Gulf Journal of Mathematics
Vol 5, Issue 4 (2017) 32-38

GENERALIZATION OF ESSENTIALLY SLIGHTLY COMPRESSIBLE MODULES

SAMRUAM BAUPRADIST

ABSTRACT. In this paper, we introduce and investigate essentially M-slightly compressible modules and essentially M-slightly compressible injective modules. It has been shown that over hereditary ring R and M is an injective right R-module. If N is an essentially M-slightly compressible module then every essential submodule A of N containing a direct summand of N. For any ring R and uniform right R-module M, we can show that N is an essentially M-slightly compressible injective module if and only if N is an M-slightly compressible injective module.

1. INTRODUCTION

Throughout this paper, all rings are associative and all modules are unitary right R-modules unless otherwise stated. Let M be a right R-module and $S = End_R(M)$, its endomorphism ring. A right R-module N is called *M-generated*, if there exists an epimorphism $M^{(I)} \to N$ for some index set I. If I is finite set, then N is called *finitely M-generated*. Following R. Wisbauer [11], $\sigma[M]$ denotes the full subcategory of $Mod - R$, whose objects are the submodules of M-generated modules. A right R-module M is called a *self-generator*, if it generates all of its submodules. A right R-module M is called a *subgenerator*, if it is a generator of $\sigma[M]$. For a submodule N of right R-module M is called *essential*, if for any non-zero submodule A of M, $N \bigcap A \neq 0$. For any unexplained terminology and all the basic results on rings and modules that are used in the sequel can be found in [1], [4] and [11].

In 1976, J. M. Zelmanowitz defined the notion of compressible. For a right R-module M is called *compressible*, if it can be imbedded in any of its non-zero submodule. For more information about compressible modules, see in [12], [13], [14], [15] and [16]. In 2005, P. F. Smith([8]), introduced the concept of slightly compressible modules, which is a generalization of compressible modules. The right R-module M is called *slightly compressible*, if for any non-zero submodule N of M, there exists a non-zero R-homomorphism from M to N. In 2006, P. F. Smith and M. R. Vedadi ([9]) generalized the concept of compressible modules

Date: Received: Apr 3, 2017; Accepted: Aug 25, 2017.
2010 *Mathematics Subject Classification.* Primary 16D50; Secondary 17D70.
Key words and phrases. Essentially M-slightly compressible modules, essentially M-slightly compressible injective modules.

and introduced essentially compressible modules and rings. An right R-module M is called *essentially compressible module*, if for any essential submodule N of M, there exists an R-monomorphism from M to N. They also studied the properties of essentially compressible modules and rings and gave some example of essentially compressible modules. Recently, A. K. Singh ([6]) introduced the concept of essentially slightly compressible modules and rings. It is generalization of essentially compressible modules and rings. An right R-module M is called *essentially slightly compressible module*, if for any essential submodule N of M, there exists an R-homomorphism from M to N. He has also pointed out some of its characterizations in terms of nonsingular modules, enough uniform modules and absolutely indecomposable modules.

In section 2, we determine a general form of essentially slightly compressible modules. It is called *essentially M-slightly compressible module* where M is a right R-module. Moreover, we provide some conditions for any right R-module to be an essentially M-slightly compressible module.

In section 3, we introduce the concept of essentially M-slightly compressible injective module. We study some property of M-slightly compressible injective module and relationship between essentially M-slightly compressible injective modules and M-slightly compressible injective modules.

2. Essentially M-Slightly Compressible Modules

Now we begin with the definition of essentially M-slightly compressible module where M is a right R-module.

Definition 2.1. Let M be a right R-module. A right R-module N is called *essentially M-slightly compressible module*, if for any essential submodule A of N, there exists a non-zero R-homomorphism from M to A.

In the case that $M = N$, N is called an *essentially slightly compressible module*. We can refer to ([6]).

Example 2.2. From [2], for right R-module M and N. N is called M-slightly compressible module, if for every non-zero submodule A of N, there exists a non-zero R-homomorphism s from M to N such that $s(M) \subset_{>} A$. It is clear that every M-slightly compressible module is an essentially M-slightly compressible module.

Proposition 2.3. *Let M be a right R-module. Then the direct sum of essentially M-slightly compressible module is an essentially M-slightly compressible module.*

Proof. Let M_i be essentially M-slightly compressible module for all $i \in I$, $N = \bigoplus_{i \in I} M_i$ and A an essential submodule of N. For $i \in I$, we have $A \cap M_i$ is an essential submodule of M_i and by assumption, there exists a non-zero R-homomorphism $\alpha_i : M \to A \cap M_i$. Choose $\alpha = \sum_{i \in I}$. Hence α is a non-zero R-homomorphism from M to A. $\square$

Proposition 2.4. *Let M and $M^{'}$ be right R-modules and N an essential submodule of M. If M is an essentially $M^{'}$-slightly compressible module then N is also essentially $M^{'}$-slightly compressible module.*

Theorem 2.5. *Let $M, M^{'}$ and N be right R-modules such that N is an essentially M-slightly compressible. We have*

1. If M is an epimorphism image of $M^{'}$ then N is an essentially $M^{'}$-slightly compressible module.

2. If M is an essentially $M^{'}$-slightly compressible module then N is an essentially $M^{'}$-slightly compressible module.

Proof. 1. Suppose that M is an epimorphism image of $M^{'}$. There exists $s \in Hom_R(M^{'}, M)$ such that $s(M^{'}) = M$. Let A be an essential submodule of N. Since N is an essentially M-slightly compressible, there exists R-homomorphism $\alpha : M \to A$ and we have $\alpha s : M^{'} \to A$. Therefore N is an essentially $M^{'}$-slightly compressible.

2. Suppose that M is an essentially $M^{'}$-slightly compressible module. Let A be an essential submodule of N. Since N is an essentially M-slightly compressible module, there exists $s \in Hom_R(M, N)$ such that $s(M) \subset_{>} A$. But M is an essentially $M^{'}$-slightly compressible module, there exists $t \in Hom_R(M^{'}, M)$ such that $t(M^{'}) \subset_{>} M$. Thus $st \in Hom_R(M^{'}, N)$ and $(st)(M^{'}) = st(M^{'}) \subset_{>} s(M) \subset_{>} A$. Therefore N is an essentially $M^{'}$-slightly compressible module. □

Theorem 2.6. *Let M be a right R-module and N an essentially M-slightly compressible right R-module. Then*

1. A is an essentially M-slightly compressible module for all essential submodule A of N.

2. N is an essentially $M^{'}$-slightly compressible module where $M^{'}$ is a non-zero submodule of M such that $Ker(s) \neq M^{'}$ for all $s \in Hom_R(M, N)$.

Proof. 1. Let A be an essential submodule of N and B an essential submodule of A. Then B is an essential submodule of N but N is an essentially M-slightly compressible module, there exists $0 \neq s \in Hom_R(M, N)$ such that $s(M) \subset_{>} B$. Hence A is an essentially M-slightly compressible module.

2. Let $M^{'}$ be a non-zero submodule of M such that $Ker(s) \neq M^{'}$ for all $s \in Hom_R(M, N)$ and A be an essential submodule of N. Since N is an essentially M-slightly compressible module, there exists $0 \neq s \in Hom_R(M, N)$ such that $s(M) \subset_{>} A$ and by assumption, we have $Ker(s) \neq M^{'}$ and hence $s\mid_{M'} \neq 0$. So $s\mid_{M'} \in Hom_R(M^{'}, N)$ such that $s\mid_{M'} \subset_{>} A$ where $s\mid_{M'}$ is an R-homomorphism with respect to $M^{'}$. Therefore N is an essentially $M^{'}$-slightly compressible module. □

Proposition 2.7. *Let M and N be right R-modules. If any non-zero factor of N is an essentially M-slightly compressible module then for every non-zero submodule of N is an M-slightly compressible module.*

Proof. Let A be a non-zero submodule of N and B a non-zero submodule of A. Since $E(B)$ is an injective R-module where $E(B)$ is an injective hull of B, the inclusion map $i : B \to E(B)$ can be extended to R-homomorphism f from

N to $E(B)$. But B is an essential submodule of $E(B)$, we have $B \bigcap f(N)$ is an essential submodule of $f(N)$. By assumption, $f(N)$ is an essentially M-slightly compressible module. Thus $Hom_R(M, B \bigcap f(N)) \neq 0$. It follows that $Hom_R(M, B) \neq 0$. Therefore A is an M-slightly compressible module. □

Theorem 2.8. *Let M and N be right R-modules. If there exists a non-zero R-homomorphism $f : M \to N$ such that $Im(f)$ is an essentially M-slightly compressible module then N is an essentially M-slightly compressible module.*

Proof. Let A be an essential submodule of N. Then $A \bigcap Im(f)$ is an essential submodule of $Im(f)$. But $Im(f)$ is an essentially M-slightly compressible module, there exists a non-zero R-homomorphism k from M to $A \bigcap Im(f)$. Thus $Hom_R(M, A) \neq 0$. □

Recall that a ring R is called *hereditary*, if all submodules of projective modules over R are again projective.

Theorem 2.9. *Let R be a right hereditary ring, M an injective right R-module and N a right R-module. If N is an essentially M-slightly compressible module then every essential submodule A of N containing a direct summand of N.*

Proof. Suppose that N is an essentially M-slightly compressible module and A is an essential submodule of N. There exists a non-zero R-homomorphism from M to A. Since M is an injective, R is hereditary ring and by Theorem 3.22 [5], $M/ker(s)$ is an injective. But $M/ker(s) \cong s(M)$, we have $s(M)$ is an injective. Then $s(M)$ is a direct summand of N. □

Corollary 2.10. *Let M be a right R-module and N a right R-module such that every essential submodule contains a non-zero direct summand of N. Then N is an essentially slightly compressible module.*

3. Essentially M-Slightly Compressible Injective Modules

Now we investigate the right essentially M-slightly compressible injective module where M is a right R-module.

Definition 3.1. Let M be a right R-module. A submodule N of M is called *essentially M-slightly compressible submodule of M*, if N is an essential submodule of M and there exists a non-zero R-homomorphism s from M to N.

Definition 3.2. Let M be a right R-module. A right R-module N is called *essentially M-slightly compressible injective module*, if for any non-zero essentially M-slightly compressible submodule A of M and for any non-zero R-homomorphism from A to N can be extended to R-homomorphism from M to N. A right R-module N is called *quasi essentially slightly compressible injective*, if N is essentially N-slightly compressible injective. For a ring R is called *right self essentially slightly compressible injective*, if it is essentially R_R-slightly compressible injective.

Example 3.3. From [2], for right R-module M and N. N is called *M-slightly compressible injective module*, if for any non-zero M-slightly compressible submodule A of M and for any R-homomorphism from A to N can be extended to

R-homomorphism from M to N. It is clear that every M-slightly compressible injective module is an essentially M-slightly compressible injective module.

First, we give a characterization of essentially M-slightly compressible injective module.

Theorem 3.4. *Let M, M_1 and M_2 be right R-modules. Then M_1 and M_2 are essentially M-slightly compressible injective modules if and only if $M_1 \oplus M_2$ is an essentially M-slightly compressible injective module.*

Proof. $\Rightarrow$ Suppose that M_1 and M_2 are essentially M-slightly compressible injective modules. Let A be an essentially M-slightly compressible submodule of M, α an R-homomorphism from A to $M_1 \oplus M_2$ and $i \in \{1, 2\}$. Since $\pi_i\alpha$ is an R-homomorphism from A to M_i where π_i is the $i^{\underline{th}}$ canonical projection map from $M_1 \oplus M_2$ to M_i and M_i is an essentially M-slightly compressible injective module, there exists an R-homomorphism $\overline{\alpha_i}$ from M to M_i such that $\overline{\alpha_i}i_A = \pi_i\alpha$ where $i_A : A \to M$ is an embedding. We can choose $\overline{\alpha} = i_1\overline{\alpha_1} + i_2\overline{\alpha_2}$ where $i_1 : M_1 \to M_1 \oplus M_2$ and $i_2 : M_2 \to M_1 \oplus M_2$ are canonical injection map.Then $\overline{\alpha}i_A = (i_1\overline{\alpha_1} + i_2\overline{\alpha_2})i_A = i_1\overline{\alpha_1}i_A + i_2\overline{\alpha_2}i_A = i_1\pi_1\alpha + i_2\pi_2\alpha = (i_1\pi_1 + i_2\pi_2)\alpha = I_{M_1 \bigoplus M_2}\alpha = \alpha$ where $I_{M_1 \bigoplus M_2}$ is an identity map on $M_1 \bigoplus M_2$. Therefore $M_1 \oplus M_2$ is an essentially M-slightly compressible injective module.
$\Leftarrow$ Suppose that $M_1 \oplus M_2$ is an essentially M-slightly compressible injective module. Let $j \in \{1, 2\}, A$ be an essentially M-slightly compressible submodule of M and α_j an R-homomorphism from A to M_j. Since $i_j\alpha_j$ is an R-homomorphism from A to $M_1 \oplus M_2$ where i_j is the canonical injective map from M_j to $M_1 \oplus M_2$ and $M_1 \oplus M_2$ is an essentially M-slightly compressible injective module, there exists R-homomorphism $\overline{\alpha}$ from M to $M_1 \oplus M_2$ such that $\overline{\alpha}i_A = i_j\alpha_j$ where i_A is an embedding. We can choose $\overline{\alpha_j} = \pi_j\overline{\alpha}$ where π_j is the $j^{\underline{th}}$ canonical projection map. Then $\overline{\alpha_j}i_A = \pi_j\overline{\alpha}i_A = \pi_j i_j\alpha_j = I_{M_j}\alpha_j = \alpha_j$ where I_{M_j} is an identity map on M_j. Therefore M_j is an essentially M-slightly compressible injective module. $\square$

Corollary 3.5. *Let M and M_i be right R-modules for all $i \in \{1, ..., n\}$ where n is a positive integer.Then M_i is an essentially M-slightly compressible injective modules for all $i \in I$ if and only if $\bigoplus_{i=1}^{n} M_i$ is an essentially M-slightly compressible module.*

Proposition 3.6. *Let M be a right R-module and N an essentially M-slightly compressible submodule of M. If N is an essentially M-slightly compressible injective module then N is a direct summand of M.*

Proof. Suppose that N is an essentially M-slightly compressible injective. There exists an R-homomorphism α from M to N such that $\alpha i_N = I_N$ where I_N is an identity map and i_N is the inclusion map. We have the short exact sequence

$$0 \to N \to^{i_N} M \to^{\pi_N} M/N \to 0$$

splits where π_N is the canonical projection map from M to M/N. Therefore N is a direct summand of M. $\square$

Proposition 3.7. *Let M, N_1 and N_2 be right R-modules. If N_1 is an essentially M-slightly compressible injective module and $N_1 \cong N_2$ then N_1 is an essentially M-slightly compressible injective module.*

Proposition 3.8. *Let M, N be right R-modules and A a direct summand of N. If N is an essentially M-slightly compressible injective module then*
1. *A is an essentially M-slightly compressible injective module.*
2. *N/A is an essentially M-slightly compressible injective module.*

Proof. Suppose that N is an essentially M-slightly compressible injective module.
1. Let B be an essentially M-slightly compressible submodule of M and α an R-homomorphism from B to A. Since A a direct summand of N, the short exact sequence

$$0 \to A \to^{i_A} N \to^{\pi_A} N/A \to 0$$

splits where i_A is the inclusion map and π_A is the canonical projection map from N to N/A. There exists an R-homomorphism α from N to A such that $\alpha i_A = I_A$ where I_A is the identity map on A. But N is an essentially M-slightly compressible injective module, there exists an R-homomorphism f from M to N such that $i_A\alpha = fi_B$ where i_B is an inclusion map. We can choose $\overline{\alpha} = \alpha f$. Then $\overline{\alpha} i_B = \alpha f i_B = \alpha(i_A\alpha) = (\alpha i_A)\alpha = I_A\alpha = \alpha$. Therefore A is an essentially M-slightly compressible injective module.
2. Since A is a direct summand of N, there exists a submodule B of N such that $N = A \bigoplus B$. Then $B \cong N/A$. From 1., we have B is an essentially M-slightly compressible injective module. By proposition 5, N/A is an essentially M-slightly compressible injective module. □

Proposition 3.9. *Every right R-module is an essentially R_R-slightly compressible module.*

Proof. By ([3], Theorem 3.2.11.) □

Theorem 3.10. *Let M be a right R-module. Then M is an injective module if and only if M is an essentially R_R-slightly compressible injective module.*

Proof. $\Rightarrow$ It is obvious
$\Leftarrow$ Suppose that M is an essentially R_R-slightly compressible injective module. Let I be a right ideal of R and α an R-homomorphism from I to M. By proposition 3.9, we have I is an essentially R_R-slightly compressible module but M is an essentially R_R-slightly compressible injective module, there exists an R-homomorphism $\overline{\alpha}$ from R to M such that $\overline{\alpha}|_I = \alpha$. By Baer 's Criterion, M is an injective module. □

Corollary 3.11. *For the ring R, we have R_R is an essentially R_R-slightly compressible injective module if and only if R is a right self-injective ring.*

Recall that a right R-module M is *uniform* if $M \neq 0$ and $L \bigcap N \neq 0$ for all nonzero submodules L and N of M.

Theorem 3.12. *Let M be an uniform right R-module and N a right R-module. Then N is an essentially M-slightly compressible injective module if and only if N is an M-slightly compressible injective module.*

Proof. $\Rightarrow$ Suppose that N is an essentially M-slightly compressible injective module. Let A be a non-zero M-slightly compressible submodule of M and α an R-homomophism from A to N. Since M is an uniform right R-module, A is an essentially M-slightly compressible submodule of M. But N is an essentially M-slightly compressible injective module, α can be extended to $\overline{\alpha}$ such that $\overline{\alpha}|_A = A$. Therefore N is an M-slightly compressible injective module.

$\Leftarrow$ clear. $\square$

Acknowledgement The authors would like to acknowledge the support and encouragement of Center of Excellence in Mathematics, CHE, Si Ayuthaya Road, Bangkok 10400, Thailand.

References

1. F.W. Anderson and K.R. Fuller, Rings and Categories of Modules, New York. NY, USA: Springer, 1974.
2. S. Baupradist, P. Janmuang, S. Asawasamrit and R. Chinram, *On slightly compressible injective modules and rings* (will be appear).
3. P. Janmuang, General form of slightly compressible modules, Thesis, Graduate School, Chulalongkorn University, Bangkok, Thailand, 2012.
4. F. Kasch, Modules and Rings, London Math Soc. Monographs **17** (C.U.P).
5. T.Y. Lam, Serre 's problem on projective modules, Springer-Verlag Berlin Heidelberg, 2006.
6. A.K. Singh, *Essentially slightly compressible modules and rings*, Asian-European Journal of Mathematics, **2** (2012), no. 5, 1-8.
7. ______, *A short survey of noncommutative geometry*, J. Math. Phys. **41** (2000), no. 6, 3832–3866.
8. P.F. Smith, *Modules with many homomorphism*, Journal of pure and applied algebra, **197** (2005), 305-321.
9. P.F. Smith and M.R. Vedadi, *Essentially compressible modules and rings*, Journal of algebra, **304** (2006), 812-831.
10. M.R. Vedadi, *Essentially retractable modules*, Journal of science, Islamic republic of Iran **18** (2007), no.4, 355-360.
11. R. Wisbauer, Foundations of Module and Ring Theory, Philadelphia, PA, USA, Gordon and Breach, 1991.
12. J. Zelmanowitz, *An extension of the Jacobson density theorem*, Bull. Amer. Math. Soc. **82** (1976), no.4, 551-553.
13. J. Zelmanowitz, *The fnite intersection property on the right annihilator ideals*, Proc. Amer. Math. Soc. **57** (1976), no.2, 213-216.
14. J. Zelmanowitz, *Dense ring of linear transformations*, in Ring Theory II (Marcel Dekker, NY) (1977), 281-294.
15. J. Zelmanowitz, *Weakly primitive rings*, Comm. Algebra **9** (1981), no.1, 23-45.
16. J. Zelmanowitz, *Weakly semisimple modules and density theory*, Comm. Algebra **21** (1993), 1785-1808.

Department of Mathematics and Computer Science,Faculty of Science, Chulalongkorn University, Bangkok 10330, Thailand
E-mail address: samruam.b@chula.ac.th