A GENERIC UNIQUENESS RESULT FOR AN INTERPOLATION PROBLEM FOR THE JOIN OF A TANGENTIAL VARIETY $\tau(X)$ AND SEVERAL COPIES OF THE VARIETY $X \subset \mathbb{P}^r$

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Abstract. Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety non-singular in codimension 1. Let $\tau(X, 2) = \tau(X) \subset \mathbb{P}^r$ be the tangential variety of $X$. Set $n := \dim(X)$. For any integer $b > 2$ let $\tau(X, b)$ be the join of $\tau(X)$ and $b - 2$ copies of $X$. Mimicking the notion of weakly $k$-degenerate varieties introduced by Chiantini and Ciliberto we give a conditions on $X$ assuring that a general $q \in \tau(X, b)$ is in the linear span of a unique scheme $Z \subset X$ with $Z = v \cup \{p_1, \ldots, p_{b-2}\}$ with $v$ connected of degree 2 and $v \subset X_{\text{reg}}$.

1. Introduction and preliminaries

Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate projective variety defined over an algebraically closed field with characteristic 0. Let $\tau(X) \subset \mathbb{P}^r$ be the tangential variety of $X$, i.e. the closure in $\mathbb{P}^r$ of the union of all linear spaces $T_pX$, $p \in X_{\text{reg}}$. Set $\tau(X, 2) := \tau(X)$. For any integer $b > 2$ let $\tau(X, b)$ be the join of $\tau(X)$ and $b - 2$ copies of $X$. Set $n := \dim(X)$. Each $\tau(X, b)$ is an integral and non-degenerate variety of dimension at most $b(n + 1) - 2$. From now on we assume that $b \geq 2$ and that $b(n + 1) - 2 < r$. For a general $q \in \tau(X)$ there is $p \in X_{\text{reg}}$ such that $q \in T_pX$ and hence there is a line $L$ tangent to $X$ at $p$ and with $q \in L$. There is a degree 2 connected zero-dimensional scheme $v \subset X$ with $v_{\text{red}} = \{p\}$ and $L = \langle v \rangle$, where $\langle \cdot \rangle$ denote the linear span. Let $Z(X, b)$ be the set of all degree $b$ schemes $v \cup \{p_1, \ldots, p_{b-2}\}$ with $v$ a degree 2 connected zero-dimensional scheme contained in $X_{\text{reg}}$ and $p_1, \ldots, p_{b-2}$ distinct points of $X \setminus v_{\text{red}}$. Call $\tau(X, b)'$ the set of all $q \in \tau(X, b)$ such that $q \in \langle Z \rangle$ for some $Z \in Z(X, b)$. By the definition of join and the case $b = 2$ just done the constructible set $\tau(X, b)'$ contains a non-empty open subset of $\tau(X, b)$. For each $q \in \tau(X, b)'$ let $Z(X, b, q)$ denote the set of all $Z \in Z(X, b)$ such that $q \in \langle Z \rangle$. The aim of this note is to give conditions assuring that $\sharp(Z(X, q, b)) = 1$ for a general $q \in \tau(X, b)'$. We were inspired by the notion of weak defectivity introduced by L. Chiantini and C. Ciliberto in [3] and used for many important generic uniqueness theorems ([4, Corollary 2.7], [5]).
Let $\mathcal{H}$ be the dual variety of $\tau(X,b)$, i.e. the closure in $\mathbb{P}^r$ of the set of all hyperplanes tangent to $\tau(X,b)_{\text{reg}}$. $\mathcal{H}$ is an irreducible variety and it is easy to check that $\dim \mathcal{H} \leq r - b$ (Lemma 2.1). Fix a general $q \in \tau(X,b)$ and take $o \in \tau(X)_{\text{reg}}$ and $(p_1,\ldots,p_{b-2}) \in X_{\text{reg}}^{b-2}$ such that $q$ is a general element of $\langle T_o \tau(X) \cup T_{p_1} X \cup \cdots \cup T_{p_{b-2}} X \rangle$. Since $o$ is general in $\tau(X)$, there is $p \in X_{\text{reg}}$ and a line $L \subset \mathbb{P}^r$ tangent to $X$ at $p$ and with $o \in L$. The contact locus of $H$ and $\tau(X)_{\text{reg}}$ contains $L \cap \tau(X)_{\text{reg}}$ (we will say that the contact locus of $H$ and $\tau(X)$ contain $L$) and the contact locus of $H$ and $X_{\text{reg}}$ contains $\{p_1,\ldots,p_{b-2}\}$.

**Definition 1.1.** $X$ is said to be **not weakly tangentially $b$-defective** if for a general $H \in \mathcal{H}$ the following conditions hold:

1. $p_1,\ldots,p_{b-2}$ are isolated points of the contact locus of $H$ and $X$;
2. $L$ is an isolated line of the contact locus of $H$ and $\tau(X)$.

If $\dim \tau(X,b) < b(n+1) - 2$, then $X$ is weakly tangentially $b$-defective (Lemma 2.2) and if $\dim \mathcal{H} < r - b$, then $X$ is weakly tangentially $b$-defective (Lemma 2.4).

We prove the following result.

**Theorem 1.2.** Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate projective variety, which is non-singular in codimension 1. Set $n := \dim(X)$. Fix an integer $b \geq 2$ and assume $r > b(n+1) - 2$ and that $X$ is not weakly tangentially $b$-defective. Fix a general $q \in \tau(X,b)$ and let $H \subset \mathbb{P}^r$ be a general hyperplane containing $T_q \tau(X,b)$. Then $Z(X,b,q) = \{Z\}$ is a singleton and $H$ is tangent to $X_{\text{reg}}$ only at the points of $Z_{\text{red}}$.

2. **Proof of Theorem 1.2**

**Lemma 2.1.** We have $\dim \mathcal{H} \leq r - b$.

**Proof.** It is sufficient to prove that for a general $q \in \tau(X,b)$ there is a $(b-1)$-dimensional linear space $V \subset \tau(X,b)$ such that $V \cap \tau(X,b)_{\text{reg}} \neq \emptyset$, $q \in V$ and $V \subset T_q \tau(X,b)$ for all $o \in V \cap \tau(X,b)_{\text{reg}}$. Fix a general $q \in \tau(X,b)$. By Terracini’s lemma for joins ([1, Corollary 1.10]) and a dimensional count there are $o \in \tau(X)_{\text{reg}}$ and $(p_1,\ldots,p_{b-2}) \in X_{\text{reg}}^{b-2}$ such that $T_q \tau(X,b) = \langle T_o \tau(X) \cup T_{p_1} X \cup \cdots \cup T_{p_{b-2}} X \rangle$ and $(o,p_1,\ldots,p_{b-2})$ is general in $(\tau(X) \times X)^{b-2}$. In particular $o \in \tau(X)'$. Since $o \in \tau(X)'$, there is a line $L \subset \tau(X)$ tangent to $X_{\text{reg}}$. $T_o \tau(X)_{\text{reg}}$ is tangent to $\tau(X)_{\text{reg}}$ at all points of $\tau(X)_{\text{reg}} \cap L$. Hence by Terracini’s lemma for joins $T_q \tau(X,b)$ is tangent to $\tau(X,b)_{\text{reg}}$ at all points of $V := \langle L \cup \{p_1,\ldots,p_{b-2}\} \rangle$. Since $r \geq b$, for a general $(o,p_1,\ldots,p_{b-2})$ we have $\dim V = b - 1$. \qedsymbol

**Lemma 2.2.** If $\dim \tau(X,b) < b(n+1) - 2$, then $X$ is weakly tangentially $b$-defective.

**Proof.** First assume $\dim \tau(X) < 2n$ and call $o$ a general point of $\tau(X)$. We get that every irreducible component of the set of all lines tangent to $X_{\text{reg}}$ and containing $o$ is positive-dimensional and so $X$ is weakly tangential 2-degenerate. If $b > 2$ a lemma of Terracini giving the tangent space of a join at its general points ([1, Corollary 1.10]) shows that condition (2) of Definition 1.1 is not satisfied. Now assume $b > 2$ and that $\dim \tau(X) = 2n$. By Terracini’s lemma we see that least one of the conditions (1) or (2) of Definition 1.1 is not satisfied. \qedsymbol
Remark 2.3. Fix $q \in \tau(X, b)'$. Since the image of algebraic set by a morphism is constructible, $Z(X, b, q)$ is constructible. Thus it makes sense to speak about the irreducible components of $Z(X, b, q)$. We have $\dim(\tau(X, b) = b(n + 1) - 2$ if and only if $Z(X, b, q)$ is finite for a general $q \in \tau(X, b)$.

**Lemma 2.4.** If $X$ is not weakly tangentially $b$-defective, we have $\dim \mathcal{H} = r - b$.

**Proof.** By Lemma 2.1 it is sufficient to prove that $\dim \mathcal{H} \geq r - b$. Since $\dim \tau(X, b) = b(n + 1) - 2$, for a general $q \in \tau(X, b)$ the set $Z(X, b, q)$ is finite. Fix a general $q \in \tau(X, b)$ and take $Z \in Z(X, b, q)$. Write $Z = \langle v \cup \{p_1, \ldots, p_{b-2}\}$ and set $L := \langle v \rangle$. Since $X$ is not weakly tangentially $b$-defective, $L$ is an isolated line in the contact locus of $H$ with $\tau(X)$ and $\{p_1, \ldots, p_{b-2}\}$ are isolated points of the contact locus of $H$ and $X$. Since $\dim \tau(X, b) = b(n + 1) - 2$, a dimensional count and Terracini’s lemma gives $\dim \mathcal{H} \geq r - b$. $\square$

Remark 2.5. Let $X \subset \mathbb{P}^r$, $r \geq \dim X + 2$, be and integral and non-degenerate variety, which is non-singular in codimension 1. Let $L$ be a general tangent line of $X_{\text{reg}}$. Call $v \subset L$ the degree 2 effective divisor with as its support the point of tangency of $L$ and $X$. Since a general curve section of $X$ is a smooth curve, [6, Theorem 3.1] gives $L \cap X = v$ as schemes.

We recall the following lemma ([2, Lemma 2.4] and Remark 2.5).

**Lemma 2.6.** Fix an integer $b \geq 2$. Let $X \subset \mathbb{P}^r$, $r \geq 1 + b + \dim X$, be an integral and non-degenerate variety non-singular in codimension 1. Fix a general $Z \in Z(X, b)$. Then $\dim(Z) = b - 1$ and $\langle Z \rangle \cap X = Z$ (scheme-theoretic intersection).

It is the use of Lemma 2.6, which force us to assume that $X$ is non-singular in codimension 1.

**Proof of Theorem 1.2:** Fix a general $q \in \tau(X, b)$. Since $\dim \tau(X, b) = b(n + 1) - 2$ and $q$ is general, $Z(X, b, q)$ is finite (Remark 2.2). Assume the existence of $Z, W \in Z(X, b, q)$ with $Z \neq W$. Since $q$ is general, we may see $Z$ and $W$ as general elements of $Z(X, b)$. Write $Z = \langle v \cup \{p_1, \ldots, p_{b-2}\}$, $W = \langle w \cup \{o_1, \ldots, o_{b-2}\}$ and set $\{p\} := v_{\text{red}}$, $L := \langle v \rangle$, $\{o\} := w_{\text{red}}$ and $R := \langle w \rangle$. Take $e \in L$ and $f \in R$ such that $q \in \langle \{e, p_1, \ldots, p_{b-2}\} \rangle \cap \langle \{f, o_1, \ldots, o_{b-2}\} \rangle$. Let $H \subset \mathbb{P}^r$ be a general hyperplane containing $T_e \tau(X) \cup T_{p_i}X \cup \cdots \cup T_{p_{b-2}}X$; $H$ exists, because $r > b(n + 1) - 2$. Note that $\mathcal{H} \subset \tau(X)'$. Call $\mathcal{H}'$ the set $\mathcal{H}$ seen as a subset of the hyperplane sections of $\tau(X)$ (i.e. we are using the set-up of [3, §2] for $\tau(X)$, not for $X$). Let $N_{\tau(X), \mathcal{H}'}$ denote the normal sheaf of $H \cap \tau(X)$ in $\mathcal{H}' \subset \mathbb{P}^r$.

**Claim 1:** $H^0(N_{\tau(X), \mathcal{H}'}) \subset H^0(\mathcal{T}_L(1))$.

**Proof of Claim 1:** Let $E \subset \mathbb{P}^r$ be a general hyperplane and let $E^\vee$ denote the dual $(r - 1)$-dimensional projective space. Call $\mathcal{H}_E \subset E^\vee$ the family induced by $\mathcal{H}$. Since $E$ is general and $\dim \mathcal{H} < r - 1$, $\tau(X, b) \cap E$ is integral, $\dim \mathcal{H}_E = \dim \mathcal{H}$ and $\mathcal{H}_E = (\tau(X, b) \cap E)'$. Since $\dim \mathcal{H} \leq r - 2$ and $E$ is general, it is easy to check that $\dim \mathcal{H}_E = \dim E$. Since $E$ is general, we have $H \neq E$ and so $H \cap E \in \mathcal{H}_E$. Let $N_{H \cap E, \mathcal{H}_E}$ denote the normal sheaf of $H \cap E$ in $\mathcal{H}_E$. By the infinitesimal Bertini’s theorem ([3, Theorem 2.2]) applied to $\mathcal{H}_E$ and $\tau(X) \cap E$ we get $H^0(N_{H \cap E, \mathcal{H}_E}) \subset H^0(E, \mathcal{T}_E \cap L(1))$. Hence $H^0(N_{\tau(X), \mathcal{H}'}) \subset H^0(\mathcal{T}_E \cap L(1))$. Since this is true for a general hyperplane $E \subset \mathbb{P}^r$, we get $H^0(N_{\tau(X), \mathcal{H}'}) \subset H^0(\mathcal{T}_L(1))$. $\square$
Call $H''$ the variety $H$ seen as a subset of the set of all hyperplane sections of $X$. Let $N_{H \cap X, H''}$ denote the normal sheaf of $H \cap X$ in $H''$.

**Claim 2:** $H^0(N_{H \cap X, H''}) \subseteq H^0(I_{\{p_1, \ldots, p_{b-2}\}}(1))$.

**Proof of Claim 2:** This is an obvious consequence of the infinitesimal Bertini’s theorem ([3, Theorem 2.2]).

By Claims 1 and 2 we have $H^0(N_{H, H}) \subseteq H^0(I_{Z}(1))$. Since $\dim \langle Z \rangle = b - 1$ and $\dim H = r - b$, we get $H^0(N_{H, H}) = H^0(I_Z(1))$. Since $\langle Z \rangle \cap X = Z$, we see that $H$ cannot be tangent to $\tau(X)_{\text{reg}}$ outside $L$ and it cannot be tangent to $X_{\text{reg}}$ outside $Z_{\text{red}}$.

Let $M$ be another general hyperplane containing $T_q(\tau, X, b)$. We just proved that $H^0(N_{M, H}) = H^0(I_Z(1))$. By Terracini’s lemma we have $\langle T_e\tau(X) \cup T_{p_1}X \cup \cdots \cup T_{p_{b-2}}X \rangle = T_q\tau(X, b) = \langle T_f\tau(X) \cup T_oX \cup \cdots \cup T_{o_{b-2}}X \rangle$. Hence using $W$ instead of $Z$ we get $H^0(I_W(1)) = H^0(I_Z(1))$. Since $\langle Z \rangle \cap H = Z$ (as schemes) by Lemma 2.6, we get $W = Z$, a contradiction.

The last assertion of Theorem 1.2 follows from the proof of the first one and the infinitesimal Bertini’s theorem ([3, Theorem 2.2]).

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**References**


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