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DIFFERENTIABLE OSTROWSKI TYPE TENSORIAL NORM INEQUALITY FOR CONTINUOUS FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Dedicated to the Professor Dragoslav Mitrinović , author of one of the most famous books on inequality theory "Analytic Inequalities", to mark the year that marks 115 years since his birth.

Abstract. In this paper several tensorial norm inequalities of Ostrowski type for continuous functions of selfadjoint operators in Hilbert spaces have been obtained. Multiple inequalities are obtained with variations due to the convexity properties of the mapping f.

1. Introduction and preliminaries

The term "tensor" was not officially used by Gibbs in 19th century when he first developed the idea of a tensor; instead, he used the term "dyadic." It can be viewed as the source of the tensor definition and its introduction to mathematics in modern language. Due to the extensive use of inequalities in mathematics, tensorial inequalities have also been found useful. Inequalities have a significant impact on mathematics and other scientific disciplines. Many types of inequalities exist, but those involving Jensen, Ostrowski, Hermite–Hadamard, and Minkowski hold particular significance among them. More about inequalities and its history can be found in these books [23, 25]. Multiple papers have been published concerning the generalizations of the said inequalities, see the following and references therein for more information [20, 28, 29, 30, 33, 1, 2, 3, 6, 7, 8, 22, 34, 18].
Since our paper is about tensorial Ostrowski type inequalities, we give the brief introduction to the topic. In 1938, A. Ostrowski [24] , proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)dt$ and the value $f(x)$, $x \in [a,b]$.

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b) and $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < +\infty$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leqslant \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f'\|_\infty (b-a),$$

for all $x \in [a,b]$ and the constant $\frac{1}{4}$ is the best possible.

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If we take $x = \frac{a+b}{2}$ we get the midpoint inequality

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leqslant \frac{1}{4}\left\| f' \right\|_\infty (b-a),$$

with $\frac{1}{4}$ as best possible constant.
Recent advances concerning the theory of inequalities in Hilbert spaces will be shown to supplement the presentation of this work. Dragomir [16] gave the following Mond-Pecarić type inequality.

Theorem 1.2. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subset [m, M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$\begin{aligned} \frac{f(m)+f(M)}{2} &\geq \langle \frac{f(A)+f((m+M)1_H - A)}{2} x, x\rangle \\ \geq \frac{f(\langle Ax,x\rangle)+f(m+M-\langle Ax,x\rangle)}{2} &\geq f\left(\frac{m+M}{2}\right), \end{aligned}$$

for each $x \in H$ with $||x|| = 1$.
In addition, if $x \in H$ with $||x|| = 1$ and $\langle Ax, x\rangle \neq \frac{m+M}{2}$, then also

$$\begin{aligned} &\frac{f(\langle Ax,x\rangle)+f(m+M-\langle Ax,x\rangle)}{2} \\ \geq &\frac{2}{\frac{m+M}{2} - \langle Ax,x\rangle}\int_{\langle Ax,x\rangle}^{m+M-\langle Ax,x\rangle} f(u)du \geq f\left(\frac{m+M}{2}\right). \end{aligned}$$

Another interesting result is the Hermite-Hadamard inequality in the selfadjoint operator sense given by Dragomir [17]

Theorem 1.3. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I we have the inequality*

$$\begin{aligned} f\left(\frac{A+B}{2}\right) &\leq \left[f\left(\frac{3A+B}{4}\right) + f\left(\frac{A+3B}{4}\right)\right] \\ &\leq \int_0^1 f((1-t)A+tB)dt \\ \leq \frac{1}{2}\left[f\left(\frac{A+B}{2}\right) + \frac{f(A)+f(B)}{2}\right] &\leq \frac{f(A)+f(B)}{2}. \end{aligned}$$

The following result has been obtained recently in a paper written by Stojiljković [32]. It features an Ostrowski Simpson type tensorial convex inequality:

Theorem 1.4. *Assume that f is continuously differentiable on I and $|f''|$ is convex and A, B are selfadjoint operators with $Sp(A), Sp(B) \subset I$, then*

$$\begin{aligned} &\Bigg|\Bigg| \frac{1}{6}\left(\exp(A)\otimes 1 + 4\exp\left(\frac{A\otimes 1 + 1\otimes B}{2}\right) + 1\otimes \exp(B)\right) \\ &-\frac{1}{4}\Bigg(\int_0^1 \exp\left(\left(\frac{1-k}{2}\right) A\otimes 1 + \left(\frac{1+k}{2}\right) 1\otimes B\right) k^{-\frac{1}{2}} dk \end{aligned}$$

$$+\int_0^1 \exp\left(\left(1-\frac{k}{2}\right)A\otimes 1+\frac{k}{2}1\otimes B\right)(1-k)^{-\frac{1}{2}}dk\right)\Bigg\|$$
$$\leq \frac{47}{360}\left\|1\otimes B-A\otimes 1\right\|^2\left(\left\|\exp(A)\right\|+\left\|\exp(B)\right\|\right).$$

In our paper we will obtain classical Ostrowski type tensorial inequalities using a different convex combination than the one used to obtain the Simpson type inequalities. Since the field of tensorial Hilbet space inequalities is relatively new, it is instrumental in the development of it that boundaries for various convex combinations are obtained.

In order to derive similar inequalities of the tensorial type, we need the following introduction and preliminaries.

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a n-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$ by following [5] , we define

$$f(A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f(\lambda_1, ..., \lambda_k) dE_1(\lambda_1) \otimes ... \otimes dE_k(\lambda_k)$$

as bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction extends the definition of Koranyi [21] for functions of two variables and have the property that

$$f(A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

Since we will be using tensorial products, we will define in the following what tensors and tensorial products are in short, for more consult the following book [19].

Let U, V and W be vector spaces over the same field F. A mapping $\xi : U\times V \to W$ is called a bilinear mapping if it is linear in each variable separately. Namely, for all $u, u_1, u_2 \in U$, $v, v_1, v_2 \in V$ and $a, b \in F$,

$\xi(au_1 + bu_2, v) = a\xi(u_1, v) + b\xi(u_2, v)$,

$\xi(u, av_1+bv_2) = a\xi(u, v_1)+b\xi(u, v_2)$. If $W = F$, a bilinear mapping $\xi : U\times V \to F$ is called a bilinear function.

Let $\otimes : U \times V \to W$ be a bilinear mapping. The pair $(W, \otimes)$ is called a tensor product space of U and V if it satisfies the following conditions:

1. Generating property $< Im\otimes >= W$;
2. Maximal span property $dim < Im\otimes >= dimU \cdot dimV$.

The member $w \in W$ is called a tensor, but not all tensors in W are products of two vectors of the form $u \otimes v$. The notation $< Im\otimes >$ denotes the span.

Example

Let $u = (x_1, .., x_m) \in \mathbb{R}^m$ and $v = (y_1, ..., y_n) \in \mathbb{R}^n$. We can view u and v as column vectors. Namely,

$$u = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}, v = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

are $m \times 1$ and $n \times 1$ matrices respectively.

We define $\otimes : \mathbb{R}^m \times \mathbb{R}^n \to M_{m,n}$,

$$u \otimes v = uv^t = \begin{bmatrix} x_1y_1 \cdots x_1y_n \\ \vdots \\ x_my_1 \cdots x_my_n \end{bmatrix},$$

an $m \times n$ matrix with entries $A_{ij} = x_iy_j$. $(M_{m,n}, \otimes)$ is a tensor product space of $\mathbb{R}^m$ and $\mathbb{R}^n$. Tensors do not need to be matrices. This is just one model given. For more consult the following book [19].

Recall the following property of the tensorial product

$$(AC) \otimes (BD) = (A \otimes B)(C \otimes D)$$

that holds for any $A, B, C, D \in B(H)$.

From the property we can deduce easily the following consequences

$$A^n \otimes B^n = (A \otimes B)^n, n \geqslant 0,$$

$$(A \otimes 1)(1 \otimes B) = (1 \otimes B)(A \otimes 1) = A \otimes B,$$

which can be extended, for two natural numbers m, n we have

$$(A \otimes 1)^n(1 \otimes B)^m = (1 \otimes B)^n(A \otimes 1)^m = A^n \otimes B^m.$$

The current research concerning tensorial inequalities can be seen in the following papers, [9, 10, 11, 12, 13, 31]. The following Lemma which we require can be found in a paper of Dragomir [14].

Lemma 1.5. *Assume A and B are selfadjoint operators with $Sp(A) \subset I, Sp(B) \subset J$ and having the spectral resolutions . Let $f; h$ be continuous on I, g, k continuous on J and ϕ and ψ continuous on an interval K that contains the sum of the intervals $f(I) + g(J); h(I) + k(J)$,then*

$$\phi(f(A) \otimes 1 + 1 \otimes g(B))\psi(h(A) \otimes 1 + 1 \otimes k(B))$$
$$= \int_I \int_J \phi(f(t) + g(s))\psi(h(t) + k(s))dE_t \otimes dF_s.$$

Definition of a well known Riemann-Liouville (RL) fractional integral is given.

Definition 1.6. Let $f \in C([a, b])$. Then the left and right sided Riemann Liouville (RL) fractional integrals of order $\alpha > 0$ with $a \geq 0$ are defined as

$$I^\alpha_{a^+} f(z) = \frac{1}{\Gamma(\alpha)} \int_a^z (z - u)^{\alpha-1} f(u)du, z > a, \tag{1.1}$$

and

$$I_{b^-}^{\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_z^b (u-z)^{\alpha-1} f(u) du, z < b, \tag{1.2}$$

where $\Gamma(.)$ denotes the Gamma function.

Definition 1.7. A real valued function $f : I \to \mathbb{R}$ is called a convex function on interval I if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1.3}$$

holds for all $t \in [0, 1]$ and for all $x, y \in I$.

In the paper written by Set [27] , the author used the following Lemma. We will utilize it to produce results in the tensorial setting.

Lemma 1.8. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $f' \in L[a, b]$, then for all $x \in [a, b]$ and $\alpha > 0$ we have:*

$$\left(\frac{(x-a)^{\alpha} + (b-x)^{\alpha}}{b-a}\right) f(x) - \frac{\Gamma(1+\alpha)}{(b-a)} \left[J_{x^-}^{\alpha} f(a) + J_{x^+}^{\alpha} f(b)\right] \tag{1.4}$$

$$= \frac{(x-a)^{\alpha+1}}{b-a} \int_0^1 t^{\alpha} f'(tx + (1-t)a) dt - \frac{(b-x)^{\alpha+1}}{b-a} \int_0^1 t^{\alpha} f'(tx + (1-t)b) dt$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-t} u^{\alpha-1} du$.

Corollary 1.9. *Setting $\alpha = 1$ reduces to the Lemma given by Alomari et al. [4] which was utilized by Dragomir [15] to obtain tensorial inequalities in Hilbert space.*

Lemma 1.10. *We will prove the following relation for fractional integrals,*

$$J_{x^+}^{\alpha} f\,(b) = \frac{(1-\lambda)^{\alpha}(b-a)^{\alpha}}{\Gamma(\alpha)} \int_0^1 f(a(1-\lambda)(1-u) + b(u + (1-u)\lambda))(1-u)^{\alpha-1} du \tag{1.5}$$

$$J_{x^-}^{\alpha} f\,(a) = \frac{\lambda^{\alpha}(b-a)^{\alpha}}{\Gamma(\alpha)} \int_0^1 u^{\alpha-1} f(a((1-u) + u(1-\lambda)) + u\lambda b) du. \tag{1.6}$$

Proof. Use the integral definition of the fractional integrals and introduce the following substitution $\zeta = ub + (1-u)x$ and $\zeta = ux + (1-u)a$ on the first and second integral, then make a substitution $x = \lambda b + (1-\lambda)a$, the result follows. □

In this work various tensorial inequalities of the Ostrowski type for the twice differentiable functions in Hilbert space have been obtained . Since this field is still developing, finding new boundaries for different convex combinations of the functions is essential for the development of it.

2. Main results

In the following Theorem, we give a fundamental result which we will use in our paper to obtain inequalities.

Theorem 2.1. *Assume that f is continuously differentiable on I, A and B are selfadjoint operators with $Sp(A), Sp(B) \subset I$, then*

$$\Big(\lambda^\alpha + (1-\lambda)^\alpha\Big) f((1-\lambda)A\otimes 1 + \lambda 1\otimes B)$$
$$-\alpha\Big((1-\lambda)^\alpha \int_0^1 f((1-\lambda)(1-u)A\otimes 1 + (u+(1-u)\lambda)1\otimes B)(1-u)^{\alpha-1}du$$
$$+\lambda^\alpha \int_0^1 u^{\alpha-1} f(((1-u)+u(1-\lambda))A\otimes 1 + u\lambda 1\otimes B)du\Big)$$
$$= \lambda^{\alpha+1}(1\otimes B - A\otimes 1)\int_0^1 u^\alpha f'((1-u\lambda)A\otimes 1 + \lambda u 1\otimes B)du$$
$$-(1-\lambda)^{\alpha+1}(1\otimes B - A\otimes 1)\int_0^1 u^\alpha f'(u(1-\lambda)A\otimes 1 + (\lambda u + 1 - u)1\otimes B)du.$$

Proof. We start with Lemma 1.8. Rewriting the fractional integral using Lemma 1.10, to be precise with (1.5)(1.6) and simplifying we obtain

$$\Big(\lambda^\alpha + (1-\lambda)^\alpha\Big) f((1-\lambda)a + \lambda b)$$
$$-\alpha\Big((1-\lambda)^\alpha \int_0^1 f((1-\lambda)(1-u)a + (u+(1-u)\lambda)b)(1-u)^{\alpha-1}du$$
$$+\lambda^\alpha \int_0^1 u^{\alpha-1} f(((1-u)+u(1-\lambda))a + u\lambda b)du\Big)$$
$$= \lambda^{\alpha+1}(b-a)\int_0^1 u^\alpha f'((1-u\lambda)a + \lambda u b)du$$
$$-(1-\lambda)^{\alpha+1}(b-a)\int_0^1 u^\alpha f'(u(1-\lambda)a + (\lambda u + 1 - u)b)du.$$

Assuming that A and B have the spectral resolutions

$$A = \int_I t dE(t) \text{ and } B = \int_I s dE_t.$$

If we take the integral $\int_I\int_I dE_t \otimes dF_s$, then we get

$$\int_I\int_I \Big(\lambda^\alpha + (1-\lambda)^\alpha\Big) f((1-\lambda)t + \lambda s) dE_t\otimes dF_s$$
$$-\int_I\int_I \alpha\Big((1-\lambda)^\alpha \int_0^1 f((1-\lambda)(1-u)t + (u+(1-u)\lambda)s)(1-u)^{\alpha-1}du$$
$$+\lambda^\alpha \int_0^1 u^{\alpha-1} f(((1-u)+u(1-\lambda))t + u\lambda s)du\Big) dE_t\otimes dF_s$$
$$= \lambda^{\alpha+1}\int_I\int_I (s-t)\int_0^1 u^\alpha f'((1-u\lambda)t + \lambda u s) du dE_t\otimes dF_s$$
$$-(1-\lambda)^{\alpha+1}\int_I\int_I (s-t)\int_0^1 u^\alpha f'(u(1-\lambda)t + (\lambda u + 1 - u)s) du dE_t\otimes dF_s.$$

By utilizing Lemma 1 for appropriate choices of the functions involved, we have successively

$$\int_I\int_I\left(\lambda^\alpha+(1-\lambda)^\alpha\right)f((1-\lambda)t+\lambda s)dE_t\otimes dF_s$$

$$=\left(\lambda^\alpha+(1-\lambda)^\alpha\right)f((1-\lambda)A\otimes 1+\lambda 1\otimes B).$$

By utilizing Lemma 1 and Fubini's Theorem for appropriate choices of the functions involved, we have successively

$$\int_I\int_I\int_0^1 f((1-\lambda)(1-u)t+(u+(1-u)\lambda)s)(1-u)^{\alpha-1}dudE_t\otimes dF_s$$

$$=\int_0^1(1-u)^{\alpha-1}\int_I\int_I f((1-\lambda)(1-u)t+(u+(1-u)\lambda)s)dE_t\otimes dF_s du$$

$$=\int_0^1(1-u)^{\alpha-1}f((1-\lambda)(1-u)A\otimes 1+(u+(1-u)\lambda)1\otimes B)du,$$

$$\int_I\int_I(s-t)\int_0^1 u^\alpha f'((1-u\lambda)t+\lambda us)dudE_t\otimes dF_s$$

$$=\int_0^1 u^\alpha\int_I\int_I(s-t)f'((1-u\lambda)t+\lambda us)dE_t\otimes dF_s du$$

$$=\int_0^1 u^\alpha(1\otimes B-A\otimes 1)f'((1-u\lambda)A\otimes 1+u\lambda 1\otimes B)du.$$

Using the similar technique in the second term, we obtain the desired equality. □

We give our first inequality of the tensorial type utilizing the tensorial equality obtained in Theorem 2.

Theorem 2.2. *Assume that f is continuously differentiable on I with $\|f'\|_{I,+\infty}:=\sup_{t\in I}|f'(t)|<+\infty$ and A,B are selfadjoint operators with $Sp(A),Sp(B)\subset I,\alpha>0,$then*

$$\Bigg\|\left(\lambda^\alpha+(1-\lambda)^\alpha\right)f((1-\lambda)A\otimes 1+\lambda 1\otimes B)$$

$$-\alpha\Bigg((1-\lambda)^\alpha\int_0^1 f((1-\lambda)(1-u)A\otimes 1+(u+(1-u)\lambda)1\otimes B)(1-u)^{\alpha-1}du$$

$$+\lambda^\alpha\int_0^1 u^{\alpha-1}f(((1-u)+u(1-\lambda))A\otimes 1+u\lambda 1\otimes B)du\Bigg)\Bigg\|$$

$$\leqslant\|1\otimes B-A\otimes 1\|\left(\frac{\lambda^{\alpha+1}}{\alpha+1}+\frac{(1-\lambda)^{\alpha+1}}{\alpha+1}\right)\|f'\|_{I,+\infty}.$$

Proof. If we take the operator norm, we get

$$\left\|\left(\lambda^{\alpha}+(1-\lambda)^{\alpha}\right)f((1-\lambda)A\otimes 1+\lambda 1\otimes B)\right.$$
$$-\alpha\left((1-\lambda)^{\alpha}\int_0^1 f((1-\lambda)(1-u)A\otimes 1+(u+(1-u)\lambda)1\otimes B)(1-u)^{\alpha-1}du\right.$$
$$\left.\left.+\lambda^{\alpha}\int_0^1 u^{\alpha-1}f(((1-u)+u(1-\lambda))A\otimes 1+u\lambda 1\otimes B)du\right)\right\|$$
$$=\left\|\lambda^{\alpha+1}(1\otimes B-A\otimes 1)\int_0^1 u^{\alpha}f'((1-u\lambda)A\otimes 1+\lambda u 1\otimes B)du\right.$$
$$\left.-(1-\lambda)^{\alpha+1}(1\otimes B-A\otimes 1)\int_0^1 u^{\alpha}f'(u(1-\lambda)A\otimes 1+(\lambda u+1-u)1\otimes B)du\right\|.$$

Using the triangle inequality and the properties of the integral and the norm, we get

$$\left\|\left(\lambda^{\alpha}+(1-\lambda)^{\alpha}\right)f((1-\lambda)A\otimes 1+\lambda 1\otimes B)\right.$$
$$-\alpha\left((1-\lambda)^{\alpha}\int_0^1 f((1-\lambda)(1-u)A\otimes 1+(u+(1-u)\lambda)1\otimes B)(1-u)^{\alpha-1}du\right.$$
$$\left.\left.+\lambda^{\alpha}\int_0^1 u^{\alpha-1}f(((1-u)+u(1-\lambda))A\otimes 1+u\lambda 1\otimes B)du\right)\right\|$$
$$\leqslant \lambda^{\alpha+1}\left\|1\otimes B-A\otimes 1\right\|\left\|\int_0^1 u^{\alpha}f'((1-u\lambda)A\otimes 1+\lambda u 1\otimes B)du\right\|$$
$$+(1-\lambda)^{\alpha+1}\left\|1\otimes B-A\otimes 1\right\|\left\|\int_0^1 u^{\alpha}f'(u(1-\lambda)A\otimes 1+(\lambda u+1-u)1\otimes B)du\right\|$$
$$\leqslant \lambda^{\alpha+1}\left\|1\otimes B-A\otimes 1\right\|\int_0^1 u^{\alpha}\left\|f'((1-u\lambda)A\otimes 1+\lambda u 1\otimes B)\right\|du$$
$$+(1-\lambda)^{\alpha+1}\left\|1\otimes B-A\otimes 1\right\|\int_0^1 u^{\alpha}\left\|f'(u(1-\lambda)A\otimes 1+(\lambda u+1-u)1\otimes B)\right\|du.$$

Observe that by Lemma 1

$$\left|f'((1-u\lambda)A\otimes 1+\lambda u 1\otimes B)\right|=\int_I\int_I\left|f'((1-u\lambda)t+\lambda us)\right|dE_t\otimes dF_s$$

for $\lambda\in[0,1]$ and $t,s\in I$. Since

$$\left|f'((1-u\lambda)t+\lambda us)\right|\leqslant\left\|f'\right\|_{I,+\infty}$$

for $\lambda\in[0,1]$ and $t,s\in I$. If we take the integral $\int_I\int_I$ over $dE_t\otimes dF_s$, then we get

$$\left|f'((1-u\lambda)A\otimes 1+\lambda u 1\otimes B)\right|$$
$$=\int_I\int_I\left|f'((1-u\lambda)t+\lambda us)\right|dE_t\otimes dF_s\leqslant\left\|f'\right\|_{I,+\infty}\int_I\int_I dE_t\otimes dF_s$$

$$= \|f'\|_{I,+\infty}$$

for $\lambda \in [0,1]$ and $t, s \in I$. This implies that

$$\|f'((1-u\lambda)A \otimes 1 + \lambda u 1 \otimes B)\| \leqslant \|f'\|_{I,+\infty}$$

for $\lambda \in [0,1]$, similarly we have

$$\|f'(u(1-\lambda)A \otimes 1 + (\lambda u + 1 - u)1 \otimes B)\| \leqslant \|f'\|_{I,+\infty}.$$

Which combined gives us the following

$$\lambda^{\alpha+1} \|1 \otimes B - A \otimes 1\| \int_0^1 u^\alpha \|f'((1-u\lambda)A \otimes 1 + \lambda u 1 \otimes B)\| \, du$$

$$+(1-\lambda)^{\alpha+1} \|1 \otimes B - A \otimes 1\| \int_0^1 u^\alpha \|f'(u(1-\lambda)A \otimes 1 + (\lambda u + 1 - u)1 \otimes B)\| \, du$$

$$\leqslant \lambda^{\alpha+1} \|1 \otimes B - A \otimes 1\| \int_0^1 u^\alpha \|f'\|_{I,+\infty} \, du$$

$$+(1-\lambda)^{\alpha+1} \|1 \otimes B - A \otimes 1\| \int_0^1 u^\alpha \|f'\|_{I,+\infty} \, du.$$

Solving the resulting integrals and simplifying, we obtain the desired result. □

Corollary 2.3. *Setting $\alpha = 1$ we obtain the inequality given by Dragomir [15], namely*

$$\left\| f((1-\lambda)A \otimes 1 + \lambda 1 \otimes B) - \int_0^1 f((1-u)A \otimes 1 + u 1 \otimes B) du \right\|$$

$$\leqslant \|f'\|_{I,+\infty} \left[\frac{1}{4} + (\lambda - \frac{1}{2})^2 \right] \|1 \otimes B - A \otimes 1\|.$$

Proof. Note that the second term on the left hand side in Theorem 2.2, before introducing the substitutions in Lemma 1.10 and separating them is exactly the second term in the inequality on the left hand side. □

Theorem 2.4. *Assume that f is continuously differentiable on I with $|f'|$ is convex on I, A and B are selfadjoint operators with $Sp(A), Sp(B) \subset I, \alpha > 0$ then*

$$\Bigg\| \Big(\lambda^\alpha + (1-\lambda)^\alpha \Big) f((1-\lambda)A \otimes 1 + \lambda 1 \otimes B)$$

$$-\alpha \Bigg((1-\lambda)^\alpha \int_0^1 f((1-\lambda)(1-u)A \otimes 1 + (u + (1-u)\lambda)1 \otimes B)(1-u)^{\alpha-1} du$$

$$+\lambda^\alpha \int_0^1 u^{\alpha-1} f(((1-u) + u(1-\lambda))A \otimes 1 + u\lambda 1 \otimes B) du \Bigg) \Bigg\|$$

$$\leqslant \|1 \otimes B - A \otimes 1\| \left(\|f'(A)\| \lambda^{\alpha+1} \left(\frac{2\alpha+3}{p(\alpha)} - \frac{2\lambda}{\alpha+2} \right) \right.$$

$$\left. + \|f'(B)\| (1-\lambda)^{\alpha+1} \left(\frac{1}{p(\alpha)} + \frac{2\lambda}{\alpha+2} \right) \right),$$

for $\lambda \in [0,1]$, *where* $p(\alpha) = (\alpha+1)(\alpha+2)$.

Proof. Since $|f'|$ is convex on I, then we get

$$\left| f'((1-u\lambda)t + \lambda us) \right| \leqslant (1-u\lambda)|f'(t)| + \lambda u|f'(s)|,$$

for all $\lambda, u \in [0,1]$ and $t, s \in I$.
If we take the integral $\int_I \int_I$ over $dE_t \otimes dF_s$, then we get

$$\begin{aligned}\left| f'\left((1-u\lambda)A \otimes 1 + \lambda u 1 \otimes B\right) \right| &= \int_I \int_I \left| f'\left((1-u\lambda)t + \lambda us)\right) \right| dE_t \otimes dF_s \\ &\leqslant \int_I \int_I ((1-u\lambda)|f'(t)| + \lambda u|f'(s)|) dE_t \otimes dF_s \\ &= (1-u\lambda)|f'(A)| \otimes 1 + u\lambda 1 \otimes |f'(B)|\end{aligned}$$

for all $u, \lambda \in [0,1]$.

If we take the norm in the inequality, we get the following

$$\begin{aligned}\left\| f'\left((1-u\lambda)A \otimes 1 + \lambda u 1 \otimes B\right) \right\| &\leqslant \|(1-u\lambda)|f'(A)| \otimes 1 + u\lambda 1 \otimes |f'(B)|\| \\ &\leqslant (1-u\lambda)\left\| |f'(A)| \otimes 1 \right\| + u\lambda \left\| 1 \otimes |f'(B)| \right\| \\ &= (1-u\lambda)\left\| f'(A) \right\| + u\lambda \left\| f'(B) \right\|.\end{aligned}$$

Similarly, we get

$$\left\| f'\left(u(1-\lambda)A \otimes 1 + (\lambda u + 1 - u) 1 \otimes B\right) \right\| \leqslant u(1-\lambda)\left\| f'(A) \right\| + (\lambda u + 1 - u)\left\| f'(B) \right\|.$$

Which when applied to the inequality obtained in the previous Theorem, we obtain the following

$$\begin{aligned}&\int_0^1 u^\alpha \left\| f'((1-u\lambda)A \otimes 1 + \lambda u 1 \otimes B) \right\| du \\ &\leqslant \int_0^1 u^\alpha ((1-u\lambda)\left\| f'(A) \right\| \otimes 1 + \lambda u \left\| f'(B) \right\|) du, \\ &\int_0^1 u^\alpha \left\| f'(u(1-\lambda)A \otimes 1 + (\lambda u + 1 - u) 1 \otimes B) \right\| du \\ &\leqslant \int_0^1 u^\alpha (u(1-\lambda)\left\| f'(A) \right\| + (\lambda u + 1 - u)\left\| f'(B) \right\|) du.\end{aligned}$$

Which when simplified after integrating the terms, we obtain the original inequality.

□

Corollary 2.5. *Setting* $\alpha = 1$ *and* $\lambda = \frac{1}{2}$ *in the previously obtained Theorem, we obtain the inequality given by Dragomir* [15]

$$\begin{aligned}&\left\| f((1-\lambda)A \otimes 1 + \lambda 1 \otimes B) - \int_0^1 f((1-u)A \otimes 1 + u 1 \otimes B) du \right\| \\ &\leqslant \frac{1}{8} \left\| 1 \otimes B - A \otimes 1 \right\| \left(\left\| f'(A) \right\| + \left\| f'(B) \right\| \right).\end{aligned}$$

We recall that the function $f : I \to \mathbb{R}$ is quasi-convex, if

$$f((1-\lambda)t+\lambda s) \leqslant \max\{f(t), f(s)\} = \frac{1}{2}(f(t)+f(s)+|f(t)-f(s)|)$$

for all $t, s \in I$ and $\lambda \in [0,1]$.

Theorem 2.6. *Assume that f is continuously differentiable on I with $|f'|$ is quasi-convex on I, A and B are selfadjoint operators with $Sp(A), Sp(B) \subset I, \alpha > 0$ then*

$$\Bigg\|\bigg(\lambda^\alpha+(1-\lambda)^\alpha\bigg)f((1-\lambda)A\otimes 1+\lambda 1\otimes B)$$
$$-\alpha\bigg((1-\lambda)^\alpha\int_0^1 f((1-\lambda)(1-u)A\otimes 1+(u+(1-u)\lambda)1\otimes B)(1-u)^{\alpha-1}du$$
$$+\lambda^\alpha\int_0^1 u^{\alpha-1}f(((1-u)+u(1-\lambda))A\otimes 1+u\lambda 1\otimes B)du\bigg)\Bigg\|$$
$$\leqslant \|1\otimes B-A\otimes 1\|\left(\frac{\lambda^{\alpha+1}}{2(\alpha+1)}\right.$$
$$(\||f'(A)|\otimes 1+1\otimes|f'(B)|\|+\||f'(A)|\otimes 1-1\otimes|f'(B)|\|)$$
$$\left.+\frac{(1-\lambda)^{\alpha+1}}{2(\alpha+1)}(\||f'(A)|\otimes 1+1\otimes|f'(B)|\|+\||f'(A)|\otimes 1-1\otimes|f'(B)|\|\right).$$

Proof. Since $|f'|$ is quasi-convex on I, then we get

$$\left|f'\left((1-u\lambda)t+\lambda us\right)\right| \leqslant \frac{1}{2}(|f'(t)|+|f'(s)|+||f'(t)-|f'(s)||)$$

for all $\lambda \in [0,1]$ and $t, s \in I$.
If we take the integral $\int_I\int_I$ over $dE_t \otimes dF_s$, then we get

$$\left|f'\left((1-u\lambda)A\otimes 1+\lambda u 1\otimes B\right)\right|$$
$$=\int_I\int_I |f'\left((1-u\lambda)t+\lambda us\right)|dE_t\otimes dF_s$$
$$\leqslant \frac{1}{2}\int_I\int_I(|f'(t)|+|f'(s)|+||f'(t)-|f'(s)||)dE_t\otimes dF_s$$
$$=\frac{1}{2}\left(|f'(A)|\otimes 1+1\otimes|f'(B)|+||f'(A)|\otimes 1-1\otimes|f'(B)||\right)$$

for all $\lambda \in [0,1]$.
If we take the norm, then we get

$$\|f'\left((1-u\lambda)A\otimes 1+\lambda u 1\otimes B\right)\|$$
$$\leqslant \frac{1}{2}\|(|f'(A)|\otimes 1+1\otimes|f'(B)|+||f'(A)|\otimes 1-1\otimes|f'(B)||)\|$$
$$\leqslant \frac{1}{2}(\||f'(A)|\otimes 1+1\otimes|f'(B)|\|+\||f'(A)|\otimes 1-1\otimes|f'(B)|\|)$$

for all $\lambda \in [0,1]$. In a similar way, we obtain

$$\begin{aligned}
&\|f'\left(u(1-\lambda)A\otimes 1+(\lambda u+1-u)1\otimes B\right)\| \\
&\leqslant \frac{1}{2}\left\|\left(|f'(A)|\otimes 1+1\otimes|f'(B)|+||f'(A)|\otimes 1-1\otimes|f'(B)||\right)\right\| \\
&\leqslant \frac{1}{2}(\||f'(A)|\otimes 1+1\otimes|f'(B)|\|+\||f'(A)|\otimes 1-1\otimes|f'(B)|\|).
\end{aligned}$$

Using these inequalities in the inequality obtained during Theorem 5, we obtain the following

$$\begin{aligned}
&\int_0^1 u^\alpha\,\|f'((1-u\lambda)A\otimes 1+\lambda u 1\otimes B)\|\,du \\
&\leqslant \int_0^1 u^\alpha \frac{1}{2}(\||f'(A)|\otimes 1+1\otimes|f'(B)|\|+\||f'(A)|\otimes 1-1\otimes|f'(B)|\|)du, \\
&\int_0^1 u^\alpha\,\|f'(u(1-\lambda)A\otimes 1+(\lambda u+1-u)1\otimes B)\|\,du \leqslant \\
&\int_0^1 u^\alpha \frac{1}{2}(\||f'(A)|\otimes 1+1\otimes|f'(B)|\|+\||f'(A)|\otimes 1-1\otimes|f'(B)|\|)du.
\end{aligned}$$

Which when simplified, we obtain the desired inequality.

□

3. Some comments

Corollary 3.1. *If A, B are selfadjoint operators with $Sp(A), Sp(B) \subset [m, M]$ and $1\otimes B - A\otimes 1$ is invertible, then by Theorem 2.2, we get*

$$\begin{aligned}
&\Bigg\|\left(\lambda^\alpha+(1-\lambda)^\alpha\right)\exp((1-\lambda)A\otimes 1+\lambda 1\otimes B) \\
&-\alpha\Bigg((1-\lambda)^\alpha\int_0^1 \exp((1-\lambda)(1-u)A\otimes 1+(u+(1-u)\lambda)1\otimes B)(1-u)^{\alpha-1}du \\
&+\lambda^\alpha\int_0^1 u^{\alpha-1}\exp(((1-u)+u(1-\lambda))A\otimes 1+u\lambda 1\otimes B)du\Bigg)\Bigg\| \\
&\leqslant \|1\otimes B-A\otimes 1\|\left(\frac{\lambda^{\alpha+1}}{\alpha+1}+\frac{(1-\lambda)^{\alpha+1}}{\alpha+1}\right)\exp(M).
\end{aligned}$$

Corollary 3.2. *Since for $f'(t)=\exp(t)$; $t\in\mathbb{R}$, is convex, then by Theorem 2.4 we get*

$$\begin{aligned}
&\Bigg\|\left(\lambda^\alpha+(1-\lambda)^\alpha\right)\exp((1-\lambda)A\otimes 1+\lambda 1\otimes B) \\
&-\alpha\Bigg((1-\lambda)^\alpha\int_0^1 \exp((1-\lambda)(1-u)A\otimes 1+(u+(1-u)\lambda)1\otimes B)(1-u)^{\alpha-1}du \\
&+\lambda^\alpha\int_0^1 u^{\alpha-1}\exp(((1-u)+u(1-\lambda))A\otimes 1+u\lambda 1\otimes B)du\Bigg)\Bigg\| \\
&\leqslant \|1\otimes B-A\otimes 1\|\Bigg(\|\exp(A)\|\,\lambda^{\alpha+1}\left(\frac{2\alpha+3}{p(\alpha)}-\frac{2\lambda}{\alpha+2}\right)
\end{aligned}$$

$$+\left\|\exp(B)\right\|(1-\lambda)^{\alpha+1}\left(\frac{1}{p(\alpha)}+\frac{2\lambda}{\alpha+2}\right)\bigg),$$

If $\alpha, \lambda = \frac{1}{2}$ *, then we get*

$$\Bigg\|\sqrt{2}\exp\left(\frac{A\otimes 1+1\otimes B}{2}\right)$$

$$-(\frac{1}{2})\Bigg(\frac{1}{\sqrt{2}}\int_0^1 \exp((\frac{1}{2})(1-u)A\otimes 1+(u+(1-u)(\frac{1}{2}))1\otimes B)(1-u)^{-\frac{1}{2}}du$$

$$+\frac{1}{\sqrt{2}}\int_0^1 u^{-\frac{1}{2}}\exp(((1-u)+u(\frac{1}{2}))A\otimes 1+u(\frac{1}{2})1\otimes B)du\Bigg)\Bigg\|$$

$$\leqslant \frac{1}{3\times\sqrt{2}}\left\|1\otimes B-A\otimes 1\right\|\bigg(\left\|\exp(A)\right\|+\left\|\exp(B)\right\|\bigg).$$

Corollary 3.3. *Choosing* $f(t)=\exp(t); t\in\mathbb{R}$*, and since* $|f'|$ *is convex, we get by Theorem 2.6*

$$\Bigg\|\bigg(\lambda^\alpha+(1-\lambda)^\alpha\bigg)\exp((1-\lambda)A\otimes 1+\lambda 1\otimes B)$$

$$-\alpha\Bigg((1-\lambda)^\alpha$$

$$\times\int_0^1 \exp((1-\lambda)(1-u)A\otimes 1+(u+(1-u)\lambda)1\otimes B)(1-u)^{\alpha-1}du$$

$$+\lambda^\alpha\int_0^1 u^{\alpha-1}\exp(((1-u)+u(1-\lambda))A\otimes 1+u\lambda 1\otimes B)du\Bigg)\Bigg\|$$

$$\leqslant \left\|1\otimes B-A\otimes 1\right\|\Bigg(\frac{\lambda^{\alpha+1}}{2(\alpha+1)}$$

$$(\||\exp(A)|\otimes 1+1\otimes|\exp(B)|\|+\||\exp(A)|\otimes 1-1\otimes|\exp(B)|\|)$$

$$+\frac{(1-\lambda)^{\alpha+1}}{2(\alpha+1)}(\||\exp(A)|\otimes 1+1\otimes|\exp(B)|\|+\||\exp(A)|\otimes 1-1\otimes|\exp(B)|\|\Bigg).$$

Setting $\alpha=\frac{1}{2}$ *we obtain the following*

$$\Bigg\|\bigg(\lambda^{\frac{1}{2}}+(1-\lambda)^{\frac{1}{2}}\bigg)\exp((1-\lambda)A\otimes 1+\lambda 1\otimes B)$$

$$-\frac{1}{2}\Bigg((1-\lambda)^{\frac{1}{2}}$$

$$\times\int_0^1 \exp((1-\lambda)(1-u)A\otimes 1+(u+(1-u)\lambda)1\otimes B)(1-u)^{-\frac{1}{2}}du$$

$$+\lambda^{\frac{1}{2}}\int_0^1 u^{-\frac{1}{2}}\exp(((1-u)+u(1-\lambda))A\otimes 1+u\lambda 1\otimes B)du\Bigg)\Bigg\|$$

$$\leqslant \left\|1\otimes B-A\otimes 1\right\|^{\frac{1}{2}}\Bigg(\frac{\lambda^{\frac{3}{2}}}{3}$$

$$(\||\exp(A)|\otimes 1+1\otimes|\exp(B)|\|+\||\exp(A)|\otimes 1-1\otimes|\exp(B)|\|)$$
$$+\frac{(1-\lambda)^{\frac{3}{2}}}{3}(\||\exp(A)|\otimes 1+1\otimes|\exp(B)|\|+\||\exp(A)|\otimes 1-1\otimes|\exp(B)|\|\Bigg).$$

4. Conclusion

Because they offer a clear mathematical foundation for posing and resolving physical problems in areas like mechanics, electromagnetism, quantum mechanics, and many others, tensors have gained significance in a variety of domains, including physics. As such inequalities are crucial in numerical aspects. Reflected in this work is the tensorial version of a Lemma given by Set [27], which as a consequence enabled us to generalize the results from Dragomir [15]. New Ostrowski type inequalities are given, as well as consequences showing our generalization. Examples of specific convex functions and their inequalities using our results are given in the section some examples and consequences. Plans for additional research can be seen in the fact that various techniques can be used to refine or generalize the inequalities found in this work. Combining the methods presented in this study with other methods for Hilbert space inequalities offers an intriguing viewpoint. One approach is the Mond-Pecaric inequality method, which is what we shall focus on.

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