PROPERTY \( (m) \) UNDER PERTURBATIONS

M.H.M. RASHID

Abstract. A Banach space operator is said to obey property \( (m) \) if the isolated points of the spectrum \( \sigma(T) \) of \( T \) which are eigenvalues of finite multiplicity are exactly those points \( \lambda \) of the spectrum for which \( T - \lambda I \) is an upper semi-Browder. In this article, we study the stability of property \( (m) \), for a bounded operator acting on a Banach space, under perturbation by finite rank operators, by nilpotent operators, quasi-nilpotent operators, Riesz operator or algebraic operators commuting with \( T \).

1. Introduction and preliminaries

Let \( \mathcal{B}(\mathcal{X}) \) denote the algebra of bounded operators acting on an infinite complex Banach space \( \mathcal{X} \). We use \( I \) to denote the identity operator on \( \mathcal{X} \), and \( \mathcal{K}(\mathcal{X}) \) to denote the ideal of all compact operators on \( \mathcal{X} \) and \( \mathcal{F}(\mathcal{X}) \) to denote the ideal of all finite rank operators on \( \mathcal{X} \). For an arbitrary operator \( T \in \mathcal{B}(\mathcal{X}) \), \( \ker(T) \) denotes its kernel and \( \mathcal{R}(T) \) denotes its range. We set \( \alpha(T) = \dim \ker(T) \) and \( \beta(T) = \dim \mathcal{X}/\mathcal{R}(T) \). Denote by

\[ SF_+(\mathcal{X}) := \{ T \in \mathcal{B}(\mathcal{X}) : \alpha(T) < \infty \text{ and } \mathcal{R}(T) \text{ is closed} \} \]

the class of all upper semi-Fredholm operators, and by

\[ SF(\mathcal{X}) := \{ T \in \mathcal{B}(\mathcal{X}) : \beta(T) < \infty \} \]

the class of all lower semi-Fredholm operators. The class of all semi-Fredholm operators is defined by \( SF(\mathcal{X}) := SF_+(\mathcal{X}) \cup SF(\mathcal{X}) \), while the class of all Fredholm operator is defined by \( F(\mathcal{X}) := SF_+(\mathcal{X}) \cap SF(\mathcal{X}) \). For a semi-Fredholm operator \( T \) we define the index, \( \text{ind}(T) \), by \( \text{ind}(T) = \alpha(T) - \beta(T) \). Let \( a := a(T) \) be the ascent of an operator \( T \); i.e., the smallest nonnegative integer \( p \) such that \( \ker(T^p) = \ker(T^{p+1}) \). If such integer does not exist we put \( a(T) = \infty \). Analogously, let \( d := d(T) \) be the descent of an operator \( T \); i.e., the smallest nonnegative integer \( q \) such that \( \mathcal{R}(T^q) = \mathcal{R}(T^{q+1}) \), and if such integer does not exist we put \( d(T) = \infty \). It is well known that if \( a(T) \) and \( d(T) \) are both finite then \( a(T) = d(T) \) \[12, \text{Proposition 38.3}\]. Moreover, \( 0 < a(T - \lambda I) = d(T - \lambda I) < \infty \) precisely when \( \lambda \) is a pole of the resolvent of \( T \), see Heuser \[12, \text{Proposition 50.2}\].

For a subset \( G \) of an arbitrary topological space, \( \overline{G} \) denotes the closure of \( G \).
Let $\mathbb{C}$ denotes the complex plane. If $K$ is a subset of $\mathbb{C}$, then $\text{iso}K$ denotes the set of all isolated points of $K$ and $\text{acc}K$ denotes the set of all points of accumulation of $K$. We use $\sigma(T)$ and $\sigma_0(T)$ to denote the spectrum and the approximate point spectrum of $T$ respectively. We use $T^*$ to denote the adjoint of $T \in \mathcal{B} (\mathcal{X})$.

Two important classes of operators are the class of all upper semi-Browder operators

$$B_+ (\mathcal{X}) := \{ T \in SF_+ (\mathcal{X}) : a(T) < \infty \}$$

and the class of all lower semi-Browder operators

$$B_- (\mathcal{X}) := \{ T \in SF_- (\mathcal{X}) : d(T) < \infty \}.$$ The class of all Browder operators is defined by $B(\mathcal{X}) := B_+ (\mathcal{X}) \cap B_- (\mathcal{X})$. Recall that a bounded linear operator $T \in \mathcal{B} (\mathcal{X})$ is said be a Weyl operator, $T \in W(\mathcal{X})$ if $T \in \mathcal{F} (\mathcal{X})$ and has index 0. Obviously, if $T \in B(\mathcal{X})$ then $T \in W(\mathcal{X})$.

These classes of operators motivate the definition of several spectra. The upper semi-Browder spectrum of $T \in \mathcal{B} (\mathcal{X})$ is defined by

$$\sigma_{ub}(T) := \{ \lambda \in \mathbb{C} : T - \lambda \notin B_+ (\mathcal{X}) \},$$

the lower semi-Browder spectrum of $T \in \mathcal{B} (\mathcal{X})$ is defined by

$$\sigma_{lb}(T) := \{ \lambda \in \mathbb{C} : T - \lambda \notin B_- (\mathcal{X}) \},$$

while the Browder spectrum of $T \in bx$ is defined by

$$\sigma_{b}(T) := \{ \lambda \in \mathbb{C} : T - \lambda \notin B(\mathcal{X}) \},$$

The Weyl spectrum of $T \in \mathcal{B} (\mathcal{X})$ is defined by

$$\sigma_{w}(T) := \{ \lambda \in \mathbb{C} : T - \lambda \notin W(\mathcal{X}) \},$$

We have that $\sigma_{w}(T) = \sigma_{w}(T^*)$, while $\sigma_{ub}(T) = \sigma_{lb}(T^*)$ and $\sigma_{ub}(T^*) = \sigma_{lb}(T)$. Evidently,

$$\sigma_{w}(T) \subseteq \sigma_{b}(T) = \sigma_{w}(T) \cup \text{acc} \sigma(T).$$

For $T \in \mathcal{B} (\mathcal{X})$, $SF^-_-(\mathcal{X}) := \{ T \in SF_+(\mathcal{X}) : \text{ind}(T) \leq 0 \}$ and $SF^+_-(\mathcal{X}) := \{ T \in SF_+(\mathcal{X}) : \text{ind}(T) \geq 0 \}$. The Weyl (or essential) approximate point spectrum is defined by $\sigma_{SF^-_-(T)} := \{ \lambda \in \mathbb{C} : T - \lambda \notin SF^+_-(\mathcal{X}) \}$. Note that $\sigma_{SF^-_-(T)}$ is the intersection of all approximate point spectra $\sigma_a(T + K)$ of compact perturbations $K$ of $T$, see [14]. The Weyl surjectivity spectrum $\sigma_{SF^+_-(T)} := \{ \lambda \in \mathbb{C} : T - \lambda \notin SF^-_+(\mathcal{X}) \}$. The spectrum $\sigma_{SF^+_-(T)}$ coincides with the intersection of all surjectivity spectra $\sigma_s(T + K)$ of compact perturbations $K$ of $T$, see [14]. Clearly, the last two spectra are dual each other, i.e., $\sigma_{SF^-_-(T)} = \sigma_{SF^+_-(T^*)}$ and $\sigma_{SF^+_-(T)} = \sigma_{SF^-_-(T^*)}$. Moreover, $\sigma_{w}(T) = \sigma_{SF^-_-(T)} \cup \sigma_{SF^+_-(T)}$. Since $a(T) < \infty$ entails $\text{ind}(T) \leq 0$ and $d(T) < \infty$ entails $\text{ind}(T) \geq 0$, we have $\sigma_{SF^-_-(T)} \subseteq \sigma_{ub}(T)$ and $\sigma_{SF^+_-(T)} \subseteq \sigma_{lb}(T)$. Hence the relationship between these spectra are given by the following equalities:

$$\sigma_{ub}(T) = \sigma_{SF^-_-(T)} \cup \text{acc} \sigma_a(T), \quad (1.1)$$

$$\sigma_{lb}(T) = \sigma_{SF^+_-(T)} \cup \text{acc} \sigma_a(T), \quad (1.2)$$
see [16].

Following [10] we say that \( T \in \mathcal{B}(\mathcal{X}) \) has the single-valued extension property (SVEP) at point \( \lambda \in \mathbb{C} \) if for every open neighborhood \( U_\lambda \) of \( \lambda \), the only analytic function \( f : U_\lambda \to \mathcal{X} \) which satisfies the equation \((T - \mu)f(\mu) = 0\) is the constant function \( f \equiv 0 \). An operator \( T \in \mathcal{B}(\mathcal{X}) \) is said to have the SVEP if \( T \) has the SVEP at every point \( \lambda \in \mathbb{C} \).

An operator \( T \in \mathcal{B}(\mathcal{X}) \) has the SVEP at every point of the resolvent \( \rho(T) := \mathbb{C} \setminus \sigma(T) \). The identity theorem for analytic functions ensures that for every \( T \in \mathcal{B}(\mathcal{X}) \), both \( T \) and \( T^* \) have the SVEP at the points of the boundary \( \partial \sigma(T) \) of the spectrum \( \sigma(T) \). In particular, that both \( T \) and \( T^* \) have the SVEP at every isolated point of \( \sigma(T) = \sigma(T^*) \). The SVEP is inherited by the restrictions to closed invariant subspaces, i.e., if \( T \in \mathcal{B}(\mathcal{X}) \) has the SVEP at \( \lambda_0 \) and \( M \) is closed \( T \)-invariant subspace then \( T|_M \) has SVEP at \( \lambda_0 \). Let \( S(T) := \{ \lambda \in \mathbb{C} : T \text{ does not have the SVEP at } \lambda \} \). Observe that \( T \in \mathcal{B}(\mathcal{X}) \) has SVEP if and only if \( S(T) = \emptyset \).

2. Property (m) under Perturbation

For a bounded operator \( T \in \mathcal{B}(\mathcal{X}) \), set
\[
\pi^0(T) : \sigma(T) \setminus \sigma_b(T) = \{ \lambda \in \sigma(T) : T - \lambda \in \mathcal{B}(\mathcal{X}) \}.
\]
Note that every \( \lambda \in \pi^0(T) \) is a pole of the resolvent and hence an isolated point of \( \sigma(T) \), see [12, Proposition 50.2]. Moreover, \( \pi^0(T) = \pi^0(T^*) \). Define
\[
E^0(T) := \{ \lambda \in iso\sigma(T) : 0 < \alpha(T - \lambda) < \infty \}.
\]
Obviously,
\[
\pi^0(T) \subseteq E^0(T) \text{ for every } T \in \mathcal{B}(\mathcal{X}).
\]

For a bounded operator \( T \in \mathcal{B}(\mathcal{X}) \), let us define
\[
E^0_a(T) := \{ \lambda \in iso\sigma_a(T) : 0 < \alpha(T - \lambda) < \infty \},
\]
and
\[
\pi^0_a(T) := \sigma_a(T) \setminus \sigma_{ub}(T) = \{ \lambda \in \sigma_a(T) : T - \lambda \in \mathcal{B}_+(\mathcal{X}) \}.
\]

**Lemma 2.1.** ([3]) For every \( T \in \mathcal{B}(\mathcal{X}) \), we have

(a) \( \pi^0(T) \subseteq \pi^0_a(T) \subseteq E^0_a(T) \) and
(b) \( E^0(T) \subseteq E^0_a(T) \).

Following Harte and W.Y. Lee [11], we shall say that \( T \) satisfies Browder’s theorem if
\[
\sigma_b(T) = \sigma_w(T),
\]
while, \( T \in \mathcal{B}(\mathcal{X}) \) is said to satisfy \( a \)-Browder’s theorem if
\[
\sigma_{SF^+_a}(T) = \sigma_{ub}(T).
\]

Obviously, \( a \)-Browder’s theorem holds for \( T \) implies Browder’s theorem holds for \( T \) and the converse is not true. Following Coburn [7], we say that Weyl’s theorem holds for \( T \in \mathcal{B}(\mathcal{X}) \) if
\[
\Delta(T) := \sigma(T) \setminus \sigma_w(T) = E^0(T).
\]
An approximate point version of Weyl’s theorem is $a$-Weyl’s theorem: according to Rakočević [17] an operator $T \in \mathcal{B}(\mathcal{X})$ is said to satisfy $a$-Weyl’s theorem if

$$\Delta_a(T) := \sigma_a(T) \setminus \sigma_{SF_+}(T) = E^0_a(T).$$

Note that

$a$-Weyl’s theorem holds for $T \implies$ Weyl’s theorem holds for $T$

while the converse in general does not hold.

**Definition 2.2.** A bounded operator $T \in \mathcal{B}(\mathcal{X})$ is said to satisfy

(i) property $(w)$ if $\Delta_a(T) = E^0(T)$ [15].

(ii) property $(t)$ if $\Delta_+(T) := \sigma(T) \setminus \sigma_{SF_+}(T) = E^0(T)$ [19].

**Definition 2.3.** ([20]) A bounded operator $T \in \mathcal{B}(\mathcal{X})$ is said to satisfy property $(m)$ if

$$\sigma(T) \setminus \sigma_{ub}(T) = E^0(T).$$

Weyl’s theorem corresponds to the half of property $(m)$, in the following sense:

**Theorem 2.4.** ([20]) If $T \in \mathcal{B}(\mathcal{X})$ then the following assertions are equivalent:

(i) property $(m)$ holds for $T$;

(ii) $T$ satisfies Weyl’s theorem and $\sigma_{ub}(T) = \sigma_w(T)$.

**Theorem 2.5.** Let $T \in \mathcal{B}(\mathcal{X})$. Then the following assertions are equivalent:

(i) Property $(t)$ holds for $T$;

(ii) $T$ satisfies property $(m)$ and $\sigma_{ub}(T) = \sigma_{SF_+}(T)$.

**Proof.** (i) $\implies$ (ii) Assume that $T$ obeys property $(t)$ then $\Delta_+(T) = E^0(T)$. If $\lambda \in \sigma(T) \setminus \sigma_{ub}(T)$, then $\lambda \in \Delta_+(T) = E^0(T)$ and so $\sigma(T) \setminus \sigma_{ub}(T) \subseteq E^0(T)$. To prove the other inclusion. Let $\lambda \in E^0(T)$ be an arbitrary given. Then $\lambda$ is an isolated in $\sigma(T)$ and hence $T$ and $T^*$ has SVEP at $\lambda$. As $T$ has property $(t)$, we have $T - \lambda \in SF_+(\mathcal{X})$ and hence $\lambda \in B_+(\mathcal{X})$. The SVEP of $T$ and $T^*$ at $\lambda$ implies by [3, Remark 1.2] that $a(T - \lambda) = d(T - \lambda) = \infty$. As $\alpha(T - \lambda) < \infty$ then it follows by [1, Theorem 3.4] that $a(T - \lambda) = \beta(T - \lambda) = \infty$ and so $\lambda \in \sigma(T) \setminus \sigma_{ub}(T)$. Therefore, $E^0(T) \subseteq \sigma(T) \setminus \sigma_{ub}(T)$ and so $T$ obeys property $(m)$.

(ii) $\implies$ (i) Suppose that $T$ obeys property $(m)$ and $\sigma_{ub}(T) = \sigma_{SF_+}(T)$. Then

$$E^0(T) = \sigma(T) \setminus \sigma_{ub}(T) = \sigma(T) \setminus \sigma_{SF_+}(T).$$

That is, $T$ obeys property $(t)$. $\blacksquare$

Let $H_{nc}(\sigma(T))$ denotes the set of all complex-valued functions $f$, dened and regular in some neighborhood of $\sigma(T)$, such that $f$ is not constant on the connected components of its domain of denition.

Recall that $T \in \mathcal{B}(\mathcal{X})$ is isolated, provided that all isolated points of $\sigma(T)$ are eigenvalues of $T$. $T \in \mathcal{B}(\mathcal{X})$ is $a$-isolated provided that all isolated points of $\sigma_a(T)$ are eigenvalues of $T$. It is well-known that $\partial \sigma(T) \subseteq \sigma_a(T)$, so all isolated points of $\sigma(T)$ are also isolated points of $\sigma_a(T)$. Now it is obvious that if $T$ is $a$-isolated, then it is also isolated.
We shall consider nilpotent perturbations of operators satisfying property \((m)\). It easy to check that if \(N\) is a nilpotent operator commuting with \(T\), then
\[
\sigma(T) = \sigma(T + N) \text{ and } \sigma_a(T) = \sigma_a(T + N) \text{ and } \sigma_{ub}(T) = \sigma_{ub}(T + N). \tag{2.1}
\]
Hence it follows from Equation (2.1)
\[
E^0(T) = E^0(T + N) \text{ and } E^0_a(T) = E^0_a(T + N), \tag{2.2}
\]
from [9, Theorem 2.13], we have
\[
\sigma_{SF^+}(T) = \sigma_{SF^+}(T + N). \tag{2.3}
\]

**Theorem 2.6.** Let \(T \in \mathcal{B}(\mathcal{X})\) and let \(N\) be a nilpotent operator commuting with \(T\). If property \((m)\) holds for \(T\) then it also holds for \(T + N\).

**Proof.** Firstly we prove that \(E^0(T) = E^0(T + N)\). It is enough to prove that if \(0 \in E^0(T)\), then \(0 \in E^0(T + N)\). Suppose that \(0 \in E^0(T)\), so \(0 < \dim \ker(T - \lambda) < \infty\).

We prove that \(\dim \ker(T + N) < \infty\). If \((T + N)x = 0\) for some \(x \neq 0\), then \(Tx = Nx\). Since \(N\) commutes with \(T\), it follows that for every positive integer \(m\): \(T^m x = (1)^m N^m x\). Let \(n\) be the smallest positive integer such that \(N^n = 0\). We get that there is some positive integer \(r, r \leq n\), such that \(T^r x = 0\). Thus \(\ker(T + N) \subseteq \ker(T^r)\) and \(\ker(T + N)\) is finite dimensional.

We prove that \(\dim \ker(T + N) > 0\). There is some \(x \neq 0\) such that \(Tx = 0\). Then \((T + N)^n x = 0\), \(0 \in \sigma_p(T + N) \subseteq \sigma(T + N)\) and \(\dim \ker(T + N) > 0\). By Eq. (2.1) we know that \(\sigma(T) = \sigma(T + N)\), so it follows that \(0 \in E^0(T + N)\). Thus, using Eq. (2.1) we get
\[
\sigma_{ub}(T + N) = \sigma_{ub}(T) = \sigma(T) \setminus E^0(T) = \sigma(T + N) \setminus E^0(T + N).
\]
Thus property \((m)\) holds for \(T + N\). \(\square\)

The following example shows that the result of Theorem 2.6 does not hold if we do not assume that the nilpotent operator commutes with \(T\).

**Example 2.7.** Let \(\mathcal{X} := l^2(\mathbb{N})\) and \(T\) and \(N\) be defined by
\[
T(x_1, x_2, \cdots) := \left(0, \frac{x_1}{2}, \frac{x_2}{3}, \cdots\right), \quad (x_n) \in \mathcal{X}
\]
and
\[
N(x_1, x_2, \cdots) := \left(0, -\frac{x_1}{2}, 0, 0, \cdots\right), \quad (x_n) \in \mathcal{X}.
\]
Clearly, \(N\) is a nilpotent operator and \(T\) is a quasinilpotent operator satisfying property \((m)\). On the other hand, it easily seen that \(0 \in E^0(T + N)\) and \(0 \notin \sigma(T + N) \setminus \sigma_{ub}(T + N)\), so that \(T + N\) does not satisfies property \((m)\).

The next result from [8] is very useful.

**Lemma 2.8.** If \(\alpha(T) = n\) and \(\dim \mathcal{R}(T) = m\), then
\[
\alpha(T + N) \leq n + m,
\]
where \(m\) and \(n\) are non-negative integers.

**Theorem 2.9.** Suppose that \(F\) is an arbitrary nite rank operator and \(TF = FT\). If \(T\) is isoloid and property \((m)\) holds for \(T\), then property \((m)\) holds for \(T + F\).
Proof. It is enough to prove that $0 \in \sigma(T + F) \setminus \sigma_{ub}(T + F)$ if and only if $0 \in E^0(T + F)$. Firstly we prove that if $0 \in \sigma(T + F) \setminus \sigma_{ub}(T + F)$, then $T + F \in B_+(\mathcal{X})$ and $0 < \alpha(T + F) < \infty$. We need to prove that $0 \in iso\sigma(T + F)$. It follows that $T \in B_+(\mathcal{X})$, so $0 \notin \sigma_{ub}(T)$. It is possible that $0 \notin \sigma(T)$. In this case we get $0 \notin \text{acc}\sigma(T)$ and hence $0 \notin \text{acc}\sigma_a(T + F)$, so $0 \in E^0(T + F)$. The second possibility is that $0 \in \sigma(T)$. Since property (m) holds for $T$, we get that $0 \notin \text{acc}\sigma(T)$ and again $0 \in E^0(T + F)$.

To prove the opposite implication, suppose that $0 \in E^0(T + F)$. Then $0 \in iso\sigma(T + F)$ and $0 < \alpha(T + F) < \infty$. Hence $0 \notin \text{acc}\sigma(T)$ and by Lemma 2.8 it follows that $0 \leq \alpha(T) < \infty$. Again we distinguish two cases. Firstly, if $0 \notin \sigma(T)$, then $T \in B_+(\mathcal{X})$ and $T + F \in B_+(\mathcal{X})$, $0 \in \sigma(T + F) \setminus \sigma_{ub}(T + F)$. On the other hand, if $0 \in \sigma(T)$ then $0 \in iso\sigma(T)$. Since $T$ is isoloid, we get that $0 < \alpha(T) < \infty$ and $0 \notin \sigma_{ub}(T)$. Now, we have $T \in B_+(\mathcal{X})$, $T + F \in B_+(\mathcal{X})$ and $0 \in \sigma(T + F) \setminus \sigma_{ub}(T + F)$. \hfill \blacksquare

Note that the operator $N$ in Example 2.7 is also a nite rank operator not commuting with $T$. In general, property (m) is also not transmitted under commuting nite rank perturbation.

Example 2.10. Let $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be an injective quasinilpotent operator, and let $U : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be dened:

$$U(x_1, x_2, \cdots) := (x_1, 0, 0, \cdots) \quad (x_n) \in \ell^2(\mathbb{N}).$$

Dene on $\mathcal{X} := \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ the operators $T$ and $F$ by

$$T := I \oplus S \quad \text{and} \quad F := U \oplus 0.$$ 

Clearly, $F$ is a nite rank operator and $TF = FT$. It is easy to check that

$$\sigma(T) = \sigma_a(T) = \sigma_w(T) = \sigma_{ub}(T) = \{0, 1\}.$$ 

Now, both $T$ and $T^*$ have SVEP, since $\sigma(T) = \sigma(T^*)$ is nite. Moreover, $E^0(T) = \sigma(T) \setminus \sigma_{ub}(T) = \emptyset$, so $T$ satises property (m). On other hand, $\sigma(T + F) = \sigma_{ub}(T + F) = \{0, 1\}$, and $E^0(T + F) = \{0\}$, so that property (m) does not hold for $T + F$.

Recall that $T \in \mathcal{B}(\mathcal{X})$ is said to be a Riesz operator if $T - \lambda \in \mathcal{F}(\mathcal{X})$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. Evidently, quasi-nilpotent operators and compact operators are Riesz operators. The proof of the following result may be found in Rakočević [18]:

**Lemma 2.11.** Let $T \in \mathcal{B}(\mathcal{X})$ and $R$ be a Riesz operator commuting with $T$. Then

(i) $T \in B_+(\mathcal{X})$ if and only if $T + R \in B_+(\mathcal{X})$.

(ii) $T \in B_-(\mathcal{X})$ if and only if $T + R \in B_-(\mathcal{X})$.

(iii) $T \in B(\mathcal{X})$ if and only if $T + R \in B(\mathcal{X})$.

Define

$$E^0_f := \{\lambda \in iso\sigma(T) : \alpha(T - \lambda) < \infty\}.$$ 

Evidently, $E^0(T) \subseteq E^0_f(T)$ for every operator $T \in \mathcal{B}(\mathcal{X})$. 
Lemma 2.12. Let $T \in \mathcal{B}(\mathcal{X})$. If $R$ is a Riesz operator that commutes with $T$, then
\[
E^0(T + R) \cap \sigma(T) \subseteq iso\sigma(T).
\] (2.4)

Proof. By [13, Lemma 2.3], we have
\[
E^0(T + R) \cap \sigma(T) \subseteq E^0(T + R) \cap \sigma(T) \subseteq iso\sigma(T).
\]

Lemma 2.13. Let $T \in \mathcal{B}(\mathcal{X})$ be an isoloid operator satisfying property (m). If $F$ is an operator that commutes with $T$ and for which there exists a positive integer $n$ such that $F^n$ is nite rank, then $E^0(T + F) = \pi^0_a(T + F)$.

Proof. Let $\lambda \in E^0(T + F)$ be an arbitrary given. We distinguish two cases. Firstly, if $\lambda \notin \sigma(T)$, then $T + F - \lambda \in B_+(\mathcal{X})$, and hence $\lambda \in \sigma_{ub}(T)$. Suppose that $\lambda \in \sigma(T)$, it follows, by Lemma 2.12, that $\lambda \in iso\sigma(T)$. Furthermore, since the operator $(T + F - \lambda)^n|_{\ker(T - \lambda)} = F^n|_{\ker(T - \lambda)}$ is both of finite-dimensional range and kernel, kernel, we obtain easily that also $\ker(T - \lambda)$ is finite-dimensional, and therefore that $\lambda \in E^0(T)$, because $T$ is isoloid. On the other hand, if $T$ obeys property (m), then $E^0(T) \cap \sigma_{ub}(T) = \emptyset$. Consequently, $T - \lambda \in B_+(\mathcal{X})$ and hence $T + F - \lambda \in B_+(\mathcal{X})$, which implies that $\lambda \in \pi^0_a(T + F)$.

To prove the other inclusion, let $\lambda \in \pi^0_a(T + F)$ be arbitrary given. Then $\lambda \in iso\sigma_a(T + F)$ and $T + F - \lambda \in B_+(\mathcal{X})$, so $\sigma(T + F - \lambda) < \infty$. Since $T + F - \lambda$ has closed range, the condition $\lambda \in \sigma_a(T + F)$ entails that $\alpha(T + F - \lambda) > 0$. Therefore, in order to show that $\lambda \in E^0(T + F)$, we need only to prove that $\lambda$ is an isolated point of $\sigma(T + F)$. We know that $\lambda \in iso\sigma_a(T)$. We have from Lemma 2.11 that $(T + F) - \lambda - F = T\lambda \in B_+(\mathcal{X})$ so that $\lambda \in \sigma_a(T) \setminus \sigma_{ub}(T) = \pi^0_a(T)$. Now, by assumption $T$ obeys property (m) so, by [20, Theorem 2.2], $\pi^0_a(T) = E^0(T)$. Moreover, $T$ satisfies Weyl’s theorem and hence
\[
E^0(T) = \pi^0_a(T) = \sigma(T) \setminus \sigma_b(T).
\]

Therefore, $T - \lambda$ is Browder and hence $T + F - \lambda$ is Browder, so
\[
0 < a(T + F - \lambda) = d(T + F - \lambda) < \infty
\]
and hence $\lambda$ is a pole of the resolvent of $T + F$. Consequently, $\lambda$ is an isolated point of $\sigma(T + F)$.

Theorem 2.14. If $T \in \mathcal{B}(\mathcal{X})$ has property (m) and $R$ is a Riesz operator for which $TR = RT$, then $E^0(T) \subseteq E^0(T + R)$.

Proof. Suppose that $T$ has property (m). Since $\sigma(T) = \sigma(T + R)$ holds for every Riesz operator commuting with $T$, we have
\[
E^0(T) = \sigma(T) \setminus \sigma_{ub}(T) = \sigma(T + R) \setminus \sigma_{ub}(T + R).
\] (2.5)

Let $\lambda \in E^0(T)$ be arbitrary given. Taking into account that $S := T + R$ commutes with $R$, by Lemma 2.12 we then have
\[
\lambda \in E^0(T) \cap \sigma(T + R) = E^0(S - R) \cap \sigma(S) \subseteq iso\sigma(S) = iso\sigma(T + R).
\]
Moreover, from (2.5) we know that \( T + R - \lambda \in B_+(\mathcal{H}) \) and hence has closed range. Since \( \lambda \in \sigma(T + R) \) it then follows that \( \lambda \) is an eigenvalue, so \( 0 < \alpha(T + R - \lambda) < \infty \), i.e., \( \lambda \in E^0(T + R) \). \( \blacksquare \)

**Lemma 2.15.** Let \( T \in \mathcal{B}(\mathcal{H}) \) be an isoloid operator satisfying property (m). If \( F \) is an operator that commutes with \( T \) and for which there exists a positive integer \( n \) such that \( F^n \) is finite rank, then \( E^0(T) = E^0(T + F) \).

**Proof.** Observe rst that \( F \) is a Riesz operator, so, by Theorem 2.14, we need only to prove the inclusion \( E^0(T + F) \subseteq E^0(T) \). Let \( \lambda \in E^0(T + F) \). Then \( \lambda \) is an isolated point of \( \sigma(T + F) \), and since \( \alpha(T + F - \lambda) > 0 \) we then have \( \lambda \in \sigma(T + F) = \sigma(T) \). Therefore, by Lemma 2.12, \( \lambda \in E^0(T + F) \cap \sigma(T) \subseteq iso\sigma(T) \). Since \( T \) is isoloid then \( \alpha(T\lambda) = 0 \). We show now that \( \alpha(T - \lambda) < \infty \). Let \( U \) denote the restriction of \( (T + F - \lambda)^n \) to \( \ker(T - \lambda) \). Clearly, if \( x \in \ker(T - \lambda) \) then \( Ux = (1)^nF^n\alpha = \mathcal{F}(\mathcal{H}) \), thus \( U \) is a finite rank operator. Moreover, since \( \lambda \in E^0(T + F) \) we have \( \alpha(T + F - \lambda) < \infty \) and hence \( \alpha(U) \leq \alpha(T + F - \lambda)^n < \infty \). By [5, Remark 2.5] it then follows that \( \ker(T - \lambda) \) is finite-dimensional. Therefore, \( \lambda \in E^0(T) \) and consequently \( E^0(T + F) \subseteq E^0(T) \). \( \blacksquare \)

**Theorem 2.16.** Let \( T \in \mathcal{B}(\mathcal{H}) \) be an isoloid operator satisfying property (m). If \( F \) is an operator that commutes with \( T \) and for which there exists a positive integer \( n \) such that \( F^n \) is finite rank, then \( T + F \) satisfies property (m).

**Proof.** Since \( F \) is a Riesz operator we have, by [18], \( \sigma_{ub}(T) = \sigma_{ub}(T + F) \), thus \( E^0(T + F) = E^0(T) = \sigma(T) \setminus \sigma_{ub}(T) = \sigma(T + F) \setminus \sigma_{ub}(T + F) \), hence \( T + F \) satisfies property (m). \( \blacksquare \)

**Example 2.17.** Generally, property (m) is not transmitted from \( T \) to a quasi-nilpotent perturbation \( T + Q \). Let \( Q : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) \) dened by

\[
Q(x_1, x_2, \cdots) = \left( \frac{x_2}{2}, \frac{x_3}{3}, \cdots \right)
\]

for all \( (x_n) \in \ell^2(\mathbb{N}) \).

Then \( Q \) is quasi-nilpotent, so \( \sigma(T) = \sigma_{ub}(T) = \{0\} \) and hence \( \{0\} = E^0(Q) \neq \sigma(Q) \setminus \sigma_{ub}(Q) = \emptyset \). Take \( T = 0 \). Clearly, \( T \) satisfies property (m) but \( T + Q = Q \) fails this property.

Recall that an operator \( T \in \mathcal{B}(\mathcal{H}) \) is nite-isoloid if isolated points of \( \sigma(T) \) are eigenvalues of \( T \) of nite multiplicity.

**Theorem 2.18.** Suppose that \( T \in \mathcal{B}(\mathcal{H}) \) is a nite-isoloid operator which obeys property (m). If \( R \) is a Riesz operator which commutes with \( T \), then \( T + R \) obeys property (m).

**Proof.** We show rst that \( E^0(T) = E^0(T + R) \). By Theorem 2.14 it sucès to prove that \( E^0(T + R) \subseteq E^0(T) \). Let \( \lambda \in E^0(T + R) \) be arbitrary given. Then \( \lambda \) is an isolated of \( \sigma(T + R) \) and \( 0 < \alpha(T + R - \lambda) < \infty \). Since \( \sigma(T) = \sigma(T + R) \) holds for every Riesz operator commuting with \( T \), we have by Lemma 2.12 that \( \lambda \in E^0(T + R) \cap \sigma(T) \subseteq iso\sigma(T) \). Since \( T \) is nite-isoloid then \( 0 < \alpha(T - \lambda) < \infty \)
and so \( \lambda \in E^0(T) \). Therefore, \( E^0(T) = E^0(T + R) \).

As \( T \) obeys property \((m)\) and \( E^0(T) = E^0(T + R) \), we have

\[
E^0(T + R) = E^0(T) = \sigma(T) \setminus \sigma_{ub}(T) = \sigma(T + R) \setminus \sigma_{ub}(T + R).
\]

That is, property \((m)\) holds for \( T + R \). ■

Since every compact operator is a Riesz operator we have:

**Corollary 2.19.** Let \( T \in \mathcal{B}(\mathcal{X}) \) be a nite-isoloid operator that obeys property \((m)\). If \( K \) is a compact operator commuting with \( T \), then \( T + K \) obeys property \((m)\).

Since every quasi-nilpotent operator is a Riesz operator we have:

**Corollary 2.20.** Let \( T \in \mathcal{B}(\mathcal{X}) \) be a nite-isoloid operator that obeys property \((m)\). If \( Q \) is a quasi-nilpotent operator commuting with \( T \), then \( T + Q \) obeys property \((m)\).

**Theorem 2.21.** Suppose that \( T \in \mathcal{B}(\mathcal{X}) \) and \( Q \) an injective quasi-nilpotent operator commuting with \( T \). If \( T \) obeys property \((m)\), then \( T + Q \) obeys property \((m)\).

**Proof.** As \( T \) obeys property \((m)\), we have

\[
\sigma(T + Q) \setminus \sigma_{ub}(T + Q) = \sigma(T) \setminus \sigma_{ub}(T) = E^0(T).
\]

To show property \((m)\) for \( T + Q \) it suces to prove that

\[
E^0(T + Q) = E^0(T) = 0.
\]

Suppose that \( E^0(T) \neq \emptyset \) and let \( \lambda \in E^0(T) \). From (2.6) we know that \( T - \lambda \in B_{\infty}(\mathcal{X}) \) and hence from [2, Lemma 2.11] it then follows that \( \alpha(T - \lambda) = 0 \), a contradiction.

To show that \( E^0(T + Q) = \emptyset \). Suppose that \( \lambda \in E^0(T + Q) \). Then \( 0 < \alpha(T + Q - \lambda) < \infty \), so there exists \( x \neq 0 \) such that \( (T + Q - \lambda)x = 0 \). Since \( Q \) commutes with \( T + Q - \lambda \), a similar argument of proof of [2, Lemma 2.11] shows that \( \lambda(T + Q - \lambda) = \infty \), a contradiction. ■

**Theorem 2.22.** Let \( T \) be an operator on \( \mathcal{X} \) that obeys property \((m)\) and such that \( \sigma_{p}(T) \cap \text{iso}\sigma(T) \subseteq E^0(T) \). If \( Q \) is a quasi-nilpotent operator that commutes with \( T \), then \( T + Q \) obeys property \((m)\).

**Proof.** As \( T \) obeys property \((m)\), we have by [20, Theorem 2.10] that \( T \) satisfies Weyl’s theorem and \( \sigma_w(T) = \sigma_{ub}(T) \). Hence by [13, Proposition 2.9], we have \( T + Q \) satisfies Weyl’s theorem. Since \( \sigma_{ub}(T + Q) = \sigma_{ub}(T) \) and \( \sigma_{w}(T) = \sigma_{w}(T + Q) \) we have \( \sigma_{ub}(T + Q) = \sigma_{w}(T + Q) \) and so \( T + Q \) obeys property \((m)\). ■

**Definition 2.23.** A bounded linear operator \( T \) is said to be algebraic if there exists a non-trivial polynomial \( h \) such that \( h(T) = 0 \).

From the spectral mapping theorem it easily follows that the spectrum of an algebraic operator is a nite set. A nilpotent operator is a trivial example of an algebraic operator. Also nite rank operators \( K \) are algebraic; more generally, if \( K^n \) is a nite rank operator for some \( n \in \mathbb{N} \) then \( K \) is algebraic. Clearly, if \( T \) is
algebraic then its dual $T^*$ is algebraic, as well as $T'$ in the case of Hilbert space operators.

A bounded operator $T \in \mathcal{B}(\mathcal{H})$ is said to be polaroid (respectively, $a$-polaroid) if $\text{iso} \sigma(T) = \emptyset$ or every isolated point of $\sigma(T)$ is a pole of the resolvent of $T$ (respectively, if $\text{iso} \sigma_a(T) = \emptyset$ or every isolated point of $\sigma_a(T)$ is a pole of the resolvent of $T$).

**Theorem 2.24.** Suppose that $T \in \mathcal{B}(\mathcal{H})$ and $K \in \mathcal{B}(\mathcal{H})$ is an algebraic operator which commutes with $T$.

(i) If $T^*$ is hereditarily polaroid and has SVEP, then $T + K$ obeys property $(m)$.

(ii) If $T$ is hereditarily polaroid and has SVEP, then $T^* + K^*$ obeys property $(m)$.

**Proof.** (i) Obviously, $K^*$ is algebraic and commutes with $T^*$. Moreover, by [5, Theorem 2.15], we have $T^* + K^*$ is polaroid, or equivalently, $T + K$ is polaroid. Since $T^*$ has SVEP then by [4, Theorem 2.14], we have $T^* + K^*$ has SVEP. Therefore, $T + K$ obeys property $(m)$ by [20, Theorem 3.3 (i)].

(ii) It follows from the proof of Theorem 2.15 of [5] that $T + K$ is polaroid and hence by duality $T^* + K^*$ is polaroid. Since $T$ has SVEP then it follows from [4, Theorem 2.14] that $T + K$ has SVEP. Therefore, $T^* + K^*$ obeys property $(m)$ by [20, Theorem 3.3 (ii)].

**Theorem 2.25.** Suppose that $T \in \mathcal{B}(\mathcal{H})$ and $K \in \mathcal{B}(\mathcal{H})$ is an algebraic operator which commutes with $T$.

(i) If $T^*$ is hereditarily polaroid and has SVEP, then $f(T + K)$ obeys property $(m)$ for all $f \in H_{nc}(\sigma(T))$.

(ii) If $T$ is hereditarily polaroid and has SVEP, then $f(T^* + K^*)$ obeys property $(m)$ for all $f \in H_{nc}(\sigma(T))$.

**Proof.** (i) We conclude from [5, Theorem 2.15] that $T + K$ is polaroid and hence by [6, Lemma 3.11], we have $f(T + K)$ is polaroid and from [4, Theorem 2.14] that $T^* + K^*$ has SVEP. The SVEP of $T^* + K^*$ entails the SVEP for $f(T^* + K^*)$ by [1, Theorem 2.40]. So, $f(T + K)$ obeys property $(m)$ by [20, Theorem 3.3 (i)].

(ii) The proof of part (ii) is analogous.

**References**


Department of Mathematics & Statistics, Faculty of Science P.O.Box(7), Mu’tah University, Jordan
E-mail address: malik_okasha@yahoo.com