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CRITICAL POINT EQUATION WITHIN THE FRAMEWORK OF VARIOUS CONTACT METRIC MANIFOLDS

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ABSTRACT. The aim of the present paper is to study the critical point equation (shortly CPE) conjecture within the framework of various contact metric manifolds. First we establish that Kenmotsu manifold satisfying the CPE either becomes an Einstein manifold or the derivative of potential function along characteristic vector field satisfy a certain relation on the distribution of η. Next we study CPE on $(\kappa,\mu)'$-almost Kenmotsu manifold and obtain that the manifold is Einstein. Later in case of 3-dimensional trans-Sasakian manifold, we get that either the manifold becomes α-Sasakian or it becomes Einstein. Finally we give examples of 3-dimensional trans-Sasakian manifold and $(\kappa,\mu)'$-almost Kenmotsu manifold to verify our outcomes.

1. Introduction

Throughout the paper we consider M as $(2n+1)$-dimensional compact (without boundary) orientable Riemannian manifold with dimension ≥ 3. Let $\mathcal{M}$ denote the set of all Riemannian metrics on M of unit volume. For given metric $g \in \mathcal{M}$ let $r_g : M \to \mathbb{R}$ and dv_g be respectively its scalar curvature and volume determined by its metric and orientation, then the total scalar curvature functional $\mathcal{S} : \mathcal{M} \to \mathbb{R}$ is defined by,

$$\mathcal{S}(g) = \int_{\mathcal{M}} r_g dv_g.$$

Einstein and Hilbert proved that the critical points of the functional $\mathcal{S}$ are Einstein metrics i.e., the critical points satisfy the Euler-Lagrange equation,

$$S_g^0 = S_g - \frac{r_g}{2n+1} g = 0, \tag{1.1}$$

where S_g^0, S_g are traceless Ricci tensor and Ricci tensor of g respectively.

As Yamabe stated that any compact manifold carries many smooth Riemannian metrics with constant scalar curvature, we define the set of metrices with constant scalar curvature as $\mathcal{C} = \{g \in M | r_g = constant\}$. Now, (1.1) restricted to $\mathcal{C}$ on M can be written as,

$$\mathcal{L}_g^*(\lambda) = S_g^0, \tag{1.2}$$

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where $\mathcal{L}_g^*(\lambda)$ is the formal L^2-adjoint of the linearized scalar curvature operator $\mathcal{L}_g(\lambda)$ and is defined as,

$$\mathcal{L}_g^*(\lambda) = -(\Delta_g\lambda)g + Hess_g\lambda - \lambda S_g, \tag{1.3}$$

where $\Delta_g\lambda$ and $Hess_g$ are Laplacian and hessian of the smooth function λ.

The function λ in (1.2) is known as the potential function. Interestingly, if λ is constant in equation (1.3), then the manifold becomes Einstein. So, now we always consider a non-constant potential function λ as a solution of (1.2). With the help of (1.1) and (1.3) we can rewrite (1.2) as,

$$S_g - \frac{r_g}{2n+1}g = (\frac{r_g}{2n}g - S_g)\lambda + Hess_g\lambda. \tag{1.4}$$

Now we consider the following definition of critical point equation (shortly CPE),

Definition 1.1. A compact oriented (without boundary) Riemannian metric manifold (M, g) of unit volume and dimension ≥ 3 with constant scalar curvature together with non-constant smooth potential function λ satisfying the equation (1.4) is called a critical point metric manifold.

In [3], Besse conjectured that a Critical Point Equation metric is always Einstein. Since then proving the conjecture has become the motivation for many mathematicians. Still date no one can prove it but some partial results are developed under some particular curvature assumptions. Lafontaine [12] proved that the conjecture is true for a locally conformally flat manifold. Then some mathematicians have improved the result, for example, Barros and Ribeiro [2] under half conformally flat assumption. In [10], Hwang concluded that the conjecture is true under the assumption $f \geqslant -1$, where f is a smooth function defined on M satisfying the critical point equation. Then he was able to prove that, if the second homology group of a 3-dimensional manifold vanishes, then it is diffeomorphic to $\mathbb{S}^3$. Again he with the help of Yun and Chang [5] provided a result stated as if (g, λ) is a non-trivial solution of the CPE on an n-dimensional compact Riemannian manifold M and satisfies the following conditions (a) Ricci tensor of g is parallel, (b) g has harmonic curvature (c) (M, g) is conformally flat, then (M, g) is isometric to a standard sphere. Then Neto [14] provided a necessary and sufficient condition on the norm of the gradient of potential function for a CPE metric to be Einstein. Most recently, Patra and Ghosh [16] has established (a) a complete K-contact metric satisfying the CPE is Einstein and is isometric to a unit sphere $\mathbb{S}^{2n+1}$, (b) if a non-Sasakian (κ, μ)-contact metric satisfies the CPE, then M^3 is flat and M^{2n+1} is locally isometric to $E^{n+1} \times \mathbb{S}^n$ if $n > 1$.

In this paper we consider the critical point equation (shortly CPE) conjecture within the framework of various contact metric manifolds. First we establish that a Kenmotsu manifold satisfying CPE is of constant scalar curvature and either the manifold is Einstein or it satisfies a particular relation on the distribution of η. Next we conclude that $(\kappa, -2)'$-almost Kenmotsu manifold with $\kappa \leq -1$ admitting CPE becomes Einstein manifold. Later we establish that a 3-dimensional trans-Sasakian manifold with constant structure functions becomes α-Sasakian or Einstein manifold, if it satisfies CPE. Finally we have constructed examples of 3-dimensional trans-Sasakian manifold and $(\kappa, -2)'$-almost Kenmotsu manifold.

2. Preliminaries

By [4], a differentiable manifold M of dimension $2n+1$ is said to has an almost contact structure or (ϕ, ξ, η) structure if M admits a $(1,1)$-tensor field ϕ, a vector field ξ, and an 1-form η satisfying,

$$\phi^2 = -I + \eta \otimes \xi, \tag{2.1}$$

$$\eta(\xi) = 1, \tag{2.2}$$

where I is the identity mapping. A Riemannian metric g is said to be compatible metric if it satisfies,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

for arbitrary vector fields X and Y on M. A manifold having almost contact structure along with compatible Riemannian metric is called almost contact metric manifold.

In a almost contact metric manifold the following conditions are satisfied,

$$\phi\xi = 0, \tag{2.4}$$

$$\eta \circ \phi = 0, \tag{2.5}$$

$$g(X, \xi) = \eta(X), \tag{2.6}$$

$$g(\phi X, Y) = -g(X, \phi Y), \tag{2.7}$$

for arbitrary $X, Y \in \chi(M)$.

2.1. **Kenmotsu manifold:** By [1] and [4], if in an almost contact metric manifold M the following condition holds for arbitrary $X, Y \in \chi(M)$,

$$(\nabla_X \phi)Y = -\eta(Y)\phi(X) - g(X, \phi Y)\xi, \tag{2.8}$$

where ∇ denotes the Levi-Civita connection of g, then the manifold M is called a Kenmotsu manifold.

In Kenmotsu manifold of dimension $(2n+1)$ the following relations hold,

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.9}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.10}$$

$$S(\phi X, \phi Y) = S(X, Y) + 2n\eta(X)\eta(Y), \tag{2.11}$$

$$S(X, \xi) = -2n\eta(X) \Leftrightarrow Q\xi = -2n\xi, \tag{2.12}$$

for arbitrary vector fields X, Y, Z, W on M, where R is Riemannian curvature tensor, S is Ricci operator and Q is Ricci tensor, defined by $S(X, Y) = g(QX, Y)$.

2.2. $(\kappa, \mu)'$ **almost Kenmotsu manifold:** Let M be a $(2n+1)$-dimensional almost contact manifold. An almost complex structure J is defined on $M \times \mathbb{R}$ by $J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$, where X is a tangent to M, t is the coordinate on $\mathbb{R}$ and f a C^∞ function on $M \times \mathbb{R}$. Clearly $J^2 = -I$. If J is integrable then the almost contact structure is said to be normal. The normality of an almost contact metric manifold is equivalent with vanishing of the tensor field $[\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis torsion tensor of ϕ (for more details see [4]).

The fundamental 2-form Φ on an almost contact metric manifold M is defined by $\Phi(X,Y) = g(X,\phi Y)$ for arbitrary $X,Y \in \chi(M)$. We recall from [7] and [17], an *almost Kenmotsu manifold* is an almost contact metric manifold where η is closed and $d\Phi = 2\eta \wedge \Phi$. Now consider two (1,1)-type tensor fields $h = \frac{1}{2}\mathcal{L}_\xi \phi$ and $h' = h \circ \phi$ and an operator $\ell = R(.,\xi)\xi$, where $\mathcal{L}_\xi \phi$ is the Lie derivative of ϕ along the direction ξ. In an almost Kenmotsu manifold M the following relations are satisfied $\forall X,Y \in \chi(M)$,

$$h\xi = h'\xi = 0, \tag{2.13}$$

$$tr(h) = tr(h') = 0, \tag{2.14}$$

$$h\phi = -\phi h, \tag{2.15}$$

$$\nabla_X \xi = X - \eta(X)\xi + h'X, \tag{2.16}$$

$$R(X,Y)\xi = \eta(X)(Y + h'Y) - \eta(Y)(X + h'X) + (\nabla_X h')Y - (\nabla_Y h')X,$$

By $(\kappa,\mu)'$-almost Kenmotsu manifold we mean almost Kenmotsu manifold where the vector field ξ satisfies the $(\kappa,\mu)'$ nullity distribution (for details see [7]), i.e.,

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)h'X - \eta(X)h'Y), \tag{2.17}$$

for any vector fields X and Y on M. On a $(\kappa,\mu)'$-almost Kenmotsu manifold M we have [7],

$$h'^2 = -(\kappa+1)[X - \eta(X)\xi], \tag{2.18}$$

for $X \in \chi(M)$. From previous relation it follows that $h' = 0$ if and only if $\kappa = -1$ and $h' \neq 0$ otherwise. Let $X \in Ker(\eta)$ be an eigenvector field of h' orthogonal to ξ w.r.t. the eigenvalue α. Then, from (2.18) we get $\alpha^2 = -(\kappa+1)$ which implies $\kappa \leq -1$. Dileo and Pastore proved that on a $(\kappa,\mu)'$-almost Kenmotsu manifold with $\kappa < -1$, we have $\mu = -2$ (Proposition 4.1 of [7]).

We recall from [17] and [7] that the following relations hold on a $(2n+1)$-dimensional $(\kappa,-2)'$-almost Kenmotsu manifold M, with $\kappa \leq -1$,

$$(\nabla_X h')Y = -g(h'X + h'^2X, Y)\xi - \eta(Y)(h'X + h'^2X), \tag{2.19}$$

$$QX = -2nX + 2n(\kappa+1)\eta(X)\xi - 2nh'(X), \tag{2.20}$$

$$2n(\kappa - 2n) = r, \tag{2.21}$$

where $X,Y \in \chi(M)$.

2.3. **Trans-Sasakian manifold:** A almost contact metric manifold M is called a trans-Sasakian manifold if $(M \times \mathbb{R}, J, G)$ [defined in section 2.2], where G is the product metric on $M \times \mathbb{R}$, belongs to the class W_4 (see [9]). If there are smooth functions α,β on an almost contact metric manifold (M,ϕ,ξ,η,g) satisfying,

$$(\nabla_X \phi)Y = \alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(\phi X,Y)\xi - \eta(Y)\phi X], \tag{2.22}$$

where $X,Y \in \chi(M)$ are arbitrary, then the manifold is called trans-Sasakian manifold of type (α,β) [11]. α,β are called structure functions of the manifold. Trans-Sasakian manifolds of type $(0,0), (\alpha,0), (0,\beta)$ are called cosymplectic, α-Sasakian, β-Kenmotsu manifolds respectively. Then form (2.22), we can deduce that,

$$\nabla_X \xi = -\alpha(\phi X) + \beta(X - \eta(X)\xi). \tag{2.23}$$

Marrero [13] showed that a trans-Sasakian manifold of dimension ≥ 5 is either cosymplectic or α-Sasakian or β-Kenmotsu. So proper trans-Sasakian manifold exists for dimension 3. In a 3-dimensional trans-Sasakian manifold the following relations hold,

$$\begin{aligned} R(X,Y)Z =& (\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2))(g(Y,Z)X - g(X,Z)Y) - g(Y,Z) \\ & [(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(X)\xi - \eta(X)(\phi D\alpha - D\beta) + (X\beta \\ & + (\phi X)\alpha)\xi] + g(X,Z)[(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(Y)\xi - \eta(Y) \\ & (\phi D\alpha - D\beta) + (Y\beta + (\phi Y)\alpha)\xi)] - [(Z\beta + (\phi Z)\alpha)\eta(Y) \\ & + (Y\beta + (\phi Y)\alpha)\eta(Z) + (\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(Y)\eta(Z)]X \\ & + [(Z\beta + (\phi Z)\alpha)\eta(X) + (X\beta + (\phi X)\alpha)\eta(Z) + (\frac{r}{2} + \xi\beta \\ & - 3(\alpha^2 - \beta^2))\eta(X)\eta(Z)]Y, \\ S(X,Y) =& (\frac{r}{2} + \xi\beta - (\alpha^2 - \beta^2))g(X,Y) - (\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(X) \\ & \eta(Y) - (Y\beta + (\phi Y)\alpha)\eta(X) - (X\beta + (\phi X)\alpha)\eta(Y), \end{aligned}$$

where Df denotes the gradient of the smooth function f and α, β are smooth functions on the manifold (for details see [15]).

Here in this paper we restricted the smooth functions α, β to be constant functions [8]. Then we got some special relations compatible to our restrictions,

$$R(X,Y)\xi = (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y), \tag{2.24}$$

$$S(X,Y) = (\frac{r}{2} - (\alpha^2 - \beta^2))g(X,Y) - (\frac{r}{2} - 3(\alpha^2 - \beta^2))\eta(X)\eta(Y). \tag{2.25}$$

3. Main Results

Lemma 3.1. *Let (g,λ) be a non-constant solution of the critical point equation on an $(2n+1)$-dimensional Riemannian manifold M. Then the Riemannian curvature tensor R can be expressed as,*

$$R(X,Y)D\lambda = (X\lambda)QY - (Y\lambda)QX + (\lambda+1)[(\nabla_X Q)Y - (\nabla_Y Q)X] + (Xf)Y - (Yf)X \tag{3.1}$$

for any vector fields $X, Y \in \chi(M)$ and $f = -r(\frac{\lambda}{2n} + \frac{1}{2n+1})$.

Proof. Since (g,λ) is a non-constant solution of the critical point equation, so, $S - \frac{r}{2n+1}g = (\frac{r}{2n}g - S)\lambda - Hess(\lambda)$. Tracing this equation we get, $\Delta_g\lambda = -r_g\frac{\lambda}{2n}$. Thus the above mentioned equation can be exhibited as,

$$\nabla_X D\lambda = (\lambda+1)QX + fX,$$

for an arbitrary vector field X, where $f = -r(\frac{\lambda}{2n} + \frac{1}{2n+1})$. Now taking covariant derivative of the above equation with respect to an arbitrary vector field Y, we obtain

$$\nabla_Y(\nabla_X D\lambda) = (Y\lambda)QX + (\lambda+1)[(\nabla_Y Q)X + Q(\nabla_Y X))] + (Yf)X + f\nabla_Y X.$$

Then we apply the expression for curvature tensor $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ to get our required result. □

Theorem 3.2. *Let $M(\phi,\xi,\eta,g)$ be a $(2n+1)$-dimensional Kenmotsu manifold. If (g,λ) is a non-constant solution of the CPE, then*

(1) *the scalar curvature of the manifold is $-2n(2n+1)$, and*
(2) *either the manifold is Einstein or $\xi\lambda = 1+\lambda$ in kernel of η.*

Proof. Plugging $Y=\xi$ in equation (3.1) and using (2.12), we acquire

$$R(X,\xi)D\lambda = [(Xf)-2n(X\lambda)]\xi+(\lambda+1)[(\nabla_X Q)\xi-(\nabla_\xi Q)X]-(\xi\lambda)QX-(\xi f)X. \tag{3.2}$$

for an arbitrary vector field X on M. From Lemma 3.1 of [6], we obtain the two relations on Kenmotsu manifold as follows, $(\nabla_X Q)\xi = -QX - 2nX$ and $(\nabla_\xi Q)X = -2QX - 4nX$, $\forall X \in \chi(M)$. Using (2.10) along with the foregoing two relations in (3.2), we achieve

$$[(Xf)-(2n+1)(X\lambda)]\xi+(\xi\lambda)[X-QX]-(\xi f)X+(\lambda+1)[QX+2nX] = 0, \tag{3.3}$$

for any vector field X in $\chi(M)$. Considering inner product of (3.3) along ξ and taking (2.12) into account, we get

$$Xf - (2n+1)(X\lambda) + (2n+1)(\xi\lambda)\eta(X) - (\xi f)\eta(X) = 0, \tag{3.4}$$

$\forall X \in \chi(M)$. Since (g,λ) is a non-constant solution of CPE, the scalar curvature r is constant and therefore $df = -\frac{r}{2n}d\lambda$, where d is the exterior derivative operator. As $n \neq 0$, using this in (3.4), we obtain

$$[r+2n(2n+1)](X\lambda - (\xi\lambda)\eta(X)) = 0. \tag{3.5}$$

As, X is an arbitrary vector field in the above relation, from here two cases arise, either $r = -2n(2n+1)$ or $D\lambda = (\xi\lambda)\xi$.

Let us consider the case $D\lambda = (\xi\lambda)\xi$. Covariant derivative along an arbitrary vector field X, yields

$$\nabla_X D\lambda = (X(\xi\lambda))\xi + (\xi\lambda)X - (\xi\lambda)\eta(X)\xi, \tag{3.6}$$

where we have used (2.9). From critical point equation (1.4), using (3.6), we acquire

$$(1+\lambda)QX = \frac{r\lambda}{2n}X + \frac{r}{2n+1}X + (X(\xi\lambda))\xi + (\xi\lambda)X - (\xi\lambda)\eta(X)\xi.$$

Considering $X=\xi$ in the last equation and using (2.2) and (2.12), we have

$$\xi(\xi\lambda) + \frac{r}{2n+1} + \frac{r\lambda}{2n} + 2n(1+\lambda) = 0. \tag{3.7}$$

Contracting X in (3.6), we get $\Delta\lambda = \xi(\xi\lambda) + 2n(\xi\lambda)$. Again, in the proof of Lemma 3.1 we have $\Delta\lambda = -r\frac{\lambda}{2n}$. Combining these last two relations we get

$$\xi(\xi\lambda) + \frac{r\lambda}{2n} + 2n(\xi\lambda) = 0. \tag{3.8}$$

Since $n \neq 0$, manipulating (3.7) using (3.8), yields

$$\xi\lambda = \frac{r}{2n(2n+1)} + 1 + \lambda. \tag{3.9}$$

Differentiating (3.9) along ξ, as r is constant, we have $\xi(\xi\lambda) = \xi\lambda$. Using this relation in (3.8), we obtain $(1+\lambda)[r+2n(2n+1)] = 0$. Since λ is a non-constant function, we can conclude that $r = -2n(2n+1)$.

Considering (3.3) along an arbitrary vector field Y and the replacing X and Y by ϕX and ϕY, respectively, we have

$$(1+\lambda-\xi\lambda)[S(X,Y)+2ng(X,Y)] = 0.$$

$\forall X, Y \in \chi(M)$, where we have used (2.3), (2.5), (2.11) and $r = -2n(2n+1)$. So, either the manifold is Einstein with Einstein constant $-2n$, or $\xi\lambda = 1+\lambda$.

Let us consider the case where $\xi\lambda = 1+\lambda$. Differentiating this relation along an arbitrary vector field X and using (2.9), we get $Hess_\lambda(X,\xi) = (\xi\lambda)\eta(X)$. Replacing Y by ξ in (1.4) and using (2.9) and the expression of hessian we get $\eta(X) = 0$. From here we can easily obtain our desired result. □

Remark 3.3. Now if we consider a ϕ-basis $\{e_i, \phi e_i, \xi\}, i = 1, 2, 3, ..., n$ of M such that $Qe_i = \rho_i e_i$. Then we have $\phi Qe_i = \rho_i \phi e_i$. Using the ϕ-basis and (2.12) we can conclude that the scalar curvature r of the manifold is

$$r = g(Q\xi,\xi) + \sum_{i=1}^{n}[g(Qe_i, e_i) + g(Q\phi e_i, \phi e_i)] = -2n + 2\sum_{i=1}^{n}\rho_i.$$

Let us suppose $\sum_{i=1}^{n}\rho_i = \kappa$. Then from the critical point equation we have,

$$\begin{aligned} f =& -r(\frac{\lambda}{2n} + \frac{1}{2n+1}) \\ =& \frac{n-\kappa}{n}\lambda + \frac{2n-2\kappa}{2n+1}. \end{aligned}$$

Theorem 3.4. *Let $M(\phi,\xi,\eta,g)$ be an $(2n+1)$-dimensional $(\kappa,-2)'$-almost Kenmotsu manifold with $\kappa \le -1$. If (g,λ) is a non-constant solution of the critical point equation, then M is an Einstein manifold.*

Proof. From (2.17), we have

$$R(\xi,Y)D\lambda = \kappa[(Y\lambda)\xi - (\xi\lambda)Y] - 2[((h'Y)\lambda)\xi - (\xi\lambda)h'Y], \tag{3.10}$$

for an arbitrary vector field Y of $\chi(M)$. Differentiating (2.20) and using (2.13), (2.16), (2.18) and (2.19), we obtain the following two relations

$$(\nabla_X Q)\xi = 2n(\kappa+2)h'X, \tag{3.11}$$

$$(\nabla_\xi Q)X = 0, \tag{3.12}$$

for any vector field X on M. Using these above two relations along with (2.20) in (3.1), we acquire

$$R(\xi,Y)D\lambda = (\xi\lambda)QY - 2n\kappa(Y\lambda)\xi - 2n(\lambda+1)(\kappa+2)h'Y + (\xi f)Y - (Yf)\xi, \tag{3.13}$$

$\forall Y \in \chi(M)$. As the scalar curvature r is constant in critical point equation, comparing (3.10) with (3.13) and using $f = -r(\frac{\lambda}{2n} + \frac{1}{2n+1})$, (2.20) and (2.21), we achieve

$$n(\kappa+1)[(\xi\lambda)\eta(Y)\xi - (Y\lambda)\xi] + ((h'Y)\lambda)\xi = [(n+1)(\xi\lambda) + n(\lambda+1)(\kappa+2)](h'Y), \tag{3.14}$$

for any $Y \in \chi(M)$. Considering scalar product with the Reeb vector field ξ, the foregoing equation reduces to

$$(h'Y)\lambda = n(\kappa+1)[(Y\lambda) - (\xi\lambda)\eta(Y)], \tag{3.15}$$

where we have used (2.13). Taking (3.15) into account, from (3.14) we have,

$$\xi\lambda = -\frac{n}{(n+1)}(\lambda+1)(\kappa+2). \tag{3.16}$$

Differentiating the above equation along the direction of the characteristic vector field ξ, we get

$$Hess_\lambda(\xi,\xi) = -\frac{n}{(n+1)}(\kappa+2)(\xi\lambda).$$

Using this relation in the critical point equation (1.4) and considering (2.20), (2.21) and (3.16), we obtain

$$4n^2(n+1)(\kappa+1) + (2n+1)(2n^2\kappa - \kappa + 2n^2)\lambda = 0.$$

Since κ is a constant, we get λ is a constant. So, we can conclude that the manifold is Einstein. □

Theorem 3.5. *Let $M(\phi,\xi,\eta,g)$ be a 3-dimensional trans-Sasakian manifold where the structure functions α,β are constant. If (g,λ) is a non-constant solution of the critical point equation then either the manifold becomes α-Sasakian manifold or the manifold becomes Einstein.*

Proof. From (2.24), we deduce

$$R(\xi,Y)D\lambda = (\alpha^2-\beta^2)[(Y\lambda)\xi - (\xi\lambda)Y], \tag{3.17}$$

for an arbitrary vector field Y on M. Again, setting $X=\xi$ in (3.1), we obtain

$$R(\xi,Y)D\lambda = (\xi\lambda)QY - (Y\lambda)Q\xi + (\lambda+1)[(\nabla_\xi Q)Y - (\nabla_Y Q)\xi] + (\xi f)Y - (Yf)\xi, \tag{3.18}$$

From (2.25) we have these following two relations,

$$QY = (\frac{r}{2} - (\alpha^2-\beta^2))Y - (\frac{r}{2} - 3(\alpha^2-\beta^2))\eta(Y)\xi, \tag{3.19}$$

$$Q\xi = 2(\alpha^2-\beta^2)\xi, \tag{3.20}$$

for any vector field Y of $\chi(M)$. As the scalar curvature r is constant in a critical point equation, taking covariant derivative of (2.25) along an arbitrary vector field X, we get

$$(\nabla_X Q)Y = -(\frac{r}{2} - 3(\alpha^2-\beta^2))[((\nabla_X\eta)Y)\xi + \eta(Y)(\nabla_X\xi)].$$

Making use of (2.23) in the above equation, yields

$$(\nabla_\xi Q)Y = 0, \tag{3.21}$$

$$(\nabla_Y Q)\xi = (\frac{r}{2} - 3(\alpha^2-\beta^2))[\alpha\phi Y - \beta Y + \beta\eta(Y)\xi], \tag{3.22}$$

$\forall Y \in \chi(M)$. Using (3.19)-(3.22) in (3.18), we acquire

$$R(\xi,Y)D\lambda =[(\xi\lambda)(\frac{r}{2}-(\alpha^2-\beta^2))+\beta(\lambda+1)(\frac{r}{2}-3(\alpha^2-\beta^2))+(\xi f)]Y- [(\frac{r}{2}-3(\alpha^2-\beta^2))((\xi\lambda)+\beta(\lambda+1))\eta(Y)+2(\alpha^2-\beta^2)(Y\lambda) - (Yf)]\xi-\alpha(\lambda+1)(\frac{r}{2}-3(\alpha^2-\beta^2))\phi(Y), \tag{3.23}$$

for any vector field Y on M. As we know $f=-r(\frac{\lambda}{2}+\frac{1}{3})$ and the scalar curvature r is constant in a critical point equation, comparing (3.17) with (3.23), we get

$$(\frac{r}{2}-3(\alpha^2-\beta^2))\big[\beta(\lambda+1)(Y-\eta(Y)\xi)-(\xi\lambda)\eta(Y)\xi+(Y\lambda)\xi-\alpha(\lambda+1)\phi Y\big]=0.$$

Taking scalar product of the aforementioned equation with arbitraty vector field X, we get

$$(\frac{r}{2}-3(\alpha^2-\beta^2))\big[\beta(\lambda+1)\{g(X,Y)-\eta(X)\eta(Y)\}-(\xi\lambda)\eta(X)\eta(Y) + (Y\lambda)\eta(X)-\alpha(\lambda+1)g(X,\phi Y)\big]=0.$$

Interchange of X and Y in the last equation, yields

$$(\frac{r}{2}-3(\alpha^2-\beta^2))\big[\beta(\lambda+1)\{g(X,Y)-\eta(X)\eta(Y)\}-(\xi\lambda)\eta(X)\eta(Y) + (X\lambda)\eta(Y)-\alpha(\lambda+1)g(\phi X,Y)\big]=0.$$

Adding the above two equations and using (2.7), we obtain

$$(\frac{r}{2}-3(\alpha^2-\beta^2))\big[2\beta(\lambda+1)\{g(X,Y)-\eta(X)\eta(Y)\}-2(\xi\lambda)\eta(X)\eta(Y) + (Y\lambda)\eta(X)+(X\lambda)\eta(Y)\big]=0. \tag{3.24}$$

Tracing (3.24) we have

$$\beta(\lambda+1)[r-6(\alpha^2-\beta^2)]=0.$$

Since λ is a non-constant function, from here two cases arise. If $\beta=0$, the manifold becomes α-Sasakian. If $r=6(\alpha^2-\beta^2)$, from (2.25) we can conclude that the manifold is Einstein with Einstein constant $2(\alpha^2-\beta^2)$. □

Remark 3.6. If we further assume $\alpha=\beta$ where α,β are constant functions. If (g,λ) is a non-constant solution of the critical point equation then either the manifold becomes α-Sasakian manifold or the manifold is scalar flat.

4. Example of 3-dimensional trans-Sasakian manifold

In this section we provide an example of trans-Sasakian manifold of dimension 3. To construct the example we consider $M=\{(x,y,x)\in\mathbb{R}^3 : y\neq 0\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. The vector fields defined as

$$e_1=e^y(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}), \qquad e_2=\frac{\partial}{\partial y}, \qquad e_3=e^y\frac{\partial}{\partial z},$$

are linearly independent at each point on M. Now we define a metric g on M as,

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1,$$
$$g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0.$$

Let η be a 1-form defined by $\eta(X) = g(X, e_2)$, for arbitrary $X \in \chi(M)$, then we have the following relations

$$\eta(e_1) = 0, \qquad \eta(e_2) = 1, \qquad \eta(e_3) = 0.$$

Let us define a (1,1)-tensor field ϕ as

$$\phi(e_1) = e_3, \qquad \phi(e_2) = 0, \qquad \phi(e_3) = -e_1,$$

then it satisfies (2.1) and (2.3). Thus for $\xi = e_2$, (ϕ, ξ, η, g) defines an almost contact metric structure on M. We can now easily conclude

$$[e_1, e_2] = -e_1, \qquad [e_2, e_3] = e_3, \qquad [e_1, e_3] = ze^y e_3 - e^{2y} e_2.$$

Let ∇ be the Levi-Civita connection of M. Then from Koszul's formula for arbitrary $X, Y, Z \in \chi(M)$ given by

$$\begin{aligned} 2g(\nabla_X Y, Z) =& Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) \\ & - g(Y, [X, Z]) + g(Z, [X, Y]), \end{aligned} \tag{4.1}$$

we can have,

$$\begin{aligned} &\nabla_{e_1} e_1 = e_2, && \nabla_{e_1} e_2 = \frac{1}{2} e^{2y} e_3 - e_1, && \nabla_{e_1} e_3 = -\frac{1}{2} e^{2y} e_2, \\ &\nabla_{e_2} e_1 = \frac{1}{2} e^{2y} e_3, && \nabla_{e_2} e_2 = 0, && \nabla_{e_2} e_3 = -\frac{1}{2} e^{2y} e_1, \\ &\nabla_{e_3} e_1 = \frac{1}{2} e^{2y} e_2 - ze^y e_3, && \nabla_{e_3} e_2 = -\frac{1}{2} e^{2y} e_1 - e_3, && \nabla_{e_3} e_3 = ze^y e_1 + e_2. \end{aligned}$$

Finally from $\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi)$ we can conclude that $\alpha = -\frac{1}{2} e^{2y}$ and $\beta = -1$ and M is a 3-dimensional trans-Sasakian Manifold.

5. Example of 5-dimensional $(\kappa, \mu)'$- almost Kenmotsu manifold

In this section we construct an example of $(\kappa, \mu)'$- almost Kenmotsu manifold of dimension 5. For this we consider the manifold as $M = \{(x, y, z, u, v) \in \mathbb{R}^5\}$, where (x, y, z, u, v) are the standard coordinates in $\mathbb{R}^5$. Let e_1, e_2, e_3, e_4, e_5 are 5 vector fields which satisfy,

$$[e_1, e_2] = 0, \qquad [e_1, e_3] = -2e_3, \qquad [e_1, e_4] = -2e_4, \qquad [e_1, e_5] = 0,$$

$$[e_i, e_j] = 0 \text{ where } i, j = 2, 3, 4, 5.$$

Now we define a metric g on M as,

$$g(e_i, e_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

Let η be a 1-form defined by $\eta(X) = g(X, e_1)$ for arbitrary $X \in \chi(M)$, then we have the following relations,

$$\eta(e_1) = 1, \qquad \eta(e_i) = 0; where\ i = 2, 3, 4, 5.$$

Let us define a $(1, 1)$-tensor field ϕ as,

$$\phi(e_1) = 0, \quad \phi(e_2) = e_3, \quad \phi(e_3) = -e_2, \quad \phi(e_4) = e_5, \quad \phi(e_5) = -e_4.$$

Then the relations (2.1) and (2.3) are satisfied. Thus for $e_1 = \xi$, (ϕ, ξ, η, g) defines an almost contact structure on M. Moreover,

$$h'\xi = 0, \quad h'e_2 = e_2, \quad h'e_3 = e_3, \quad h'e_4 = 0, \quad h'e_5 = 0.$$

Let ∇ be the Levi-Civita connection of M. Then from Koszul's formula (4.1), we have,

$$\begin{array}{lllll}
\nabla_\xi \xi = 0, & \nabla_\xi e_2 = 0, & \nabla_\xi e_3 = 0, & \nabla_\xi e_4 = 0, & \nabla_\xi e_5 = 0, \\
\nabla_{e_2} \xi = 0, & \nabla_{e_2} e_2 = 0, & \nabla_{e_2} e_3 = 0, & \nabla_{e_2} e_4 = 0, & \nabla_{e_2} e_5 = 0, \\
\nabla_{e_3} \xi = 2e_3, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_3 = -2\xi, & \nabla_{e_3} e_4 = 0, & \nabla_{e_3} e_5 = 0, \\
\nabla_{e_4} \xi = 2e_4, & \nabla_{e_4} e_2 = 0, & \nabla_{e_4} e_3 = 0, & \nabla_{e_4} e_4 = -2\xi, & \nabla_{e_4} e_5 = 0, \\
\nabla_{e_5} \xi = 0, & \nabla_{e_5} e_2 = 0, & \nabla_{e_5} e_3 = 0, & \nabla_{e_5} e_4 = 0, & \nabla_{e_5} e_5 = 0.
\end{array}$$

From the above relations we can easily conclude that the relation $\nabla_X \xi = -\phi^2 X + h'X$ holds for arbitrary $X \in \chi(M)$. So M is an almost Kenmotsu manifold.

By the above results, we can easily calculate the components of the curvature tensor R as follows

$$\begin{array}{lll}
R(\xi, e_3)\xi = 4e_3, & R(\xi, e_4)\xi = 4e_4, & R(\xi, e_3)e_3 = -4\xi, \\
R(\xi, e_4)e_4 = -4\xi, & R(e_3, e_4)e_3 = 4e_4, & R(e_3, e_4)e_4 = -4e_3.
\end{array}$$

From here we can conclude that the characteristic vector field ξ belongs to the $(\kappa, \mu)'$-nullity distribution with $\kappa = -2$ and $\mu = -2$. So the manifold is a $(\kappa, \mu)'$-almost Kenmotsu manifold.

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