Rightarrow C_{A^\theta}(xyx^{-1}) \geq C_{A^\theta}(y) \forall x, y \in X$. Hence, A^θ is normal. □

Remark 3.5. Let $A, B, C \in MG(X)$ such that $C \subseteq B \subseteq A$.

(i) If C is a characteristic submultigroup of B and B is a characteristic submultigroup of A, then C is a characteristic submultigroup of A.
(ii) If C is a characteristic submultigroup of B and B is a normal submultigroup of A, then C is a normal submultigroup of A.

Theorem 3.6. *Let A be a normal submultigroup of $B \in MG(X)$ and let θ be a homomorphism of X into itself which leaves invariant the subgroup A^*. Then θ induces a homomorphism $\bar{\theta}$ of B/A into itself define by*

$$\bar{\theta}(xA) = \theta(x)A \, \forall x \in X.$$

Proof. First, we show that $\bar{\theta}$ is well defined. To show this, suppose $x, y \in X$ such that $xA = yA$. Then it suffices to prove that

$$\theta(x)A = \theta(y)A.$$

Since $xA = yA$, we have

$$C_{xA}(x) = C_{yA}(x) \, and \, C_{xA}(y) = C_{yA}(y)$$

$\Rightarrow$

$$C_A(e) = C_A(y^{-1}x) \, and \, C_A(x^{-1}y) = C_A(e)$$

$\Rightarrow$

$$C_A(y^{-1}x) = C_A(x^{-1}y) = C_A(e)$$

$\Rightarrow$

$$y^{-1}x, x^{-1}y \in A^*.$$

Since by hypothesis, $\theta(A^*) = A^*$, we say $\theta(y^{-1}x)$ and $\theta(x^{-1}y)$ also belong to A^*. Thus, we have

$$C_A(\theta(y^{-1}x)) = C_A(\theta(x^{-1}y)).$$

Now, let $g \in X$. Then

$$\begin{aligned} C_{\theta(x)A}(g) &= C_A(\theta(x^{-1})g) \\ &= C_A(\theta(x^{-1})\theta(y)\theta(y^{-1})g) \\ &\geq C_A(\theta(x^{-1})\theta(y)) \wedge C_A(\theta(y^{-1})g) \\ &= C_A(\theta(x^{-1}y)) \wedge C_A(\theta(y^{-1})g) \\ &= C_{\theta(y)A}(g). \end{aligned}$$

Thus, $C_{\theta(x)A}(g) \geq C_{\theta(y)A}(g)$. Similarly, $C_{\theta(y)A}(g) \geq C_{\theta(x)A}(g)$. Since $g \in X$ is arbitrary, we have

$$C_{\theta(x)A}(g) = C_{\theta(y)A}(g),$$

that is, $\theta(x)A = \theta(y)A$. Therefore, $\bar{\theta}$ is well defined. Next, we prove that $\bar{\theta}$ is a homomorphism. Let $x, y \in X$. Since θ is a homomorphism, $\theta(xy) = \theta(x)\theta(y)$. Thus, $\theta(xy)A = \theta(x)A\theta(y)A$. Hence, $\bar{\theta}(xyA) = \theta(x)A\theta(y)A$. So,

$$\bar{\theta}(xAyA) = \bar{\theta}(xA)\bar{\theta}(yA).$$

Therefore, $\bar{\theta}$ is a homomorphism. □

Corollary 3.7. *With the same hypothesis as in Theorem 3.6, the homomorphism $\bar{\theta}$ is an automorphism if θ is an automorphism and X is finite.*

Proof. Since X has finite order, it is easy to see that θ has finite order. Suppose that θ has order n. Then $\theta^n = i$, where i depicts identity map. Now, since by Theorem 3.6, $\bar{\theta}$ is a homomorphism, it remains to prove that θ is one-to-one. Then, let $x, y \in X$ such that $\bar{\theta}(xA) = \bar{\theta}(yA)$. Then $\theta(x)A = \theta(y)A$. Thus,

$$\bar{\theta}(\theta(x)A) = \bar{\theta}(\theta(y)A).$$

This implies by the definition of $\bar{\theta}$ that $\theta^2(x)A = \theta^2(y)A$. Continue in this process, we obtain

$$i(x)A = \theta^n(x)A = \theta^n(y)A = i(y)A$$

and so, $xA = yA$. Hence, $\bar{\theta}$ is one-to-one. The proof is complete. □

Corollary 3.8. *With the same hypothesis as in Theorem 3.6, the function θ is an automorphism of X if $\bar{\theta}$ is an automorphism and $A^* = \{e\}$.*

Proof. Let $x, y \in X$ such that $\theta(x) = \theta(y)$. Then it follows that $\theta(x)A = \theta(y)A$. That is, $\bar{\theta}(xA) = \bar{\theta}(yA)$. Hence, $xA = yA$ since $\bar{\theta}$ is one-to-one by the hypothesis. Thus,

$$C_{xA}(y) = C_{yA}(y) \Rightarrow C_A(x^{-1}y) = C_A(e).$$

Therefore, $x^{-1}y \in A^*$ and so, $x^{-1}y = e$ since $A^* = \{e\}$ by the hypothesis. Thus,

$$C_A(x) = C_A(y) \Rightarrow x = y.$$

Hence, θ is one-to-one. The result follows. □

Theorem 3.9. *Let $B \in MG(X)$ and A a normal submultigroup of B. If θ is an automorphism of X such that $A^\theta = A$, then θ induces an automorphism $\bar{\theta}$ of B/A define by $\bar{\theta}(xA) = \theta(x)A \, \forall x \in X$.*

Proof. Let $x, y \in X$. Then we have

$$xA = yA \Leftrightarrow xA^\theta = yA^\theta$$

$$\Leftrightarrow$$

$$C_{xA^\theta}(g) = C_{yA^\theta}(g) \, \forall g \in X$$

$$\Leftrightarrow$$

$$C_{A^\theta}(x^{-1}g) = C_{A^\theta}(y^{-1}g) \, \forall g \in X$$

$$\Leftrightarrow$$

$$C_A(\theta(x^{-1}g)) = C_A(\theta(y^{-1}g)) \, \forall g \in X$$

$$\Leftrightarrow$$

$$C_A(\theta(x^{-1})\theta(g)) = C_A(\theta(y^{-1})\theta(g)) \, \forall g \in X$$

$$\Leftrightarrow$$

$$C_{\theta(x)A}(\theta(g)) = C_{\theta(y)A}(\theta(g)) \, \forall g \in X$$

$$\Leftrightarrow$$

$$\theta(x)A = \theta(y)A$$

$$\Leftrightarrow$$

$$\bar{\theta}(xA) = \bar{\theta}(yA).$$

Thus, $\bar{\theta}$ is well defined and one-to-one. Clearly, $\bar{\theta}$ maps B/A onto itself. The proof of the fact that $\bar{\theta}$ is a homomorphism is analogue to the corresponding part of the proof of Theorem 3.6. The result follows. □

Theorem 3.10. *Let $\theta : X \to Y$ be a homomorphism. Let A and B be multigroups of X and Y, respectively. Then $\theta(A) \subseteq B$ if and only if $C_B(\theta(x)) \geq C_A(x)$ for every $x \in X$.*

Proof. Let $\theta(A) \subseteq B$ and $x \in X$. Then we have

$$\begin{aligned} C_B(\theta(x)) &\geq C_{\theta(A)}(\theta(x)) \\ &= \bigvee\{C_A(z) \mid \theta(z) = \theta(x)\} \\ &= C_A(x). \end{aligned}$$

Conversely, let $C_B(\theta(x)) \geq C_A(x)$ for every $x \in X$. Then we have

$$\begin{aligned} C_{\theta(A)}(y) &= \bigvee\{C_A(x) \mid \theta(x) = y\} \\ &\leq \bigvee\{C_B(\theta(x)) \mid \theta(x) = y\} \\ &= C_B(y) \end{aligned}$$

for every $y \in Y$. Hence, $\theta(A) \subseteq B$. □

Proposition 3.11. *Let X be a cyclic group and $A, B \in MG(X)$ such that $A \subseteq B$. Then A is fully invariant.*

Proof. Let $X =< x >$ and A be a submultigroup of $B \in MG(X)$. Then $C_A(x^m) \geq C_A(x)$ for every $m \in \mathbb{Z}^+$. This implies that $C_A(\theta(x)) \geq C_A(x)$ for every $\theta \in Aut(X)$. Hence, it follows from Theorem 3.10 that, A is fully invariant. □

4. Conclusion

We have introduced the notion of characteristic submiultgroups and some of its homomorphic properties. Some related results were established with respect to factor multigroups and normal submultigroups. The work could be extended to the analogous of Cayley theorem in multigroup setting.

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[1] Department of Mathematics, Ahmadu Bello University, Zaria, Nigeria
E-mail address: amibrahim@abu.edu.ng

[2] Department of Mathematics/Statistics/Computer Science, University of Agriculture, P.M.B. 2373, Makurdi, Nigeria
E-mail address: ejegwa.augustine@uam.edu.ng