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EXISTENCE OF SOLUTIONS FOR A NONLOCAL $(P_1(x), P_2(x))$-LAPLACE EQUATION WITH DEPENDENCE ON THE GRADIENT

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Abstract. The object of this work is to study the existence of solutions for a nonlocal $(p_1(x), p_2(x))$ Laplace equation with dependence on the gradient. We establish our results by using the degree theory for operators of (S_+) type in the framework of variable exponent Sobolev spaces.

1. Introduction

In this paper, we consider the nonlocal $(p_1(x), p_2(x))$ Laplace boundary value problem:

$$\begin{aligned} -M_1(L_1(u))\ \mathrm{div}(|\nabla u|^{p_1(x)-2}\nabla u) - M_2(L_2(u))\ \mathrm{div}(|\nabla u|^{p_2(x)-2}\nabla u) & \\ = f(x,u,\nabla u)|u|^{t(x)}_{s(x)} \quad \text{in } \Omega & \qquad (1.1)\\ u = 0 \qquad \text{on } \partial\Omega, & \end{aligned}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and $p_i(x) \in C(\overline{\Omega})$ with $p_i(x) > 1$ for any $x \in \overline{\Omega}$ i=1,2 , $L_i(u) = \int_\Omega \frac{1}{p_i(x)}|\nabla u|^{p_i(x)}\,dx$, and M_i, f are functions that satisfy conditions which will be stated later.
The problem (1.1) is related to the stationary problem of a model introduced by Kirchhoff [37]. Accurately, he introduced a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \qquad (1.2)$$

which extends the classical D'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. After the work of Lions [34], where an abstract framework to the problem (1.1) was given, various problems of Kirchhoff type have been studied by many authors, we refer to [2, 7, 15] . The study of Kirchhoff type equations has already been extended to the case involving the p-Laplacian (see [12, 13, 19, 20]) and $p(x)$-Laplacian (see

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[10, 16, 26, 18, 24, 46]). For the physical and biological meaning of the nonlocal coefficients we refer the readers to [11, 14, 8, 9] and the references therein.
In a recent paper, the authors in [6], have dealt with the $p(x)$-Kirchhoff type equation

$$\begin{aligned} -M\Big(\int_\Omega (A(x,\nabla u)+\frac{1}{p(x)}|u|^{p(x)})\,dx\Big)\Big[\operatorname{div}(a(x,\nabla u))-|u|^{p(x)-2}u\Big] \\ = f(x,u)|u|_{s(x)}^{t(x)} \quad \text{in } \Omega, \\ u = constant \quad \text{on } \partial\Omega, \\ \int_{\partial\Omega} a(x,\nabla u).\nu d\Gamma = 0. \end{aligned}$$

by topological methods.
Since the nonlinearity f depends on the gradient of the solution, the problem (1.1) has no variational structure, so the most usual variational techniques can not be applied directly. For the case $M_1 = M_2 = 1$ and $p_1 = p_2 :=$ constant , $t(x) = 0$ there have been several works , using method of sub and supersolutions, topological degree, fixed point theorems, theory of pseudomonotone operators, Morse theory and approximation techniques; see, among other papers [27, 25, 36, 39, 40, 44]. In [29], the authors used the idea developed in [28]. This idea consists of regaining some variational aspect for (1.1) by freezing the term ∇u appearing as argument of f. After this, an iterative scheme is performed to obtain a nontrivial solution for the initial problem. Similarly, in the very recent paper [4], the authors established the existence of a positive solution. The case $M_1 = M_2$ and $p_1 = p_2 :=$ constant , $t(x) = 0$, was studied in [30, 38, 1, 35]. In [32, 33, 42, 47] the authors consider the problem (1.1), with $M_1 = M_2 = 1$, $p_1 \neq p_2; p_1, p_2$ continuous functions and $f = f(x, u)$, they showed existence of solutions via variational methods. As far as we know, there is few papers that deals with nonlocal problem involving $(p_1(x), p_2(x))$ Laplace operator (see [3]).
Our aim is to establish the existence results for problem (1.1) through the degree theory. First, we will establish the existence of a weak solution via the degree theory for (S_+) type mapping in terms of the Hammerstein equation. This method was recently introduced in [31] in order to study a quasilinear equation governed by the p-Laplacian operator $-\Delta_p$ when the nonlinear term depends on the gradient of the solution, i.e. when it is of the form $f = f(x, u, \nabla u)$. Here we adapt this technique in order to consider an equation with the nonlocal $(p_1(x), p_2(x))$ Laplace operator. Next, we will get the existence of a weak solution of the problem (1.1) by employing directly the degree theory for (S_+) type mapping.
To the best of our knowledge, problem (1.1) has not been studied via the degree theory for operators of generalized (S_+) type . The results obtained in the present paper are also new for the case that $p(x)$ is a constant.
This paper is organized as follows. In Section 2, we recall some basic facts about the variable Lebesgue–Sobolev spaces and the degree theory for operators of (S_+) type . Sections 3, is devoted to the main results of the paper.

2. Preliminaries

To discuss problem (1.1), we need some theory on $W_0^{1,p(x)}(\Omega)$ which is called variable exponent Sobolev space. Firstly we state some basic properties of spaces $W_0^{1,p(x)}(\Omega)$ which will be used later (for details, see [23]). Denote by $\mathbf{S}(\Omega)$ the set of all measurable real functions defined on Ω. Two functions in $\mathbf{S}(\Omega)$ are considered as the same element of $\mathbf{S}(\Omega)$ when they are equal almost everywhere. Write

$$C_+(\overline{\Omega}) = \{h : h \in C(\overline{\Omega}), h(x) > 1 \text{ for any } x \in \overline{\Omega}\},$$
$$h^- := \min_{\overline{\Omega}} h(x), \quad h^+ := \max_{\overline{\Omega}} h(x) \quad \text{for every } h \in C_+(\overline{\Omega}).$$

Define

$$L^{p(x)}(\Omega) = \{u \in \mathbf{S}(\Omega) : \int_\Omega |u(x)|^{p(x)}\, dx < +\infty \text{ for } p \in C_+(\overline{\Omega})\}$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}\, dx \le 1\},$$

and

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Proposition 2.1 ([23]). *The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

Proposition 2.2 ([23]). *Set $\rho(u) = \int_\Omega |u(x)|^{p(x)}\, dx$. For any $u \in L^{p(x)}(\Omega)$, then*

(1) *for $u \neq 0$, $|u|_{p(x)} = \lambda$ if and only if $\rho(\frac{u}{\lambda}) = 1$;*
(2) *$|u|_{p(x)} < 1$ $(= 1; > 1)$ if and only if $\rho(u) < 1$ $(= 1; > 1)$;*
(3) *if $|u|_{p(x)} > 1$, then $|u|_{p(x)}^{p^-} \le \rho(u) \le |u|_{p(x)}^{p^+}$;*
(4) *if $|u|_{p(x)} < 1$, then $|u|_{p(x)}^{p^+} \le \rho(u) \le |u|_{p(x)}^{p^-}$;*
(5) *$\lim_{k\to+\infty} |u_k|_{p(x)} = 0$ if and only if $\lim_{k\to+\infty} \rho(u_k) = 0$;*
(6) *$\lim_{k\to+\infty} |u_k|_{p(x)} = +\infty$ if and only if $\lim_{k\to+\infty} \rho(u_k) = +\infty$.*

Proposition 2.3 ([23]). *In $W_0^{1,p(x)}(\Omega)$ the Poincaré inequality holds; that is, there exists a positive constant C_0 such that*

$$|u|_{L^{p(x)}(\Omega)} \le C_0 |\nabla u|_{L^{p(x)}(\Omega)}, \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

So, $|\nabla u|_{L^{p(x)}(\Omega)}$ is a norm equivalent to the norm $\|u\|$ in the space $W_0^{1,p(x)}(\Omega)$. We will use the equivalent norm in the following discussion and write $\|u\|_p = |\nabla u|_{L^{p(x)}(\Omega)}$ for simplicity.

Proposition 2.4 ([21, 23]). *If $q \in C_+(\overline{\Omega})$ and $q(x) \le p^*(x)$ $(q(x) < p^*(x))$ for $x \in \overline{\Omega}$, then there is a continuous (compact) embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, where*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \ge N. \end{cases}$$

Proposition 2.5 ([22, 23]). *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{q(x)}(\Omega)$, where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$ holds a.e. in Ω. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have the following Hölder-type inequality*

$$\Big| \int_\Omega uv\, dx \Big| \le (\frac{1}{p^-} + \frac{1}{q^-})|u|_{p(x)}|v|_{q(x)}.$$

We write

$$L(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx.$$

Proposition 2.6 ([22]). *The functional $L : X \to \mathbb{R}$ is convex. The mapping $L' : X \to X^*$ is a strictly monotone, bounded homeomorphism, and is of (S_+) type, namely*

$$u_n \rightharpoonup u \text{ and } \limsup_{n\to+\infty} L'(u_n)(u_n - u) \le 0 \text{ implies } u_n \to u,$$

where $X = W_0^{1,p(x)}(\Omega)$, X^* is the dual space of X.

We recall some definitions and basic properties of the the Berkovits degree theory for demicontinuous operators of generalized (S_+) type in real reflexive Banach spaces. Let X be a real reflexive Banach space with dual space X^*. We can assume that X and X^* are locally uniformly convex (Troyanski's theorem [45]). The symbol $\langle ., . \rangle_X$ denotes the usual dual paring between X^* and X in this order. In the reflexive case where the bidual space X^{**} is identified with X, we sometimes write $\langle y, x \rangle$ for $\langle x, y \rangle_{X^*}$ for $x \in X$ and $y \in X^*$.
Let Ω be a nonempty subset of X, $(u_\nu)_{\nu\ge1} \subseteq \Omega$ and $F : \Omega \subseteq X \to X^*$. Then the mapping F is

(1) demicontinuous if $u_\nu \to u$ implies $F(u_\nu) \rightharpoonup F(u)$
(2) of class (S_+) if $u_\nu \rightharpoonup u$ and $\limsup_{\nu\to+\infty} \langle Fu_\nu, u_\nu - u\rangle \le 0$ yield $u_\nu \to u$
(3) Quasimonotone if $u_\nu \rightharpoonup u$ implies $\limsup_{\nu\to+\infty} \langle Fu_\nu, u_\nu - u\rangle \le 0$

For any operator $F : \Omega \subseteq X \to X$ and any bounded operator $T : \Omega_1 \subseteq X \to X^*$ such that $\Omega \subseteq \Omega_1$, we say that F satisfies condition $(S_+)_T$ if for any sequence (u_ν) in Ω with $u_\nu \rightharpoonup u$, $Tu_\nu \rightharpoonup y$ and $\limsup_{\nu\to+\infty} \langle Fu_\nu, Tu_\nu - y\rangle \le 0$, we have $u_\nu \to u$

Remark 2.7. It is known that
a)If a mapping is compact in a set, then it is quasi-monotone in that set.
b) If the mapping is demi-continuous and satisfies the condition (S_+) in a set, then it is quasimonotone in that set.

Now, we consider the following sets

$$\mathfrak{F}_1(\Omega) = \Big\{ F : \Omega \subseteq X \to X^*, F \quad \text{is bounded, demicontinuous and of class } (S_+) \Big\}$$

$$\mathfrak{F}_T(\Omega) = \Big\{ F : \Omega \subseteq X \to X^*, F \quad \text{is demicontinuous and of class } (S_+)_T \Big\}$$

for any $\Omega \subseteq D_F$ and any $T \in \mathfrak{F}_1(\Omega)$, where D_F denotes the domain of F.
We need the following result related to the Hammerstein operator of the form $I + S \circ T$

Lemma 2.8 ([31, Lemma 2.3]). *Suppose that $T \in \mathfrak{F}_1(\overline{\Omega})$ is continuous and $S : D_S \subseteq X^* \to X$ demicontinuous such that $T(\overline{\Omega}) \subseteq D_S$, where Ω is a bounded open set in the real reflexive Banach space X. Then the following statements are true:*

(1) *a) If S is quasimonotone, then $I + S \circ T \in \mathfrak{F}_T(\overline{\Omega})$ where I denotes the identity operator.*
(2) *b) If S satisfies conditions (S_+), then $S \circ T \in \mathfrak{F}_T(\overline{\Omega})$.*

Next, we present a result which pertains to the existence of the topological degree for a class of demicontinuous operator satisfying condition $(S_+)_T$.

Theorem 2.9 ([31, Theorem3.4]). *Let $\mathcal{M} = \Big\{ (F, \Omega, h)/\Omega \in \ \mathcal{O}, T \in \mathfrak{F}_1(\overline{\Omega}), F \in \mathfrak{F}_T(\overline{\Omega}), h \notin F(\partial\Omega) \Big\}$*
where $\mathcal{O}$ denotes the collection of all bounded open sets in X.
There exists one (Hammerstein type)degree function

$$d : \mathcal{M} \to \mathbb{Z}$$

which has the following properties:

a) *(Existence) If $d(F, \Omega, h) \neq 0$ then the equation $Fu = h$ has a solution in Ω*
b) *(Additivity) Let $F \in \mathfrak{F}_T(\overline{\Omega})$. For every disjoint open sets $\Omega_1, \Omega_2 \subseteq \Omega$ and every $h \notin F(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2))$ we have*

$$d(F, \Omega, h) = d(F, \Omega_1, h) + d(F, \Omega_2, h)$$

c) *(Normalization) $d(I, \Omega, h) = 1$, for every $h \in \Omega$.*
d) *(Invariance under homotopy) Let $T \in \mathfrak{F}_1(\overline{\Omega})$ be continuous and $F, S \in \mathfrak{F}_T(\overline{\Omega})$. An admissible affine homotopy (from $\overline{\Omega}$ into X) is a mapping*

$$H : [0, 1] \times \overline{\Omega} \to X$$

defined by

$$H(t, u) = (1 - t)Fu + tSu; \qquad (t, u) \in [0, 1] \times \overline{\Omega}$$

(so H satisfies condition $(S_+)_T$ by Lemma 2.5 in [31]).
If $H : [0, 1] \times \overline{\Omega} \to X$ is an admissible affine homotopy, with the common "support" T and $h : [0, 1] \to X$ is a continuous curve in X such that $h(t) \notin H(t, \partial\Omega)$ for all $t \in [0, 1]$, the value $d(H(t, .), \Omega, h(t))$ is constant for all $t \in [0, 1]$

Regarding the functions M_1, M_2, we assume that

(M_0) $M_i : [0, +\infty[\to]m_0, +\infty[$ $(i = 1, 2)$ are continuous and nondecreasing functions with $m_0 > 0$.

We give the properties of the nonlocal $(p_1(x), p_2(x))$-Laplace operator

$$-M_1(L_1(u))\ \mathrm{div}(|\nabla u|^{p_1(x)-2}\nabla u) - M_2(L_2(u))\ \mathrm{div}(|\nabla u|^{p_2(x)-2}\nabla u)$$

We denote

$$p_M(x) = \max\{p_1(x), p_2(x)\} \quad , p_m(x) = \min\{p_1(x), p_2(x)\}$$

It is easy to prove that $p_M, p_m \in C_+(\overline{\Omega})$.
Consider the following functional

$$\Phi(u) = \widehat{M_1}\Big(L_1(u)\Big) + \widehat{M_2}\Big(L_2(u)\Big) := \Phi_1(u) + \Phi_2(u), \quad \forall u \in X$$

where $X := W_0^{1,p_1(x)}(\Omega) \bigcap W_0^{1,p_2(x)}(\Omega)$, with its norm given by

$$\|u\| := \|u\|_{p_1(x)} + \|u\|_{p_2(x)}, \quad \text{for all} \quad u \in X \quad ,$$

$L_i(u) = \int_\Omega \frac{1}{p_i(x)}|\nabla u|^{p_i(x)}\,dx$ and $\widehat{M_i}(s) = \int_0^s M_i(t)\,dt$.
It is well known that $\Phi_i, i = 1, 2$ are well defined and continuously Gâteaux differentiable whose Gâteaux derivatives at point $u \in X$ are given by

$$\langle \Phi_i'(u), v\rangle = M_i\Big(\int_\Omega \frac{1}{p_i(x)}|\nabla u|^{p_i(x)}\,dx\Big)\int_\Omega |\nabla u|^{p_i(x)-2}\nabla u \cdot \nabla v\,dx$$

Thus

$$\langle \Phi'(u), v\rangle = \langle \Phi_1'(u), v\rangle + \langle \Phi_2'(u), v\rangle$$

Therefore, the nonlocal $(p_1(x), p_2(x))$-Laplace operator is the derivative operator of Φ in the weak sense.

Remark 2.10. The following statements hold:
i)$(X, \|\cdot\|)$ is a separable and reflexive Banach space .
ii) For $q(x) \in C_+(\overline{\Omega})$ such that $q(x) < p_M^*(x)$ for any $x \in \overline{\Omega}$, we have $X := W_0^{1,p_1(x)}(\Omega) \bigcap W_0^{1,p_2(x)}(\Omega) = W_0^{1,p_M(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, the embedding is continuous and compact.

Proposition 2.11. *Assume that* (M_0) *holds. Then*

(i) $\Phi' : X \to X^*$ *is a continuous, bounded and strictly monotone operator;*
(ii) Φ' *is of type* (S_+)*, i.e. if* $u_\nu \rightharpoonup u$ *in* X *and*

$$\limsup_{\nu\to+\infty}\langle \Phi'(u_n) - \Phi'(u), u_\nu - u\rangle = 0,$$

then $u_\nu \to u$ *in* X*;*
(iii) Φ' *is coercive. Furthermore,* Φ' *is a homeomorphism.*

Proof. i)It is obvious that the mapping $\Phi' : X \to X^*$ is a continuous and bounded. Consider the functional $L_i : X \to \mathbb{R}$, defined by

$$L_i(u) = \int_\Omega \frac{1}{p_i(x)}|\nabla u|^{p_i(x)}\,dx. \quad \text{for all } u \in X,$$

whose Gâteaux derivative at point $u \in X$ is given by

$$\langle L_i'(u), v\rangle = \int_\Omega |\nabla u|^{p_i(x)-2}\nabla u \nabla \varphi \, dx, \quad \text{for all } v \in X,$$

Taking into account the inequality [41, (2.2)] for $p > 1$ there exists a positive constant C_p such that

$$\langle |x|^{p-2}x - |y|^{p-2}y, x-y\rangle \geq \begin{cases} C_p|x-y|^p & \text{if } p \geq 2 \\ C_p \frac{|x-y|^2}{(|x|+|y|)^{p-2}}, \ (x,y) \neq (0,0) & \text{if } 1 < p < 2, \end{cases} \tag{2.1}$$

for all $x, y \in \mathbb{R}^N$. Therefore,

$$\langle L_i'(u) - L_i'(v), u - v\rangle > 0,$$

for all $u \neq v \in X$ which means that L_i' is strictly monotone. So, by [48, Prop. 25.10], L is strictly convex. Moreover, since M_i is nondecreasing, $\widehat{M_i}$ is convex in $[0, +\infty[$. Thus, for every $u, v \in X$ with $u \neq v$, and every $s, t \in (0,1)$ with $s + t = 1$, one has

$$\widehat{M_i}(L_i(su + tv)) < \widehat{M_i}(sL_i(u) + tL_i(v)) \leq s\widehat{M_i}(L_i(u)) + t\widehat{M_i}(L_i(v)).$$

This shows that $\Phi_i, i = 1, 2$ is strictly convex, and so $\Phi' = \Phi_1' + \Phi_2'$ is strictly monotone.

ii) Arguing as in [26] we can prove that Φ_1' and Φ_2' are of type (S_+) then their sum Φ' also has the property (S_+).

iii) Since Φ' is a strictly monotone operator in X, Φ' is an injection. For $u \in X$ with $\|u\| > 1$, we have

$$\begin{aligned} \frac{\langle \Phi'(u), u\rangle}{\|u\|} &= \\ &\frac{M_1\left(L_1(u)\right)\int_\Omega |\nabla u|^{p_1(x)}\,dx + M_2\left(L_2(u)\right)\int_\Omega |\nabla u|^{p_2(x)}\,dx}{\|u\|} \\ &\geq \frac{m_0\Big(\int_\Omega |\nabla u|^{p_1(x)}\,dx + \int_\Omega |\nabla u|^{p_2(x)}\,dx\Big)}{\|u\|} \\ &\geq \frac{m_0(\|u\|^{p_1^-} + \|u\|^{p_2^-})}{\|u\|} \geq m_0\|u\|^{p_m^- - 1} \end{aligned}$$

therefore, Φ' is coercive. Clearly Φ' is also demicontinuous. In view of the Minty-Browder theorem [48, Theorem 26A], the operator Φ' is a surjection, and hence the inverse $(\Phi')^{-1} : X^* \to X$ of Φ' exists. It suffices then to show the continuity of $(\Phi')^{-1}$. Let (g_n) be a sequence of X^* such that $g_n \to g$ in X^*. Let $u_n = (\Phi')^{-1}(g_n)$, $u = (\Phi')^{-1}(g)$, then $\Phi'(u_n) = g_n$, $\Phi'(u) = g$. By the coercivity of Φ', we deduces that (u_n) is bounded in X, up to subsequence, we can assume that $u_n \rightharpoonup u$. Since $g_n \to g$,

$$\lim_{n\to+\infty} \langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle = \lim_{n\to+\infty} \langle g_n - g, u_n - u\rangle = 0.$$

Since Φ' is of type (S_+), $u_n \to u$, so $(\Phi')^{-1}$ is continuous. □

From now on we deal with the properties for the superposition operator induced by the function f in (1.1). We assume that
(F_1) $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy the Carathéodory condition in the sense that $f(., u, \xi)$ is measurable for all $(u, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and $f(x, ., .)$ is continuous for almost all $x \in \Omega$
(F_2) $|f(x, u, \xi)| \le k(x) + |u|^{\eta(x)} + |\xi|^{\delta(x)}$ a.e. $x \in \Omega$, all $(u, \xi) \in \mathbb{R} \times \mathbb{R}^N$, where $k : \mathbb{R} \to \mathbb{R}^+$, $k \in L^{p'_M(x)}(\Omega)$ and $0 \le \eta(x) < p^-_m - 1, 0 \le \delta(x) < (p^-_m - 1)/p'^+_M$.

Lemma 2.12. *Under assumption $(F1)$ the operator $S : W^{1,p_M(x)}_0(\Omega) \to (W^{1,p_M(x)}_0(\Omega))^*$ given by*

$$\langle Su, v\rangle = -\int_\Omega f(x, u, \nabla u)|u|^{t(x)}_{s(x)} v \, dx$$

is continuous and compact.

Proof. Here, we follow the lines of Theorem 3.1 in [6]. We split the proof in two steps.
1.- S is well defined. Indeed, using (F_1) and $t \in C(\overline{\Omega})$, for all u, v in $W^{1,p_M(x)}_0$ we have

$$\begin{aligned}
|\langle Su, v\rangle| &\le \int_\Omega |f(x, u, \nabla u)||u|^{t(x)}_{s(x)}|v| \, dx \\
&\le C|f(x, u, \nabla u)|_{p'_M(x)}|v|_{p_M(x)} \\
&\le C\Big(\int_\Omega |k(x)|^{p'_M(x)} \, dx + \int_\Omega |u(x)|^{\eta(x)p'_M(x)} \, dx + \int_\Omega |\nabla u(x)|^{\delta(x)p'_M(x)} \, dx\Big)^{1/\alpha} \|v\| \\
&\le C\Big(|k|^\tau_{p'_M(x)} + |u|^\beta_{\eta(x)p'_M(x)} + |\nabla u|^\theta_{\delta(x)p'_M(x)}\Big)^{1/\alpha} \|v\| \\
&\le C(1 + \|u\|^\beta + \|u\|^\theta)^{1/\alpha}\|v\| < \infty,
\end{aligned} \tag{2.2}$$

where

$$\alpha = \begin{cases} p'^-_M, & \text{if } |f(x, u, \nabla u)|_{p'_M(x)} > 1, \\ p'^+_M, & \text{if } |f(x, u, \nabla u)|_{p'_M(x)} \le 1, \end{cases} \quad , \quad \tau = \begin{cases} p'^-_M, & \text{if } |k|_{p'_M(x)} > 1, \\ p'^+_M, & \text{if } |k|_{p'_M(x)} \le 1, \end{cases}$$

$$\beta = \begin{cases} (\eta p'_M)^+, & \text{if } |u|_{\eta(x)p'_M(x)} > 1, \\ (\eta p'_M)^-, & \text{if } |u|_{\eta(x)p'_M(x)} \le 1, \end{cases} \quad , \quad \text{and } \theta = \begin{cases} (\delta p'_M)^+, & \text{if } |\nabla u|_{\delta(x)p'_M(x)} > 1, \\ (\delta p'_M)^-, & \text{if } |\nabla u|_{\delta(x)p'_M(x)} \le 1, \end{cases}.$$

We note that $0 \le \eta(x) < p_M(x) - 1$ implies $0 \le \eta(x)p'_M(x) < p_M(x)$ and $0 \le \delta(x) < (p_M(x) - 1)/p'_M(x)$ implies $0 \le \delta(x)p'_M(x) < p_M(x) - 1 < p_M(x)$. So, there exist $C_1, C_2 > 0$ such that

$$|u|_{\eta(x)p'_M(x)} \le C_1|u|_{p_M(x)} \quad \text{and} \quad |\nabla u|_{\delta(x)p'_M(x)} \le C_2|\nabla u|_{p_M(x)}$$

respectively.
2.- S is continuous on X.
Let $u_\nu \to u$ in X. So, up to a subsequence we deduce

$$u_\nu \to u \qquad \text{and} \qquad \nabla u_\nu \to \nabla u \quad , and \tag{2.3}$$

$$|u_\nu(x)|^{p_M(x)} \le \hat{k}(x) \ , \ |\partial u_\nu(x)/\partial x_j|^{p_M(x)} \le \hat{w}_j(x) \quad \text{for some} \quad \hat{k}, \hat{w}_j \in L^1(\Omega) \tag{2.4}$$

for almost all $x \in \Omega$.
Since $t \in C(\overline{\Omega})$

$$|u_\nu|^{t(x)}_{s(x)} \to |u|^{t(x)}_{s(x)} \quad \text{a.e.} \quad x \in \Omega.$$

Furthermore, since the function f is a Caratheodory function

$$f(x, u_\nu, \nabla u_\nu) \to f(x, u, \nabla u) \quad \text{a.e.} \quad x \in \Omega,$$

Thus , we get

$$fx, u_\nu, \nabla u_\nu)|u_\nu|^{t(x)}_{s(x)} \to f(x, u, \nabla u)|u|^{t(x)}_{s(x)} \quad \text{a.e.} \quad x \in \Omega.$$

It follows from (F_1) and (2.4) that

$$\begin{aligned}
&\left|f(x, u_\nu, \nabla u_\nu)|u_\nu|^{t(x)}_{s(x)} - f(x, u, \nabla u)|u|^{t(x)}_{s(x)}\right|^{p'_M(x)} \\
&\leq C2^{p'^+_M}\left[|f(x, u_\nu, \nabla u_\nu)|^{p'^+_M} + |f(x, u_\nu, \nabla u_\nu)|^{p'^+_M}\right] \\
&\leq C(1 + \hat{k}(x) + |\hat{w}|^{\alpha(x)}), \qquad \hat{w} = (\hat{w}_1, \hat{w}_2, \cdots, \hat{w}_n)
\end{aligned}$$

But, $C(1 + k(x) + |\hat{w}|^{\alpha(.)}) \in L^1(\Omega)$; thus, applying the Dominated Convergence Theorem with (2.3), we obtain

$$\lim_{\nu\to\infty} \int_\Omega \left|f(x, u_\nu, \nabla u_\nu)|u_\nu|^{t(x)}_{s(x)} - f(x, u, \nabla u)|u|^{t(x)}_{s(x)}\right|^{p'_M(x)} dx = 0$$

This implies that

$$\lim_{\nu\to\infty} \left|f(x, u_\nu, \nabla u_\nu)|u_\nu|^{t(x)}_{s(x)} - f(x, u, \nabla u)|u|^{t(x)}_{s(x)}\right|_{p'_M(x)} = 0 \tag{2.5}$$

Hence by direct computations we get

$$\begin{aligned}
|\langle Su_\nu, v\rangle - \langle Su, v\rangle| &\leq \int_\Omega \left|f(x, u_\nu, \nabla u_\nu)|u_\nu|^{t(x)}_{s(x)} - f(x, u, \nabla u)|u|^{t(x)}_{s(x)}\right||v|\, dx \\
&\leq C\left|f(x, u_\nu, \nabla u_\nu)|u_\nu|^{t(x)}_{s(x)} - f(x, u, \nabla u)|u|^{t(x)}_{s(x)}\right|_{p'_M(x)} \|v\|
\end{aligned}$$

therefore,from (2.5)

$$|Su_\nu - Su| \leq C\left|f(x, u_\nu, \nabla u_\nu)|u_\nu|^{t(x)}_{s(x)} - f(x, u, \nabla u)|u|^{t(x)}_{s(x)}\right|_{p'_M(x)} \to 0 \tag{2.6}$$

Therefore, by the convergence principle in Banach spaces , $Su_\nu \to Su$ in X^*.
Now , repeating the same arguments as in steps 1) and 2), we get that the superposition operator $\Upsilon : L^{p_M(x)}(\Omega) \times L^{p_M(x)}(\Omega, \mathbb{R}^n) \to L^{p'_M(x)}(\Omega)$ defined by

$$\Upsilon(u, v)(x) = f(x, u, v)|u|^{t(x)}_{s(x)}$$

is bounded and continuous. Moreover, the linear operator $J : X \to L^{p_M(x)}(\Omega) \times L^{p_M(x)}(\Omega, \mathbb{R}^n)$ defined by

$$J(u) = (u, \nabla u)$$

is bounded. The canonical linear embedding $I : X \to L^{p_M(x)}(\Omega)$ is compact by Rellich's embedding theorem, so the adjoint operator $I^* : L^{p'_M(x)}(\Omega) \to X^*$ given by

$$(I^*v)(u) = \int_\Omega vu\,dx$$

is a compact embedding. From the relation $S = I^* \circ \Upsilon \circ J$ it follows that $S : X \to X^*$ is compact. Thus, S is quasimonotone. □

3. Main results

We now state our main results. We shall prove the solvability of problem (1.1) by using of the degree theory.
Denote $Y = W_0^{1,p_1(x)}(\Omega) \bigcap W_0^{1,p_2(x)}(\Omega)$. Define the mappings $F : Y \to Y^*$ and $S : Y \to Y^*$ by

$$\langle Fu, v\rangle = M_1\Big(L_1(u)\Big)\int_\Omega |\nabla u|^{p_1(x)-2}\nabla u\cdot\nabla v\,dx + M_2\Big(L_2(u)\Big)\int_\Omega |\nabla u|^{p_2(x)-2}\nabla u\cdot\nabla v\,dx$$

and

$$\langle Su, v\rangle = -\int_\Omega f(x,u,\nabla u)|u|_{s(x)}^{t(x)}v\,dx$$

for any $u, v \in Y$.
Then $u \in Y$ is a weak solution of (1.1) if and only if

$$Fu = -Su.$$

Theorem 3.1. *Let (M_0), (F_1) and (F_2) hold. Then problem (1.1) has a weak solution.*

Proof. First, we consider the problem

$$\begin{gathered} -M_1(L_1(u))\ \mathrm{div}(|\nabla u|^{p_1(x)-2}\nabla u) - M_2(L_2(u))\ \mathrm{div}(|\nabla u|^{p_2(x)-2}\nabla u) = v \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

From theorem 3.2 in [32], this problem admits a unique weak solution for each $v \in Y^*$. By Proposition 2.11 the operator $F : Y \to Y^*$ is bounded, continuous, uniformly monotone, coercive and satisfies condition (S_+).
Set $X = Y^*$ and identify X^* with Y thanks to the reflexivity. Hence the Minty-Browder theorem implies that the inverse operator $T := F^{-1} : X \to X^*$ exists and is bounded . Besides, it is continuous and satisfies condition (S_+) because F is continuous and satisfies condition (S_+), and T is bounded. Furthermore, from Lemma 2.12 $S : X^* \to X$ is bounded, continuous and quasimonotone. Therefore $u \in Y$ is a weak solution of (1.1) if and only if

$$u = Tv \quad \text{and } v + (S \circ T)v = 0 \quad \text{for each } v \in Y^*. \tag{3.1}$$

We shall use the topological degree theory in Section 2 to solve (3.1). Let us consider the set

$$B = \Big\{v \in X \ : \exists \lambda \in [0,1] \text{ such that } v + \lambda(S \circ T)v = 0\Big\}$$

Next, we prove that B is bounded in X.
For $v \in B$, i.e. $v \in X$and $v + \lambda(S \circ T)v = 0$ for some $\lambda \in [0,1]$ we have

$$\langle Fu, u\rangle = \langle v, Tv\rangle = -\lambda\langle (S \circ T)v, Tv\rangle$$

Hence

$$\begin{aligned} m_0 C_1 \int_\Omega (|\nabla u|^{p_1(x)} + |\nabla u|^{p_2(x)})\, dx &\leq \lambda \int_\Omega f(x,u,\nabla u)|u|_{s(x)}^{t(x)} u\, dx \\ &\leq \int_\Omega \big(k(x)|u| + |u|^{\eta(x)+1} + |\nabla u|^{\delta(x)}|u|\big)\, dx \end{aligned}$$

Then

$$\|u\|^\sigma \leq C\big(\|u\| + \|u\|^\beta + \|u\|^{(\vartheta/\gamma)+1}\big) \tag{3.2}$$

where

$$\sigma = \begin{cases} p_m^-, & \text{if } \|u\| > 1, \\ p_M^+, & \text{if } \|u\| \leq 1, \end{cases} \quad , \quad \gamma = \begin{cases} (p'_M)^-, & \text{if } ||\nabla u|^{\delta(x)}|_{p'_M(x)} > 1, \\ (p'_M)^+, & \text{if } ||\nabla u|^{\delta(x)}|_{p'_M(x)} \leq 1 \end{cases} .$$

From hypothesis (F_2) and (3.2), it follows that $\{u = Tv : v \in B\}$ is bounded. Since the operator S is bounded, we infer that there exists R such that

$$\|v\|_X \leq R$$

So, $Iv + \lambda(S \circ T)v \neq 0 \quad \text{on} \quad \partial B(0,R) \ \ \forall \lambda \in [0,1]$
Furthermore, from Lemma 2.8 we can claim that

$$I + \lambda(S \circ T) \in \mathfrak{F}_T(\overline{B(0,R)}) \quad \text{and} \quad I = F \circ T \in \mathfrak{F}_T(\overline{B(0,R)})$$

Now, we define the homotopy $H : [0,1] \times \overline{B(0,R)} \to X$ given by

$$H(\lambda, v) = Iv + \lambda(S \circ T)v \quad \text{for} \quad (\lambda, v) \in [0,1] \times \overline{B(0,R)}$$

Hence, by the homotopy invariance and normalization properties in Theorem 2.9 we see that

$$d(I + (S \circ T), \overline{B(0,R)}, 0) = d(I, \overline{B(0,R)}, 0) = 1$$

Thus, by Theorem 2.9(a), we can find $v \in \overline{B(0,R)}$ such that

$$v + (S \circ T)v = 0$$

Therefore, we have that $u = Tv$ is a weak solution of (1.1). The proof is now complete. □

Next, we use the degree theory of (S_+) type mappings to prove the second result of this paper.
For (S_+) mapping theory, including the degree theory and the surjection theorem, we refer the reader to [5, 43, 48].

Theorem 3.2. *Assume that* (M_0)*and* (H_0) *hold. If* $\beta < p_m^-$ *and* $\frac{\theta}{\gamma} + 1 < p_m^-$ *problem* (1.1) *has a weak solution .*

Proof. The mappings J and $S : X \to X^*$ defined by

$$\langle Ju, v\rangle = M_1(L_1(u)) \int_\Omega |\nabla u|^{p_1(x)-2}\nabla u \cdot \nabla v \, dx + M_2(L_2(u)) \int_\Omega |\nabla u|^{p_2(x)-2}\nabla u \cdot \nabla v \, dx$$

$$\langle Su, v\rangle = -\int_\Omega f(x, u, \nabla u)|u|_{s(x)}^{t(x)} v \, dx, \quad u, v \in X.$$

are respectively of type (S_+) and compact.
It is clear that $u \in X$ is a solution of (1.1) if an only if $I(u) := J(u) + S(u) = 0$. From the above analysis, it is obvious that $I : X \to X^*$ is continuous and bounded. Noting that the sum of an (S_+) type mapping and a compact mapping is of type (S_+) , it follows that the mapping $I = J + S$ is of type (S_+) .
Then, proceeding similarly as in the proof of Lemma 2.12, for $\|u\| > 1$ we have that

$$\langle I(u), u\rangle \geq m_0\|u\|^{p_m^-}\left(1 - c_1\|u\|^{1-p_m^-} + c_2\|u\|^{\beta-p_m^-} + c_3\|u\|^{(\vartheta/\gamma)+1-p_m^-}\right) \tag{3.3}$$

Choosing $\|u\| = R$ large enough, we deduce from (3.3) that

$$\langle I(u), u\rangle > 0 \text{ for all } u \in X \text{ such that } \|u\| = R.$$

By the topological degree theory for (S_+) type mappings, for $R > 0$ we conclude that

$$deg(L, B(0, R), 0) = 1$$

Therefore the equation $I(u) = 0$ has at least one solution $u \in B(0, R)$. Furthermore

$$\lim_{\|u\|\to+\infty} \frac{\langle I(u), u\rangle}{\|u\|} \geq c_9\|u\|^{p_m^- - 1} = +\infty$$

So, the mapping I is coercive, and hence, by the surjection theorem for the pseudomonotone mappings (see [48, theorem 27.A]), the mapping I is surjective. This completes the proof. □

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