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# THE WEAKLY ANNIHILATING-IDEAL GRAPH OF A COMMUTATIVE RING

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Abstract. Let $R$ be a commutative ring with non-zero identity which is not an integral domain. An ideal $I$ of a ring $R$ is called an annihilating ideal if there exists $r \in R\backslash\{0\}$ such that $Ir = (0)$. Let $\mathbb{A}(R)$ denote the set of all annihilating ideals of $R$ and $\mathbb{A}(R)^* = \mathbb{A}(R)\backslash\{(0)\}$. In this article, we introduce and investigate the weakly annihilating-ideal graph of $R$ denoted by $\mathbb{WAG}(R)$. It is the undirected graph whose vertex set is $\mathbb{A}(R)^*$ and two distinct vertices $I, J$ are adjacent in this graph if and only if there exist non-zero ideals $A, B$ of $R$ with $A \subseteq ann(I)$ and $B \subseteq ann(J)$ such that $AB = (0)$. The aim of this article is to study the interplay between the ring-theoretic properties of $R$ and the graph-theoretic properties of $\mathbb{WAG}(R)$. We discuss some results regarding the connectedness of $\mathbb{WAG}(R)$ and determine its diameter and girth. Moreover, we provide some conditions under which $\mathbb{WAG}(R)$ and $\mathbb{AG}(R)$ are identical.

## 1. Introduction

The study of algebraic structures is a prominent area of research. There are many advanced techniques available to study the algebraic properties of the algebraic objects. The use of graph theory is one of the most interesting techniques to explore the algebraic structures. In the literature, there are many articles on assigning a graph to different algebraic structures. The purpose of this article is to study the structure of commutative rings with the help of graph theory. The graphs associated with commutative rings those motivate the research work presented in this article are the zero-divisor graphs of commutative rings, the annihilating-ideal graphs of commutative rings, and the weakly zero-divisor graphs of commutative rings.

Let $R$ be a commutative ring with non-zero identity which is not an integral domain. Let $Z(R)$ denote the set of all zero-divisors of $R$ and $Z(R)^* = Z(R)\backslash\{0\}$. The idea of associating a graph to a ring was initiated by Beck in [7] but he was mainly interested in colorings. Anderson and Livingston modified Beck's definition and introduced the notion of zero-divisor graph in [4]. Recall from [4] that the *zero-divisor graph* of a ring $R$ denoted by $\Gamma(R)$ is an undirected graph whose vertex set is $Z(R)^*$ and two distinct vertices $x, y$ are adjacent in this graph

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if and only if $xy = 0$. A lot of research articles on the zero-divisor graphs of commutative rings were published by eminent researchers.

The generalization of the zero-divisor graph was introduced by Behboodi and Rakeei in [8]. Recall from [8] that an ideal $I$ of $R$ is said to be an *annihilating ideal* if $Ir = (0)$ for some $r \in R\backslash\{0\}$. As in [8], we denote by $\mathbb{A}(R)$, the set of all annihilating ideals of $R$ and by $\mathbb{A}(R)^* = \mathbb{A}(R)\backslash\{(0)\}$. Recall from [8] that the *annihilating-ideal graph* of a ring $R$ denoted by $\mathbb{AG}(R)$ is an undirected graph whose vertex set is $\mathbb{A}(R)^*$ and two distinct vertices $I, J \in \mathbb{A}(R)^*$ are adjacent in this graph if and only if $IJ = (0)$. Out of many published research articles, some interesting and inspiring results related to annihilating-ideals of commutative rings were proved in [1–3, 8, 9, 12–15].

In [11], Nikmehr et al. introduced the concept of the weakly zero-divisor graph of a commutative ring. Recall from [11] that the *weakly zero-divisor graph* of a ring $R$ denoted by $W\Gamma(R)$ is an undirected (simple) graph whose vertex set is $Z(R)^*$ and two distinct vertices $x$ and $y$ are adjacent in $W\Gamma(R)$ if and only if there exist $r \in ann(x)$ and $s \in ann(y)$ such that $rs = 0$. It is easy to verify that $\Gamma(R)$ is a spanning subgraph of $W\Gamma(R)$. Several interesting results on $W\Gamma(R)$ were proved in [11].

Motivated by the work done on $W\Gamma(R)$ mentioned above, in this article, we generalize this concept by introducing the *weakly annihilating-ideal graph* of $R$ denoted by $\mathbb{WAG}(R)$. It is the undirected graph whose vertex set is $\mathbb{A}(R)^*$ and two distinct vertices $I, J$ are adjacent in this graph if and only if there exist non-zero ideals $A, B$ of $R$ with $A \subseteq ann(I)$ and $B \subseteq ann(J)$ such that $AB = (0)$. The main objective of this article is to study some properties of $\mathbb{WAG}(R)$. Before we give a brief summary of the results proved in this article on $\mathbb{WAG}(R)$, it is desirable to recall the following results from commutative ring theory.

For an ideal $I$ of $R$, the *annihilator* of $I$ in $R$ denoted by $ann_R(I)$ or $ann(I)$ is $\{r \in R \mid Ir = (0)\}$. A ring $R$ is said to be *quasilocal* if $R$ has a unique maximal ideal. By a *local ring*, we mean a Noetherian quasilocal ring. A local ring $R$ with unique maximal ideal $\mathfrak{m}$ is denoted by $(R, \mathfrak{m})$. An element $x$ of a ring $R$ is said to be *nilpotent* if there exists a positive integer $n$ such that $x^n = 0$. The *nilradical* of a ring $R$ is the set of all nilpotent elements in the ring $R$. We use $nil(R)$ to denote the nilradical of a ring $R$. Clearly, $nil(R)$ is an ideal of $R$. A ring $R$ is said to be *reduced* if it has no non-zero nilpotent elements. Note that $R$ is reduced if $nil(R) = (0)$. An ideal $I$ of a ring $R$ is said to be a *nilpotent ideal* if there exists $k \in \mathbb{N}$ such that $I^k = (0)$. Recall that a local ring $(R, \mathfrak{m})$ is said to be a *special principal ideal ring (SPIR)* if $R$ is a principal ideal ring and $\mathfrak{m}$ is nilpotent. We denote the set of all minimal prime ideals of $R$ by $Min(R)$. Let $n \in \mathbb{N}$ with $n \geq 2$. We denote the ring of integer modulo $n$ by $\mathbb{Z}_n$. The cardinality of a set $A$ is denoted by $|A|$. For any ring $R$, the set of all nilpotent ideals of $R$ is denoted by $\mathbb{NI}(R)$. Any unexplained terminology from ring theory can be referred from the standard textbooks like [5, 10].

We next recall the following definitions from graph theory which are useful in this article. The graphs considered in this article are undirected. Let $G = (V, E)$ be a graph. A graph $G$ is said to be *connected* if for each pair of distinct vertices $a, b \in V$, there exists at least one path in $G$ between $a$ and $b$. Let $G = (V, E)$ be a

connected graph. Recall that for distinct vertices $a, b$ of $G$, the *distance* between $a$ and $b$ denoted by $d_G(a, b)$ or $d(a, b)$ is defined as the length of a shortest path between $a$ and $b$ in $G$. We define $d(a, a) = 0$ and define the *diameter* of $G$ denoted by $diam(G)$ as $diam(G) = sup\{d(a, b) \mid a, b \in V\}$. Let $G = (V, E)$ be a connected graph. Let $v \in V$. Recall that the *eccentricity of* $v$ denoted by $E(v)$ is defined as $E(v) = sup\{d(v, w) \mid w \in V\}$. The *radius of* $G$ denoted by $r(G)$ is defined as $r(G) = min\{E(v) \mid v \in V\}$. A vertex $v \in V$ with minimum eccentricity is called a *center of* $G$. Recall that the *girth* of $G$ denoted by $girth(G)$ or $gr(G)$ is defined as the length of a shortest cycle in $G$. If $G$ does not contain any cycle, then we set $gr(G) = \infty$.

A simple graph $G$ is said to be *complete* if every pair of distinct vertices of $G$ are adjacent in $G$. Let $n \geq 1$. A complete graph on $n$ vertices is denoted by $K_n$. A *bipartite graph* is a graph $G$ whose vertex set can be partitioned into two subsets $A$ and $B$ such that no edge has both ends in any one subset. A *complete bipartite graph* is a graph $G$ which may be partitioned into two disjoint non-empty vertex sets $A$ and $B$ such that two distinct vertices are adjacent if and only if they are in distinct vertex sets. A complete bipartite graph with parts $A$ and $B$ such that $|A| = m$ and $|B| = n$ is denoted by $K_{m,n}$ (we allow $m$ and $n$ to be infinite cardinals). A *star graph* is a complete bipartite graph $K_{1,n}$ and $K_{1,\infty}$ denotes a star graph with infinitely many vertices.

Given a graph $G$, a *subgraph* $H$ of $G$ is a graph whose vertex set and edge set are subsets of those of $G$. A subgraph $H$ of $G$ is called an *induced subgraph* if all edges of $G$ joining two vertices in $H$ are also edges of $H$. A *spanning subgraph* $H$ of $G$ is a subgraph containing all the vertices of $G$. A *vertex coloring* of $G$ is a map $f : V \to S$, where $S$ is a set of distinct colors and a vertex coloring $f$ is said to be *proper vertex coloring* if adjacent vertices of $G$ receive distinct colors of $S$; that is, if $u, v$ are adjacent in $G$, then $f(u) \neq f(v)$. The *chromatic number* of $G$ denoted by $\chi(G)$ is the minimum number of colors needed for a proper vertex coloring of $G$. Further concepts of graph theory can be referred from [6].

We now give brief account of the results proved in this article. This article consists of three sections. Section 1 is on introduction. In Section 2, some basic properties of $\mathbb{WAG}(R)$ are proved with several examples (see Examples 2.5, 2.9, and 2.11). It is shown in Theorem 2.3 that $\mathbb{WAG}(R)$ is connected and $diam(\mathbb{WAG}(R)) \leq 2$. Moreover, if $\mathbb{WAG}(R)$ contains a cycle, then $gr(\mathbb{WAG}(R)) \leq 4$. It is proved in Lemma 2.4 that if $R$ is a reduced ring with $|Min(R)| = 2$, then $\mathbb{WAG}(R) = \mathbb{AG}(R)$ is a complete bipartite graph. Let $R$ be a ring which is not reduced. It is shown in Lemma 2.6 that $r(\mathbb{WAG}(R)) = 1$ and $\mathbb{WAG}(R)$ has at least $|\mathbb{NI}(R)^*|$ centers. Moreover, the subgraph of $\mathbb{WAG}(R)$ induced on $\mathbb{NI}(R)^*$ is complete. Let $R$ be a ring and let $\mathbb{WAG}(R)$ contains a cycle. It is proved in Lemma 2.10 that $gr(\mathbb{WAG}(R)) = 4$ if and only if $R$ is reduced with $|Min(R)| = 2$. Let $R$ be a reduced ring which is not an integral domain and let $Z(R)$ be an ideal of $R$. Then it is verified in Theorem 2.12 that $\mathbb{WAG}(R) \neq \mathbb{AG}(R)$ and $gr(\mathbb{WAG}(R)) = 3$. Let $R$ be a ring which is not reduced. If $Z(R)$ is an annihilator ideal, then it is shown in Lemma 2.13 that $\mathbb{WAG}(R)$ is complete. Let $R$ be a ring which is not an integral domain. It is proved in Lemma 2.14 that if $gr(\mathbb{WAG}(R)) = \infty$, then $\mathbb{WAG}(R) = \mathbb{AG}(R)$.

In Section 3, we focus on answering the question of when $\mathbb{WAG}(R)$ is identical to $\mathbb{AG}(R)$. Let $R$ be a reduced ring which is not an integral domain. Then it is proved in Theorem 3.1 that $\mathbb{WAG}(R) = \mathbb{AG}(R)$ if and only if $|Min(R)| = 2$. For a reduced ring $R$, several equivalent statements for $\mathbb{WAG}(R)$ and $\mathbb{AG}(R)$ are provided in Corollary 3.2. For a non-reduced ring $R$, it is proved in Theorem 3.3 that the statements 1) $\mathbb{WAG}(R) = \mathbb{AG}(R)$ and 2) $\mathbb{AG}(R)$ is complete are equivalent. For a non-reduced ring $R$, several equivalent statements for $\mathbb{WAG}(R)$ and $\mathbb{AG}(R)$ are provided in Corollary 3.4. Moreover, some examples are given to illustrate the results proved in this section (see Examples 3.5 and 3.6).

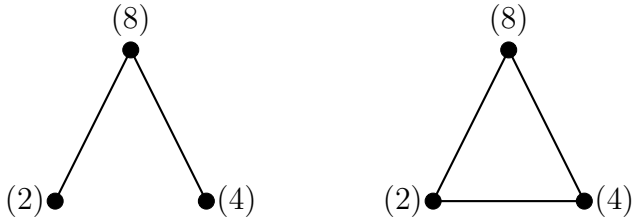
## 2. Basic Properties of $\mathbb{WAG}(R)$

Let $R$ be a ring such that $|\mathbb{A}(R)^*| \geq 1$. The aim of this section is to state and prove some basic properties of $\mathbb{WAG}(R)$. We prove that $\mathbb{WAG}(R)$ is connected and $diam(\mathbb{WAG}(R)) \leq 2$. If $|\mathbb{A}(R)^*| \geq 3$, then we prove that $gr(\mathbb{WAG}(R)) \leq 4$. We start this section with the following lemma.

**Lemma 2.1.** *Let $R$ be a ring such that $\mathbb{A}(R)^* \neq \emptyset$. Then $\mathbb{AG}(R)$ is a spanning subgraph of $\mathbb{WAG}(R)$.*

*Proof.* Let $I, J \in \mathbb{A}(R)^*$ with $I \neq J$. Suppose that $I$ and $J$ are adjacent in $\mathbb{AG}(R)$. Therefore, $IJ = (0)$. Note that $I \subseteq ann(J)$ and $J \subseteq ann(I)$. Hence, $I$ and $J$ are adjacent in $\mathbb{WAG}(R)$. As $V(\mathbb{AG}(R)) = V(\mathbb{WAG}(R))$, it follows that $\mathbb{AG}(R)$ is a spanning subgraph of $\mathbb{WAG}(R)$. □

*Remark* 2.2. 1) We know that $\mathbb{AG}(R)$ is connected with $diam(\mathbb{AG}(R)) \leq 3$ [8, Theorem 2.1]. Moreover, it is clear from Lemma 2.1 that $\mathbb{AG}(R)$ is a spanning subgraph of $\mathbb{WAG}(R)$. This concludes that $\mathbb{WAG}(R)$ is connected with $diam(\mathbb{WAG}(R)) \leq 3$. However, we prove in Theorem 2.3 that $\mathbb{WAG}(R)$ is connected and $diam(\mathbb{WAG}(R)) \leq 2$.
2) The converse of Lemma 2.1 is not true in general. For example, let $R = \mathbb{Z}_{16}$. Then $\mathbb{A}(R)^* = \{(2), (4), (8)\}$. It is easy to verify from Lemma 2.1 that (2) and (8) are adjacent in $\mathbb{WAG}(R)$. Similarly, (4) and (8) are adjacent in $\mathbb{WAG}(R)$. Note that $ann((2)) = \{0, 8\}$, $ann((4)) = \{0, 4, 8, 12\}$, and $(8) = \{0, 8\}$. As $(8) \subseteq ann((2)) \cap ann((4))$ and $(8)(8) = (0)$, it follows that (2) and (4) are adjacent in $\mathbb{WAG}(R)$. From $(2)(4) \neq (0)$, we get that (2) and (4) are not adjacent in $\mathbb{AG}(R)$. Observe that $\mathbb{WAG}(R)$ satisfies weaker conditions for adjacency of vertices than $\mathbb{AG}(R)$ (see Figure 1).




$\mathbb{AG}(\mathbb{Z}_{16}) \cong K_{1,2}$ $\mathbb{WAG}(\mathbb{Z}_{16}) \cong K_3$

Figure 1. $\mathbb{AG}(\mathbb{Z}_{16})$ and $\mathbb{WAG}(\mathbb{Z}_{16})$

**Theorem 2.3.** *Let $R$ be a ring which is not an integral domain. Then $\mathbb{WAG}(R)$ is connected and $diam(\mathbb{WAG}(R)) \leq 2$. Moreover, if $\mathbb{WAG}(R)$ contains a cycle, then $gr(\mathbb{WAG}(R)) \leq 4$.*

*Proof.* Let $I, J \in \mathbb{A}(R)^*$ be such that $I \neq J$. Suppose that $I$ and $J$ are not adjacent in $\mathbb{WAG}(R)$. Then $ann(I)ann(J) \neq (0)$. Observe that $I - ann(I)ann(J) - J$ is a path of length two between $I$ and $J$ in $\mathbb{AG}(R)$ and so, in $\mathbb{WAG}(R)$. Hence, $diam(\mathbb{WAG}(R)) \leq 2$.

Now, we are assuming that $\mathbb{WAG}(R)$ contains a cycle. Let $I_1 - I_2 - I_3 - \cdots - I_n - I_1$ be a cycle in $\mathbb{WAG}(R)$. In view of [8, Theorem 2.1], we can assume that at least one edge of this cycle is not an edge of $\mathbb{AG}(R)$. Without loss of generality, we can assume that $I_1I_2 \neq (0)$. Either $n = 3$ or $n > 3$. If $n > 3$, then $I_1$ and $I_3$ are not adjacent in $\mathbb{WAG}(R)$. Hence, $ann(I_1)ann(I_3) \neq (0)$. We claim that $ann(I_1)ann(I_3) \notin \{I_1, I_2, I_3\}$. We know that $ann(I_1)ann(I_3) \subseteq ann(I_1) \cap ann(I_3)$. Suppose that $ann(I_1)ann(I_3) = I_1$. This implies that $I_1 \subseteq ann(I_1) \cap ann(I_3)$. Hence, $I_1I_3 = (0)$. This shows that $I_1$ and $I_3$ are adjacent in $\mathbb{AG}(R)$ and so, in $\mathbb{WAG}(R)$, a contradiction. Similarly, we can prove that $ann(I_1)ann(I_3) \neq I_3$. Suppose that $ann(I_1)ann(I_3) = I_2$. This implies that $I_2 \subseteq ann(I_1) \cap ann(I_3)$. Hence, $I_1I_2 = (0)$, a contradiction. Thus, $ann(I_1)ann(I_3) \notin \{I_1, I_2, I_3\}$. Observe that $I_1 - I_2 - I_3 - ann(I_1)ann(I_3) - I_1$ is a cycle of length four in $\mathbb{WAG}(R)$. This proves that if $\mathbb{WAG}(R)$ contains a cycle, then $gr(\mathbb{WAG}(R)) \leq 4$. □

**Lemma 2.4.** *Let $R$ be a reduced ring with $|Min(R)| = 2$. Then $\mathbb{WAG}(R) = \mathbb{AG}(R)$ is a complete bipartite graph.*

*Proof.* Let $Min(R) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$. Since $R$ is reduced, it follows from [5, Proposition 1.8] and [10, Theorem 10] that $(0) = nil(R) = \mathfrak{p}_1 \cap \mathfrak{p}_2$. Observe that $Z(R) = \mathfrak{p}_1 \cup \mathfrak{p}_2$. Let $K \in \mathbb{A}(R)^*$. Then there exists a unique $i \in \{1, 2\}$ such that $K \subseteq \mathfrak{p}_i$. We know from the proof of (4) $\Rightarrow$ (3) of [9, Theorem 2.3] that $\mathbb{AG}(R)$ is a complete bipartite graph with vertex partition $\mathbb{A}(R)^* = V_1 \cup V_2$, where $V_1 = \{I \in \mathbb{A}(R)^* \mid I \subseteq \mathfrak{p}_1\}$ and $V_2 = \{J \in \mathbb{A}(R)^* \mid J \subseteq \mathfrak{p}_2\}$. Moreover, it is clear from Lemma 2.1 that $\mathbb{AG}(R)$ is a spanning subgraph of $\mathbb{WAG}(R)$. Hence, it is enough to prove that every edge of $\mathbb{WAG}(R)$ is an edge of $\mathbb{AG}(R)$. Let $I_1, I_2 \in \mathbb{A}(R)^*$ with $I_1 \neq I_2$. Suppose that $I_1$ and $I_2$ are adjacent in $\mathbb{WAG}(R)$. Therefore, there exist non-zero ideals $A, B$ of $R$ with $A \subseteq ann(I_1)$ and $B \subseteq ann(I_2)$ such that $AB = (0)$. We prove that $I_1$ and $I_2$ are adjacent in $\mathbb{AG}(R)$. As $AB = (0)$, it follows that $A$ and $B$ cannot be contained in the same $\mathfrak{p}_i$. Without loss of generality, we can assume that $A \subseteq \mathfrak{p}_1$ and $B \subseteq \mathfrak{p}_2$. Then $I_1 \subseteq ann(ann(I_1)) \subseteq ann(A) \subseteq \mathfrak{p}_2$. Similarly, it follows that $I_2 \subseteq ann(ann(I_2)) \subseteq ann(B) \subseteq \mathfrak{p}_1$. Therefore, $I_1I_2 = (0)$ and so, $I_1$ and $I_2$ are adjacent in $\mathbb{AG}(R)$. This proves that $\mathbb{WAG}(R) = \mathbb{AG}(R)$ is a complete bipartite graph. □

**Example 2.5.** Let $R = \mathbb{Z}_2 \times \mathbb{Z}$. Observe that $R$ is reduced with $Min(R) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$, where $\mathfrak{p}_1 = \mathbb{Z}_2 \times (0)$ and $\mathfrak{p}_2 = (0) \times \mathbb{Z}$. Therefore, we obtain from Lemma 2.4 that $\mathbb{WAG}(R) = \mathbb{AG}(R)$ is a complete bipartite graph. Moreover, it

is clear that $\mathbb{WAG}(R) = \mathbb{AG}(R)$ is a star graph with center $\mathfrak{p}_1 = \mathbb{Z}_2 \times (0)$. Hence, $diam(\mathbb{WAG}(R)) = diam(\mathbb{AG}(R)) = 2$ and $gr(\mathbb{WAG}(R)) = gr(\mathbb{AG}(R)) = \infty$.

**Lemma 2.6.** *Let $R$ be a ring which is not reduced. Then $r(\mathbb{WAG}(R)) = 1$ and $\mathbb{WAG}(R)$ has at least $|\mathbb{NI}(R)^*|$ centers. Moreover, the subgraph of $\mathbb{WAG}(R)$ induced on $\mathbb{NI}(R)^*$ is complete.*

*Proof.* By hypothesis, $R$ is not reduced. Hence, it is possible to find $x \in R\backslash\{0\}$ such that $x^2 = 0$. It is clear that $Rx \in \mathbb{NI}(R)^*$. Hence, $\mathbb{NI}(R)^* \neq \emptyset$. Let $I \in \mathbb{NI}(R)^*$. Let $J \in V(\mathbb{WAG}(R))$ with $I \neq J$. Let $A$ be a non-zero ideal of $R$ with $A \subseteq ann(J)$. Since $I \in \mathbb{NI}(R)^*$, we obtain that there exists $n \in \mathbb{N}$ such that $I^n = (0)$ and $I^i \neq (0)$ for all $i \in \{1, 2, \ldots, n-1\}$. Note that $I^{n-1} \subseteq ann(I)$. If $I^{n-1}A = (0)$, then $I$ and $J$ are adjacent in $\mathbb{WAG}(R)$. If $I^{n-1}A \neq (0)$, then $I^{n-1}A \subseteq ann(I) \cap ann(J)$ and $I^{n-1}AI^{n-1}A = (0)$. Hence, $I$ and $J$ are adjacent in $\mathbb{WAG}(R)$. This shows that $E(I) = 1$ in $\mathbb{WAG}(R)$ and so, $r(\mathbb{WAG}(R)) = 1$. It is shown that $I$ is a center of $\mathbb{WAG}(R)$ for each $I \in \mathbb{NI}(R)^*$. Therefore, $\mathbb{WAG}(R)$ has at least $|\mathbb{NI}(R)^*|$ centers. As $E(I) = 1$ for all $I \in \mathbb{NI}(R)^*$, it follows that the subgraph of $\mathbb{WAG}(R)$ induced on $\mathbb{NI}(R)^*$ is complete. □

**Corollary 2.7.** *Let $(R, \mathfrak{m})$ be a SPIR. Then $\mathbb{WAG}(R)$ is complete.*

*Proof.* Let $m \in \mathfrak{m}$ be such that $\mathfrak{m} = Rm$. Let $n \geq 2$ be least with the property that $\mathfrak{m}^n = (0)$. Then it follows from the proof of $iii) \Rightarrow i)$ of [5, Proposition 8.8] that $A = \{\mathfrak{m}^i = Rm^i \mid i \in \{1, 2, \ldots, n-1\}\}$ is the set of all non-zero proper ideals of $R$. Note that $A$ is the set of all nilpotent ideals of $R$. Now, the proof is clear from Lemma 2.6. □

**Corollary 2.8.** *Let $R$ be a local Artinian ring which is not a field. Then $r(\mathbb{WAG}(R)) = 1$. In particular, if $R$ is a finite local ring which is not a field, then $r(\mathbb{WAG}(R)) = 1$.*

*Proof.* Let $\mathfrak{m}$ denote the unique maximal ideal of the local Artinian ring $R$. Since $R$ is not a field, it follows that $\mathfrak{m} \neq (0)$. We know from [5, Corollary 8.2 and Proposition 8.4] that $\mathfrak{m}$ is nilpotent. Therefore, we obtain from Lemma 2.6 that $E(\mathfrak{m}) = 1$. Therefore, $r(\mathbb{WAG}(R)) = 1$.

Let $R$ be a finite local ring which is not a field. Since any finite ring is Artinian, it follows from the previous paragraph that $r(\mathbb{WAG}(R)) = 1$. □

**Example 2.9.** Let $R = \mathbb{Z}_{p^n}$ for some prime $p \geq 2$ and positive integer $n \geq 2$. Note that $\mathbb{A}(R)^* = \mathbb{NI}(R)^* = \{(p), (p^2), \ldots, (p^{n-1})\}$. Hence, it is clear from Lemma 2.6 that $\mathbb{WAG}(R)$ is complete.

**Lemma 2.10.** *Let $R$ be a ring and let $\mathbb{WAG}(R)$ contains a cycle. Then $gr(\mathbb{WAG}(R)) = 4$ if and only if $R$ is reduced with $|Min(R)| = 2$.*

*Proof.* Suppose that $gr(\mathbb{WAG}(R)) = 4$. We first claim that $R$ is reduced. Suppose that $R$ is not reduced. Then there exists $x \in R\backslash\{0\}$ such that $x^2 = 0$. Let $I = Rx$. It is clear that $I \in \mathbb{A}(R)^*$. Since $I$ is nilpotent, it follows from Lemma 2.6 that $gr(\mathbb{WAG}(R)) = 3$, a contradiction. This proves that $R$ is reduced. We next claim that $\mathbb{WAG}(R) = \mathbb{AG}(R)$. Suppose that $\mathbb{WAG}(R) \neq \mathbb{AG}(R)$. Then there exist distinct $I, J \in \mathbb{A}(R)^*$ such that $I$ and $J$ are adjacent in $\mathbb{WAG}(R)$ but they

are not adjacent in $\mathbb{AG}(R)$. Hence, there exist non-zero ideals $A, B$ of $R$ with $A \subseteq ann(I)$ and $B \subseteq ann(J)$ such that $AB = (0)$. As $R$ is reduced, it follows that $A \neq B$ and $A, B \notin \{I, J\}$.

Suppose that $(0) \neq T \subseteq ann(I) \cap ann(J)$. Then we obtain a cycle $T - I - J - T$ of length 3 in $\mathbb{WAG}(R)$. Hence, $gr(\mathbb{WAG}(R)) = 3$, a contradiction. Suppose that $ann(I) \cap ann(J) = (0)$. We claim that $I$, $IJ$, and $J$ are pairwise distinct. Suppose that $I = IJ$. It is clear that $B \subseteq ann(J) \subseteq ann(IJ) = ann(I)$. As $I$ and $J$ are adjacent in $\mathbb{WAG}(R)$, it follows that $B^2 = (0)$. Since $R$ is reduced, we get that $B = (0)$, a contradiction. Thus, $I$ and $IJ$ are pairwise distinct. Similarly, we can show that $J$ and $IJ$ are pairwise distinct. Since $A \subseteq ann(I) \subseteq ann(IJ)$ and $AB = (0)$, we get that $IJ$ and $J$ are adjacent in $\mathbb{WAG}(R)$. Moreover, $IJ$ and $I$ are adjacent in $\mathbb{WAG}(R)$, since $B \subseteq ann(J) \subseteq ann(IJ)$ and $AB = (0)$. Thus, we obtain a cycle $IJ - I - J - IJ$ of length 3 in $\mathbb{WAG}(R)$, again a contradiction. Hence, $\mathbb{WAG}(R) = \mathbb{AG}(R)$. This proves that $gr(\mathbb{AG}(R)) = 4$. This implies that $\mathbb{AG}(R)$ is a complete bipartite graph [3, Theorem 2.4]. Hence, $|Min(R)| = 2$ [9, Corollary 2.5].

Conversely, suppose that $R$ is reduced with $|Min(R)| = 2$. Therefore, we obtain from Lemma 2.4 that $\mathbb{WAG}(R)$ is complete bipartite. By hypothesis, $\mathbb{WAG}(R)$ contains a cycle and so, $gr(\mathbb{WAG}(R)) = 4$. □

**Example 2.11.** Let $R = K[X, Y]/(XY)$, where $K$ is an arbitrary field. It is easy to verify that $\mathbb{AG}(R)$ is complete bipartite with two infinite parts $V_1 = \{(\bar{f}) \mid f \in K[X], f(0) = 0\}$ and $V_2 = \{(\bar{g}) \mid g \in K[Y], g(0) = 0\}$. Observe that $R$ is reduced with $|Min(R)| = 2$. Hence, it follows from Lemma 2.10 that $gr(\mathbb{WAG}(R)) = 4$. Moreover, it is clear from Lemma 2.4 that $\mathbb{WAG}(R) = \mathbb{AG}(R)$ is complete bipartite.

**Theorem 2.12.** *Let $R$ be a reduced ring which is not an integral domain and let $Z(R)$ be an ideal of $R$. Then $\mathbb{WAG}(R) \neq \mathbb{AG}(R)$ and $gr(\mathbb{WAG}(R)) = 3$.*

*Proof.* By hypothesis, $R$ is reduced and $Z(R)$ is an ideal of $R$. Then $Z(R)$ is necessarily a prime ideal of $R$ and let us denote it by $\mathfrak{p}$. Since $R$ is not an integral domain, there exist $a, b \in R \backslash \{0\}$ such that $ab = 0$. Let us denote $Ra$ by $I$. Note that $I$ is a non-zero annihilating ideal of $R$. Let us denote $ann(I)$ by $J$. As $x^2 \neq 0$ for any non-zero $x \in R$, it follows that $\mathfrak{p} \neq ann(x)$ for any non-zero $x \in R$. Therefore, we obtain from [14, Lemma 2.1] that $\mathfrak{p} \nsubseteq I \cup J$. Hence, there exists $x \in \mathfrak{p}$ such that $Ix \neq (0)$ and $Jx \neq (0)$. It is clear that $Rx \in \mathbb{A}(R)^*$. Note that either $I + Rx \notin \mathbb{A}(R)$ or $I + Rx \in \mathbb{A}(R)$. If $I + Rx \notin \mathbb{A}(R)$, then $d(I, Rx) \geq 3$ in $\mathbb{AG}(R)$. Suppose that $I + Rx \in \mathbb{A}(R)$. We know from [12, Lemma 2.3] that $I + J \notin \mathbb{A}(R)$. Therefore, $I + Rx + J \notin \mathbb{A}(R)$. From $Jx \neq (0)$, it follows that $(I + Rx)J \neq (0)$. Therefore, $d(I + Rx, J) \geq 3$ in $\mathbb{AG}(R)$. Hence, we obtain from [8, Theorem 2.1] that $diam(\mathbb{AG}(R)) = 3$. We know from Theorem 2.3 that $diam(\mathbb{WAG}(R)) \leq 2$ and so, $\mathbb{WAG}(R) \neq \mathbb{AG}(R)$.

Now, we claim that $gr(\mathbb{WAG}(R)) = 3$. Since $R$ is not an integral domain, there exist $x, y \in Z(R)^*$ such that $xy = 0$. From $x^2 \neq 0$, it follows that $\mathfrak{p} \neq ann(x)$. From $y^2 \neq 0$, it follows that $\mathfrak{p} \neq ann(y)$. Observe that $ann(x) \cup ann(y) \subseteq \mathfrak{p}$. Note that $\mathfrak{p} \nsubseteq ann(x) \cup ann(y)$. Let $z \in \mathfrak{p}$ be such that $zx \neq 0$ and $zy \neq 0$. Let

$w \in R$, $w \neq 0$ be such that $zw = 0$. Observe that $Rzx - Rw - Rzy - Rzx$ is a cycle of length 3 in $\mathbb{AG}(R)$. Therefore, $gr(\mathbb{AG}(R)) = 3$ and so, $gr(\mathbb{WAG}(R))$ is also 3. □

**Lemma 2.13.** *Let $R$ be a ring which is not reduced. If $Z(R)$ is an annihilator ideal, then $\mathbb{WAG}(R)$ is complete.*

*Proof.* By hypothesis, $R$ is not reduced. Since $Z(R)$ is an annihilator ideal, there exists $x \in R\backslash\{0\}$ such that $Z(R) = ann(x)$. Let us denote $Rx$ by $K$. It is clear that $K \in \mathbb{A}(R)^*$. Note that $x \in Z(R) = ann(x)$ and so, $x^2 = 0$. Let $I, J \in \mathbb{A}(R)^*$ be such that $I \neq J$. Note that $Ix = (0)$ and $Jx = (0)$. Suppose that $K \in \{I, J\}$. Without loss of generality, we can assume that $K = I$. In this case, $IJ = KJ = (Rx)(J) = (0)$. Thus, $I$ and $J$ are adjacent in $\mathbb{AG}(R)$ and so, in $\mathbb{WAG}(R)$. Suppose that $K \notin \{I, J\}$. In this case, $K \subseteq ann(I) \cap ann(J)$ and $K^2 = (Rx)^2 = (0)$. Hence, $I$ and $J$ are adjacent in $\mathbb{WAG}(R)$. This proves that $\mathbb{WAG}(R)$ is complete. □

**Lemma 2.14.** *Let $R$ be a ring which is not an integral domain. If $gr(\mathbb{WAG}(R)) = \infty$, then $\mathbb{WAG}(R) = \mathbb{AG}(R)$.*

*Proof.* By hypothesis, $gr(\mathbb{WAG}(R)) = \infty$. It is clear from Lemma 2.1 that $\mathbb{AG}(R)$ is a spanning subgraph of $\mathbb{WAG}(R)$. Hence, it is enough to prove that every edge of $\mathbb{WAG}(R)$ is an edge of $\mathbb{AG}(R)$. Let $I, J \in \mathbb{A}(R)^*$ with $I \neq J$. Suppose that $I$ and $J$ are adjacent in $\mathbb{WAG}(R)$. We claim that $I$ and $J$ are adjacent in $\mathbb{AG}(R)$ under the assumption that $\mathbb{WAG}(R)$ has no cycle. Suppose that $I$ and $J$ are not adjacent in $\mathbb{AG}(R)$. Then $d(I, J) = 2$ in $\mathbb{AG}(R)$ or $d(I, J) = 3$ in $\mathbb{AG}(R)$. If $d(I, J) = 2$ in $\mathbb{AG}(R)$, then there exists $K \in \mathbb{A}(R)^*\backslash\{I, J\}$ such that $I - K - J$ is a path of length 2 between $I$ and $J$ in $\mathbb{AG}(R)$. Therefore, it is clear from Lemma 2.1 that $I - K - J - I$ is a cycle of length 3 in $\mathbb{WAG}(R)$. This is impossible. If $d(I, J) = 3$ in $\mathbb{AG}(R)$, then there exist distinct $K_1, K_2 \in \mathbb{A}(R)^*\backslash\{I, J\}$ such that $I - K_1 - K_2 - J$ is a path of length 3 between $I$ and $J$ in $\mathbb{AG}(R)$. Therefore, it is again clear from Lemma 2.1 that $I - K_1 - K_2 - J - I$ is a cycle of length 4 in $\mathbb{WAG}(R)$. This is in contradiction to the assumption that $\mathbb{WAG}(R)$ has no cycle. Therefore, $I$ and $J$ are adjacent in $\mathbb{AG}(R)$. □

## 3. When is $\mathbb{WAG}(R)$ identical to $\mathbb{AG}(R)$?

Let $R$ be a ring which is not an integral domain. We know from Lemma 2.1 that $\mathbb{AG}(R)$ is a spanning subgraph of $\mathbb{WAG}(R)$. A natural question is posed: When are $\mathbb{WAG}(R)$ and $\mathbb{AG}(R)$ identical? In this section, we completely answer this question. We start this section with the following theorem.

**Theorem 3.1.** *Let $R$ be a reduced ring which is not an integral domain. Then $\mathbb{WAG}(R) = \mathbb{AG}(R)$ if and only if $|Min(R)| = 2$.*

*Proof.* Suppose that $\mathbb{WAG}(R) = \mathbb{AG}(R)$. If $|Min(R)| \geq 3$, then $diam(\mathbb{AG}(R)) = 3$ [9, Lemma 1.8]. This contradicts the fact that $diam(\mathbb{WAG}(R)) \leq 2$. As $R$ is not an integral domain, it follows that $|Min(R)| \neq 1$. Hence, $|Min(R)| = 2$.

Conversely, suppose that $R$ is reduced with $|Min(R)| = 2$. Hence, it is clear from Lemma 2.4 that $\mathbb{WAG}(R) = \mathbb{AG}(R)$. □

**Corollary 3.2.** *Let $R$ be a reduced ring. Then the following statements are equivalent:*
1) *$R$ has exactly two minimal prime ideals.*
2) *$\mathbb{WAG}(R) = \mathbb{AG}(R)$ is complete bipartite with two non-empty parts.*
3) *$\mathbb{WAG}(R) = \mathbb{AG}(R)$ is bipartite with two non-empty parts.*
4) *$\chi(\mathbb{AG}(R)) = 2$.*

*Proof.* 1) $\Rightarrow$ 2) is clear from Theorem 3.1 and Lemma 2.4. Moreover, 2) $\Rightarrow$ 3) $\Rightarrow$ 4) $\Rightarrow$ 1) are clear from [9, Corollary 2.5]. □

We next study non-reduced rings $R$ whose $\mathbb{WAG}(R)$ and $\mathbb{AG}(R)$ are identical.

**Theorem 3.3.** *Let $R$ be a ring which is not reduced. Then the following statements are equivalent:*
1) *$\mathbb{WAG}(R) = \mathbb{AG}(R)$.*
2) *$\mathbb{AG}(R)$ is complete.*

*Proof.* 1) $\Rightarrow$ 2): By hypothesis, $R$ is not reduced. Hence, it is possible to find $x \in R\backslash\{0\}$ such that $x^2 = 0$. Let us denote $Rx$ by $I$. It is clear that $I \in \mathbb{NI}(R)^*$. We know from Lemma 2.6 that $E(I) = 1$ in $\mathbb{WAG}(R)$. This fact together with $\mathbb{WAG}(R) = \mathbb{AG}(R)$ implies that $Z(R)$ is an annihilator ideal [8, Theorem 2.2 and Corollary 2.3]. Therefore, it is clear from Lemma 2.13 that $\mathbb{WAG}(R) = \mathbb{AG}(R)$ is complete.
2) $\Rightarrow$ 1): It is clear from Lemma 2.1 that $\mathbb{AG}(R)$ is a spanning subgraph of $\mathbb{WAG}(R)$. This concludes that $\mathbb{WAG}(R) = \mathbb{AG}(R)$. □

**Corollary 3.4.** *Let $R$ be a ring which is not reduced. Then the following statements are equivalent:*
1) *$\mathbb{WAG}(R)$ is a star graph.*
2) *$gr(\mathbb{WAG}(R)) = \infty$.*
3) *$\mathbb{WAG}(R) = \mathbb{AG}(R)$ and $gr(\mathbb{AG}(R)) = \infty$.*
4) *$|\mathbb{A}(R)^*| = |\mathbb{NI}(R)^*| = 2$.*
5) *$\mathbb{WAG}(R) = \mathbb{AG}(R) \cong K_{1,1}$.*

*Proof.* 1) $\Rightarrow$ 2): It is clear.
2) $\Rightarrow$ 3): As $gr(\mathbb{WAG}(R)) = \infty$, it follows from Lemma 2.14 that $\mathbb{WAG}(R) = \mathbb{AG}(R)$ and so, $gr(\mathbb{AG}(R)) = \infty$.
3) $\Rightarrow$ 4): It is clear from Theorem 3.3 that $\mathbb{WAG}(R)$ and $\mathbb{AG}(R)$ are complete. Since $gr(\mathbb{AG}(R)) = \infty$ and $R$ is not reduced, we get that $|\mathbb{A}(R)^*| = |\mathbb{NI}(R)^*| = 2$.
4) $\Rightarrow$ 5): It is clear from Lemma 2.6.
5) $\Rightarrow$ 1): It is clear. □

Now, we provide following examples to illustrate results proved in this section.

**Example 3.5.** Let $R = D \times F$, where $D$ is an integral domain which is not a field and $F$ is a field. Let $\mathfrak{p}_1 = (0) \times F$ and $\mathfrak{p}_2 = D \times (0)$. It is clear that $R$ is reduced and $Min(R) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$. Hence, it is clear from Corollary 3.2 that $\mathbb{WAG}(R) = \mathbb{AG}(R)$ is a complete bipartite graph.

**Example 3.6.** Let $(R, \mathfrak{m})$ be a SPIR with $\mathfrak{m}^2 \neq (0)$ but $\mathfrak{m}^3 = (0)$. Note that $R$ is non-reduced and $\mathbb{A}(R)^* = \{\mathfrak{m}, \mathfrak{m}^2\}$. Hence, it is clear from Corollary 3.4 that $\mathbb{AG}(R) = \mathbb{WAG}(R) \cong K_{1,1}$.

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