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TOTALLY (m, n)-PARANORMAL OPERATORS

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ABSTRACT. In this paper, we investigate the class of totally (m, n)-paranormal operators on Hilbert space and prove some additional properties of totally (m, n)-paranormal operators on Hilbert space. We also demonstrate the applicability of Weyl's theorem to operators in this class. Additionally, we demonstrate that the spectral mapping theorem is satisfied by the Weyl spectrum for totally (m, n)-paranormal operators.

1. Introduction and preliminaries

Throughout the article, $\mathcal{H}$ will denote an infinite dimensional separable complex Hilbert space with the inner product $\langle , \rangle$, unless specified otherwise and $B(\mathcal{H})$ denote the C^*-algebra of bounded linear operators on the Hilbert space $\mathcal{H}$. In this paper the set of all complex numbers denoted by $\mathbb{C}$. We define $ker(T)$ and $ran(T)$ as the null space and range space of T, respectively, for T in $B(\mathcal{H})$. Additionally, $\alpha(T)$ and $\beta(T)$ will be used to represent the dimensions of $ker(T)$ and $ker(T^*)$, respectively.

[2, 9, 17] For a bounded linear operator $T \in B(\mathcal{H})$, the spectrum of T and the point spectrum of T are defined by $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not invertible}\}$ and $\sigma_p(T) = \{\lambda \in \mathbb{C} : Tx = \lambda x \text{ for some non-zero vector } x\}$, respectively. Equivalently, the Weyl spectrum of T and the set of all isolated points of spectrum of T, which are eigen values of finite multiplicity are defined by $\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$ and $\pi_{00}(T) = \{\lambda \in iso\sigma(T) : 0 < \alpha(T - \lambda) < \infty\}$, respectively, where $iso\sigma(T)$ is the set of all isolated points of spectrum of T.

An operator $T \in B(\mathcal{H})$ is called Fredholm if the following hold.

- $ker(T)$ is finite dimensional.
- $ran(T)$ is closed.
- $ker(T^*)$ is finite dimensional.

$\alpha(T) - \beta(T)$ gives the index of the Fredholm operator T and If operator T in $B(\mathcal{H})$ is Fredholm of index zero, it is said to be Weyl.

T satisfies the Weyl's theorem if $\sigma(T) \backslash \sigma_w(T) = \pi_{00}(T)$. An operator T is said to be positive if $\langle Tx, x \rangle \geq 0 \; \forall \; x$ in $\mathcal{H}$ and also strictly positive if $\langle Tx, x \rangle > 0 \; \forall$

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x in $\mathcal{H}$. If an operator T is strictly positive, that is, $T > 0$, then T is invertible. The formula for the spectral radius of T is

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

If $r(T) = \|T\|$, then we say that T is normaloid. If $\|Tx\|^{n+1} \leq m\|T^{n+1}x\|\|x\|^n$ for all $x \in \mathcal{H}$, where m is a positive real number and n is a positive integer, then $T \in B(\mathcal{H})$ is (m,n)-paranormal [3].

Definition 1.1. An operator T is said to be a totally (m,n)-paranormal operator if $\|(T-\lambda)x\|^{n+1} \leq m\|(T-\lambda)^{n+1}x\|\|x\|^n$, for all $x \in \mathcal{H}$ and all $\lambda \in \mathbb{C}$.

Let the ascent and descent of an operator T be denoted, respectively, by $p(T)$ and $q(T)$. If there is at least one non-zero integer m_1 such that $ker(T^{m_1}) = ker(T^{m_1+1})$, then T has finite ascent and $p(T) = m_1$. Additionally, if such an integer doesn't exist, then $p(T) = \infty$.

Similar to this, if there is at least one non-negative integer m_2 such that $ran(T^{m_2}) = ran(T^{m_2+1})$, we claim that T has finite descent and that $q(T) = m_2$. Also, $q(T) = \infty$ if such an integer does not exist. When $p(T)$ and $q(T)$ are both finite, $p(T) = q(T)$ [11, Proposition 38.3]. Additionally, given a complex number λ, $0 < p(T-\lambda) = q(T-\lambda) < \infty$ if and only if λ is a pole of the resolvent.

Definition 1.2. If an analytic function $f : G \to \mathcal{H}$ is the only one that satisfies the equation $(T-\gamma I)f(\gamma) = 0$ for all $\gamma \in G$, then an operator T is said to have single valued extension property (abbreviated as SVEP) at $\gamma_0 \in \mathbb{C}$, where G is open neighborhood of γ_0. If an operator T has SVEP at each $\gamma \in \mathbb{C}$, then T has SVEP.

Let $T \in B(\mathcal{H})$ and λ_0 be an isolated point of $\sigma(T)$. Then there exists a small enough positive real number $r > 0$ such that $\{\lambda \in \mathbb{C} : |\lambda - \lambda_0| \leq r\} \cap \sigma(T) = \{\lambda_0\}$. Let

$$E = \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=r}(\lambda - T)^{-1}d\lambda.$$

E is called the Riesz idempotent with respect to λ_0. Then it is well known that $E^2 = E$, $ET = TE$, $\sigma(T|_{ran(E)}) = \{\lambda_0\}$ and $ker(T-\lambda_0) \subseteq ran(E)$. Riesz idempotent E is not orthogonal projection. The necessary and sufficient condition for E to be orthogonal is that E is self-adjoint [11]. Moreover, we refer more articles to understand the background this article[17, 18, 4, 5, 14].

2. Weyl's theorem for totally (m,n)-paranormal operators

In this section, we study some properties of totally (m,n)-paranormal operators. We begin with the following theorem which is a characterization of the class of totally (m,n)-paranormal operators [1, 3].

Theorem 2.1. *Let $T \in B(\mathcal{H})$. Then T is totally (m,n)-paranormal for each positive integer n and positive real number m if and only if*

$$m^{\frac{2}{n+1}}(T-\lambda)^{*n+1}(T-\lambda)^{n+1} - (n+1)a^n(T-\lambda)^*(T-\lambda) + m^{\frac{2}{n+1}}na^{n+1}I \geq 0, \quad (2.1)$$

for each $a > 0$ and all $\lambda \in \mathbb{C}$.

Proof. For each $x \in \mathcal{H}$, from the definition of totally (m,n)-paranormal operator, then operator T satisfies

$$\|(T-\lambda)x\|^{n+1} \le m\|(T-\lambda)^{n+1}x\|\|x\|^n,$$

this gives

$$\|(T-\lambda)x\|^2 \le m^{\frac{2}{n+1}}\|(T-\lambda)^{n+1}x\|^{\frac{2}{n+1}}\|x\|^{\frac{2n}{n+1}}. \tag{2.2}$$

Equation (2.2) yields the extended arithmetic-geometric mean inequality, which gives

$$\begin{aligned}\frac{m^{\frac{2}{n+1}}a^{-n}}{n+1}\|(T-\lambda)^{n+1}x\|^2 + \frac{nm^{\frac{2}{n+1}}a}{n+1}\|x\|^2 \\ &\ge m^{\frac{2}{n+1}}(a^{-n}\|(T-\lambda)^{n+1}x\|^2)^{\frac{1}{n+1}}(a\|x\|^2)^{\frac{n}{n+1}} \\ &= m^{\frac{2}{n+1}}\|(T-\lambda)^{n+1}x\|^{\frac{2}{n+1}}\|x\|^{\frac{2n}{n+1}} \\ &\ge \|(T-\lambda)x\|^2.\end{aligned}$$

It follows that

$$m^{\frac{2}{n+1}}(T-\lambda)^{*n+1}(T-\lambda)^{n+1} - (n+1)a^n(T-\lambda)^*(T-\lambda) + m^{\frac{2}{n+1}}na^{n+1}I \ge 0,$$

for each $a > 0$.

Conversely, suppose that (2.1) holds. If $\|(T-\lambda)^{n+1}x\| = 0$ for some $x \in \mathcal{H}$, then on multiplying by a^{-n}, we obtain

$$-(n+1)(T-\lambda)^*(T-\lambda) + m^{\frac{2}{n+1}}naI \ge 0.$$

Now taking the limit $a \to 0$ in the above equation, we have $\|(T-\lambda)x\| = 0$. Therefore,

$$\|(T-\lambda)x\|^{n+1} \le m\|(T-\lambda)^{n+1}x\|\|x\|^n.$$

Also, if for some $x \in \mathcal{H}$, $\|(T-\lambda)^{n+1}x\| > 0$, then for $a = \left(\frac{\|(T-\lambda)^{n+1}x\|}{\|x\|}\right)^{\frac{2}{n+1}}$ in (2.1), we obtain

$$\begin{aligned}\frac{m^{\frac{2}{n+1}}}{n+1}\|(T-\lambda)^{n+1}x\|^{\frac{2}{n+1}}\|x\|^{\frac{2n}{n+1}} + m^{\frac{2}{n+1}}\frac{n}{n+1}\|(T-\lambda)^{n+1}x\|^{\frac{2}{n+1}}\|x\|^{\frac{2n}{n+1}} \\ \ge \|(T-\lambda)x\|^2,\end{aligned}$$

which implies

$$\|(T-\lambda)x\|^2 \le m^{\frac{2}{n+1}}\|(T-\lambda)^{n+1}x\|^{\frac{2}{n+1}}\|x\|^{\frac{2n}{n+1}},$$

and hence

$$\|(T-\lambda)x\|^{n+1} \le m\|(T-\lambda)^{n+1}x\|\|x\|^n.$$

Hence, T is totally (m,n)-paranormal. □

The following theorem shows that if T is totally (m,n)-paranormal on $\mathcal{H}$, then T is also totally (m,n)-paranormal on every invariant subspace W of $\mathcal{H}$.

Proposition 2.2. *Let $T \in B(\mathcal{H})$ be a totally (m,n)-paranormal operator on $\mathcal{H}$. Then T is also totally (m,n)-paranormal on an invariant subspace $\mathcal{W}$ of $\mathcal{H}$.*

Proof. Since T is totally (m,n)-paranormal on $\mathcal{H}$, so

$$\|(T-\lambda)x\|^{n+1} \leq m\|(T-\lambda)^{n+1}x\|\|x\|^n.$$

The invariance of $\mathcal{W}$ under λI implies that for $y \in \mathcal{W}$, we have

$$\begin{aligned}\|(T|_{\mathcal{W}}-\lambda)y\|^{n+1} &= \|(T-\lambda)|_{\mathcal{W}}y\|^{n+1}\\ &= \|(T-\lambda)y\|^{n+1}\\ &\leq m\|(T-\lambda)^{n+1}y\|\|y\|^n\\ &= m\|(T-\lambda)^{n+1}|_{\mathcal{W}}y\|\|y\|^n.\end{aligned}$$

Thus, T is totally (m,n)-paranormal on $\mathcal{W}$. □

Proposition 2.3. *If T is totally (m,n)-paranormal, then $T-\alpha$ and αT are also totally (m,n)-paranormal.*

Proof. Let T be totally (m,n)-paranormal. Then

$$\|(T-\lambda)x\|^{n+1} \leq m\|(T-\lambda)^{n+1}x\|\|x\|^n \tag{2.3}$$

for all $\lambda \in \mathbb{C}$ and $x \in \mathcal{H}$. Now, we shall show that $T-\alpha$ is totally (m,n)-paranormal. For $x \in \mathcal{H}$ and all $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}\|[(T-\alpha)-\lambda]x\|^{n+1} &= \|(T-(\alpha+\lambda))x\|^{n+1}\\ &\leq m\|(T-(\alpha+\lambda))^{n+1}x\|\|x\|^n.\end{aligned}$$

This implies that $T-\alpha$ is totally (m,n)-paranormal.

Now, we shall shows that αT is totally (m,n)-paranormal. First, we assume that $\alpha=0$, then $\alpha T=0$. By using (2.3), αT is also totally (m,n)-paranormal for $m \leq 1$.

In the second case, we assume that $\alpha \neq 0$, then for all $x \in \mathcal{H}$, we have

$$\begin{aligned}\|(\alpha T-\lambda)x\|^{n+1} &= |\alpha|^{n+1}\|(T-\frac{\lambda}{\alpha})x\|^{n+1}\\ &\leq m|\alpha|^{n+1}\|(T-\frac{\lambda}{\alpha})^{n+1}x\|\|x\|^n\\ &= m\|(\alpha T-\lambda)^{n+1}x\|\|x\|^n\end{aligned}$$

that is, $\|(\alpha T-\lambda)x\|^{n+1} \leq m\|(\alpha T-\lambda)^{n+1}x\|\|x\|^n$. Therefore, αT is totally (m,n)-paranormal. □

If we put $\lambda=0$ in totally (m,n)-paranormal, then it reduces to a (m,n)-paranormal. This implies that totally (m,n)-paranormal is a proper subclass of (m,n)-paranormal operators, but converse is not true. The following example shows the same.

Example 2.4. Let T be an operator defined on $\mathcal{H}=\mathbb{R}\oplus\mathbb{R}$ as matrix representation $T=\begin{bmatrix}2 & 2\\ 0 & 2\end{bmatrix}$. By [3, Theorem 2.1] T is $(2^{\frac{3}{2}},2)$-paranormal if and only if

$$2T^{*3}T^3 - 3a^2T^*T + 4a^3I \geq 0,$$

for each $a > 0$. Consider the operator equation

$$2T^{*3}T^3 - 3a^2T^*T + 4a^3I = \begin{bmatrix} 128 - 12a^2 + 4a^3 & 384 - 12a^2 \\ 384 - 12a^2 & 1280 - 24a^2 + 4a^3 \end{bmatrix}.$$

The above operator is positive for each $a > 0$. Now, we shall show that T is not totally (m,n)-paranormal. We know that T is totally $(2^{\frac{3}{2}}, 2)$-paranormal if and only if

$$2(T-\lambda)^{*3}(T-\lambda)^3 - 3a^2(T-\lambda)^*(T-\lambda) + 4a^3I \geq 0,$$

for each $a > 0$. Consider the operator equation for $\lambda = 1$

$$2(T-\lambda)^{*3}(T-\lambda)^3 - 3a^2(T-\lambda)^*(T-\lambda) + 4a^3I$$
$$= \begin{bmatrix} 2 - 3a^2 + 4a^3 & 12 - 6a^2 \\ 12 - 6a^2 & 74 - 15a^2 + 4a^3 \end{bmatrix}.$$

The above operator is not positive for $a = 3$. Therefore, T is not totally $(2^{\frac{3}{2}}, 2)$-paranormal.

The example below demonstrates how T is totally (m,n)-paranormal operator and (m,n)-paranormal operator as well.

Example 2.5. Let $T = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ be an operator on $\mathcal{H} = \mathbb{R} \oplus \mathbb{R}$. By [3, Theorem 2.1] T is $(2^{\frac{3}{2}}, 2)$-paranormal if and only if

$$2T^{*3}T^3 - 3a^2T^*T + 4a^3I \geq 0,$$

for each $a > 0$. Now, Consider the operator equation

$$2T^{*3}T^3 - 3a^2T^*T + 4a^3I = \begin{bmatrix} 128 - 12a^2 + 4a^3 & 0 \\ 0 & 128 - 12a^2 + 4a^3 \end{bmatrix},$$

which is positive for each $a > 0$. It follows that T is $(2^{\frac{3}{2}}, 2)$-paranormal. Now, we shall show that T is also totally (m,n)-paranormal. Choose $m = 2^{\frac{3}{2}}$, $n = 2$. The operator T is totally $(2^{\frac{3}{2}}, 2)$-paranormal if and only if

$$2(T-\lambda)^{*3}(T-\lambda)^3 - 3a^2(T-\lambda)^*(T-\lambda) + 4a^3I \geq 0,$$

for each $a > 0$. Consider

$$2(T-\lambda)^{*3}(T-\lambda)^3 - 3a^2(T-\lambda)^*(T-\lambda) + 4a^3I$$
$$= \begin{bmatrix} 2|2+\lambda|^6 - 3a^2|2+\lambda|^2 + 4a^3 & 0 \\ 0 & 2|2+\lambda|^6 - 3a^2|2+\lambda|^2 + 4a^3 \end{bmatrix}.$$

The above operator is positive for each $a > 0$ and for all complex number λ. Therefore, T is also totally $(2^{\frac{3}{2}}, 2)$-paranormal.

The proof of the following theorem is straightforward.

Theorem 2.6. *Let $T \in B(\mathcal{H})$ with matrix representation as $T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}$ on $\mathcal{H} \oplus \mathcal{H}$ be totally (m,n)-paranormal. Then T_1 and T_2 are also totally (m,n)-paranormal.*

Theorem 2.7. *For a totally (m,n)-paranormal operator T on $\mathcal{H}$, T has finite ascent.*

Proof. Since T is totally (m,n)-paranormal

$$\|(T-\lambda)x\|^{n+1} \leq m\|(T-\lambda)^{n+1}x\|\|x\|^n, \tag{2.4}$$

for all $x \in \mathcal{H}$ and all $\lambda \in \mathbb{C}$. It is obvious that $ker(T-\lambda) \subseteq ker(T-\lambda)^{n+1}$. Suppose that $x \in ker(T-\lambda)^{n+1}$, then (2.4) implies $(T-\lambda)x = 0$. Thus $ker(T-\lambda) = ker(T-\lambda)^{n+1}$.

Now, we shall shows that $ker(T-\lambda) = ker(T-\lambda)^2$ for all complex number λ. It is clear that $ker(T-\lambda) \subset ker(T-\lambda)^2$. Let $x \in ker(T-\lambda)^2$, then $(T-\lambda)^2x = 0$, so by using equation (2.4) and n is a positive integer, therefore we get that $(T-\lambda)x = 0$. Thus, $ker(T-\lambda) = ker(T-\lambda)^2$ for all complex number λ. Equivalently, we can prove that $ker(T-\lambda) = ker(T-\lambda)^i$ for $1 \leq i \leq n+1$. This implies that $T-\lambda$ has finite ascent for all complex number λ implying T has finite ascent. □

Corollary 2.8. *If T is a totally (m,n)-paranormal operator, then T has SVEP.*

Proof. By [15, Proposition 1.8], T has SVEP. □

Remark 2.9. If T is a totally (m,n)-paranormal operator for positive integer $n \geq 1$, then $ker(T) = ker(T)^i$ for $1 \leq i \leq n+1$.

Proposition 2.10. *If T is totally (m,n)-paranormal, where $m \leq 1$. Then $r(T) = \|T\|$, that is, T is normaloid.*

Proof. Since T is totally (m,n)-paranormal, hence for $x \neq 0$ we have

$$\|(T-\lambda)x\|^{n+1} \leq m\|(T-\lambda)^{n+1}x\|\|x\|^n.$$

It follows that

$$\|(T-\lambda)\left(\frac{x}{\|x\|}\right)\|^{n+1} \leq \|(T-\lambda)^{n+1}\left(\frac{x}{\|x\|}\right)\|.$$

This implies that

$$\|(T-\lambda)\|^{n+1} \leq \|(T-\lambda)^{n+1}\|. \tag{2.5}$$

It is easy to sees that

$$\|(T-\lambda)^{n+1}\| \leq \|(T-\lambda)\|^{n+1}. \tag{2.6}$$

From (2.5) and (2.6), we obtain that $\|(T-\lambda)^{n+1}\| = \|(T-\lambda)\|^{n+1}$ for all complex numbers λ. Thus, $T-\lambda$ is normaloid for all complex number λ [12, Proposition 1]. Therefore T is normaloid. □

Proposition 2.11. *Let T be a totally (m,n)-paranormal operator with $m \leq 1$ and $\sigma(T) = \{\lambda\}$. Then $T = \lambda I$.*

Proof. Since T is totally (m,n)-paranormal, so we have

$$\|(T-\lambda)x\|^{n+1} \leq m\|(T-\lambda)^{n+1}x\|\|x\|^n$$

for all $x \in \mathcal{H}$ and for all complex number λ. Let $\sigma(T) = \{\lambda\}$.

Case 1:- Let $\lambda = 0$. Then $\sigma(T) = \{0\}$. By Proposition 2.10, T is a normaloid operator. Therefore $r(T) = \|T\| = 0$, it follows that $\|T\| = 0$, implying $T = 0$. Hence, $T = \lambda I$.

Case 2:- Let $\lambda \neq 0$. Assume that $A_1 = \lambda^{-1}T$, this implies that $\sigma(A_1) = \{1\}$, that is, by [16, Theorem 1.5.14], A_1 is an identity operator. Therefore, $A_1 = I$, that is, $T = \lambda I$. This completes the required result. □

We give the following lemma which will be useful to prove Weyl's theorem for a totally (m, n)-paranormal operator.

Lemma 2.12. *For $m \leq 1$, let T be totally (m, n)-paranormal and λ_0 be an isolated point of $\sigma(T)$. Then the Riesz idempotent E with respect to λ_0 satisfies $ran(E) = ker(T - \lambda_0)$. Therefore λ_0 is an eigen value of T.*

Proof. By definition of Riesz idempotent E, it is clear that $ker(T-\lambda_0) \subseteq ran(E)$. Now, We have to proof that $ran(E) \subseteq ker(T-\lambda_0)$. By general rule of Riesz idempotent $\sigma(T|_{ran(E)}) = \{\lambda_0\}$ and Proposition 2.2, T is totally (m, n)-paranormal on $ran(E)$. Assume that $\lambda_0 = 0$, thus $\sigma(T|_{ran(E)}) = \{0\}$, by Proposition 2.11, we get $T|_{ran(E)} = 0$. This implies that $ker(T) = ran(E)$.

Now, let $\lambda \neq 0$. Let $A_1 = {\lambda_0}^{-1}T$, again by proposition 2.11 and Proposition 2.3, A_1 is totally (m, n)-paranormal operator for $m \leq 1$ on $ran(E)$. It follows that $\sigma(A_1|_{ran(E)}) = \{1\}$. Hence, we obtain $A_1 = I$ on $ran(E)$. Therefore, $ker(T-\lambda_0) = ran(E)$. This is the required result. □

Regarding local spectral theory [6, 7, 13], recall that if $T \in B(\mathcal{H})$ and F be a closed subset of the complex plane $\mathbb{C}$, we define a global spectral subspace $H_T(F)$ as follows:

$$H_T(F) = \{x \in \mathcal{H} : \text{there exists an analytic function } f : \mathbb{C}\backslash F \to \mathcal{H} \text{ such that } (T - \lambda)f(\lambda) = x\}$$

Theorem 2.13. *If T is a totally (m, n)-paranormal operator with $m \leq 1$, Weyl's theorem holds for T, that is, $\sigma(T)\backslash\sigma_w(T) = \pi_{00}(T)$.*

Proof. Let $\lambda \in \sigma(T)\backslash\sigma_w(T)$. This implies that $\lambda \in \sigma(T)$ and $\lambda \notin \sigma_w(T)$. It follows that $T-\lambda$ is a Fredholm operator of index zero and $T-\lambda$ is not invertible. Also, $ker(T - \lambda)$ is non-zero finite dimensional subspace. Now, we represent T by the matrix $T = \begin{bmatrix} \lambda & 0 \\ 0 & T_1 \end{bmatrix}$ on $\mathcal{H} = ker(T - \lambda) \oplus ker(T - \lambda)^{\perp}$. Consider $T - \lambda = \begin{bmatrix} 0 & 0 \\ 0 & T_1 - \lambda \end{bmatrix}$. $ker(T - \lambda)$ is non-zero and finite dimensional subspace, so it is clear that operator *zero* on $ker(T - \lambda)$ is a Fredholm operator of index zero. Thus, by [8, Corollary 5], $ind(T - \lambda) = ind(0) + ind(T_1 - \lambda) = ind(T_1 - \lambda)$. That is, $ind(T - \lambda) = ind(T_1 - \lambda) = 0$. Now, we shall show that $ker(T_1 - \lambda) = 0$. It is clear that by decomposition of $\mathcal{H}$, $ker(T_1 - \lambda) \subseteq ker(T - \lambda)^{\perp}$. Moreover, it is obvious that $ker(T_1 - \lambda) \subseteq ker(T - \lambda)$, therefore, we obtain $ker(T_1 - \lambda) = 0$ and $ker(T_1 - \lambda)^* = 0$. Thus, $T_1 - \lambda : ker(T - \lambda)^{\perp} \to ker(T - \lambda)^{\perp}$ is one to one map implying the ontoness of the map $T_1 - \lambda$. Hence $T_1 - \lambda$ is an invertible operator. Thus, $\lambda \notin \sigma(T_1)$. By [8, Corollary 8], it is easy to get $\sigma(T) = \{\lambda\} \cup \sigma(T_1)$, so λ is an isolated point of $\sigma(T)$. Hence $\lambda \in \pi_{00}$.

Conversely, assume that $\lambda \in \pi_{00}$. Then $ker(T-\lambda)$ is non-zero finite dimensional subspace of $\mathcal{H}$. It is given that λ is an isolated point of spectrum of T, by Lemma 2.12, $ran(E) = ker(T-\lambda)$ and λ is an eigen value of T. It follows that $\lambda \notin \sigma(T|_{ran(I-E)})$. By the general theory of Riesz integral, we take

$ran(T-\lambda) = (T-\lambda)(ran(E)) + (T-\lambda)(ran(I-E))$. This implies that $ran(T-\lambda) = (T-\lambda)ran(I-E) = ran(I-E)$. It is clear that $ran(I-E)$ is closed set, so $ran(T-\lambda)$ is closed set. Thus, $(T-\lambda)$ is a semi Fredholm operator. Consider

$\alpha(T-\lambda)^* = dim\left(\frac{\mathcal{H}}{ran(T-\lambda)}\right) = dim\left(\frac{\mathcal{H}}{ran(I-E)}\right) = dim\ ran(E) = \alpha(T-\lambda)$. Therefore, $(T-\lambda)$ is a Fredholm operator of index zero, hence $\lambda \notin \sigma_w(T)$. It follows that $\lambda \in \sigma(T)\backslash\sigma_w(T)$. This completes the proof. □

Theorem 2.14. *Let M be a subspace of $\mathcal{H}$ such that $T(M) \subseteq (M)$ and $T(M^\perp) \subseteq (M^\perp)$. Then T is a totally (m,n)-paranormal operator on $\mathcal{H}$ if and only if T is a totally (m,n)-paranormal operator on M and $M^\perp$.*

Proof. Let T be totally (m,n)-paranormal on $\mathcal{H} = M \oplus M^\perp$. Then

$$\|(T-\lambda)x\|^{n+1} \leq m\|(T-\lambda)^{n+1}x\|\|x\|^n$$

for all $x \in \mathcal{H}$ and for all complex numbers λ. Let $T|_M = T_1$ and $T|_{M^\perp} = T_2$.

Note that $T = T_1$ and $T^* = T_1^*$ on M. For $x \in M$, we have

$$\begin{aligned}\|(T_1-\lambda)x\|^{n+1} &= \|(T-\lambda)x\|^{n+1}\\ &\leq m\|(T-\lambda)^{n+1}x\|\|x\|^n\\ &= m\|(T_1-\lambda)^{n+1}x\|\|x\|^n.\end{aligned}$$

That is, T is totally (m,n)-paranormal on M. Again for $x \in M^\perp$, we have

$$\begin{aligned}\|(T_2-\lambda)x\|^{n+1} &= \|(T-\lambda)x\|^{n+1}\\ &\leq m\|(T-\lambda)^{n+1}x\|\|x\|^n\\ &= m\|(T_2-\lambda)^{n+1}x\|\|x\|^n\end{aligned}$$

This shows that T_2 is totally (m,n)-paranormal on $M^\perp$.

Conversely, assume that T is totally (m,n)-paranormal on M and $M^\perp$. Let $x = x_1 + x_2$ in $\mathcal{H}$, so $x_1 \in M$ and $x_2 \in M^\perp$. Thus, for all $x \in \mathcal{H}$ and all complex number λ, we obtain

$$\begin{aligned}\|(T-\lambda)x\|^{n+1} &= \|(T-\lambda)x_1 + (T-\lambda)x_2\|^{n+1}\\ &= \|(T_1-\lambda)x_1 + (T_2-\lambda)x_2\|^{n+1}\\ &\leq \|(T_1-\lambda)x_1\|^{n+1} + \|(T_2-\lambda)x_2\|^{n+1}\\ &\leq m\|(T_1-\lambda)^{n+1}x_1\|\|x_1\|^n + m\|(T_2-\lambda)^{n+1}x_2\|\|x_2\|^n\\ &= m\|(T-\lambda)^{n+1}x_1\|\|x_1\|^n + m\|(T-\lambda)^{n+1}x_2\|\|x_2\|^n\\ &\leq m\|(T-\lambda)^{n+1}x\|\|x\|^n\end{aligned}$$

This implies that T is totally (m,n)-paranormal on $\mathcal{H}$. □

For $T \in B(\mathcal{H})$, a complex number $\lambda \in \sigma(T)$ is said to be a regular point if there exists some $S \in B(\mathcal{H})$ such that $T-\lambda = (T-\lambda)S(T-\lambda)$. An operator T is said to be reguloid if every point in $\sigma(T)$ is regular point. It is well known that from [10, Theorem 4.6.4 and Theorem 8.4.4] that $T-\lambda = (T-\lambda)S(T-\lambda)$ if and only if $ran(T-\lambda)$ is closed set.

Theorem 2.15. *For $m \le 1$, let T be totally (m,n)-paranormal, then T is reguloid.*

Proof. Let T be totally (m,n)-paranormal. Then

$$\|(T-\lambda)x\|^{n+1} \le m\|(T-\lambda)^{n+1}x\|\|x\|^n$$

for all $x \in \mathcal{H}$ and all $\lambda \in \mathbb{C}$. Let α be an isolated point of $\sigma(T)$. Now, we can represent T as matrix representation: $T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}$ on $ker(T-\alpha) \oplus ker(T-\alpha)^{\perp}$ with $\sigma(T_1) = \{\alpha\}$ and $\sigma(T_2) = \sigma(T)\backslash\{\alpha\}$. By Theorem 2.6, T_1 is totally (m,n)-paranormal for $m \le 1$, so by Proposition 2.11, we get $T_1 = \alpha$. Consider,

$T-\alpha = \begin{bmatrix} 0 & 0 \\ 0 & T_2-\alpha \end{bmatrix} = 0 \oplus T_2 - \alpha$. It follows that

$ran(T-\alpha) = (T-\alpha)\mathcal{H} = (0 \oplus T_2-\alpha)\mathcal{H} = (T_2-\alpha)(ker(T-\alpha))^{\perp}$. This implies that $ran(T-\alpha)$ is closed set. Therefore, T is a reguloid operator. ☐

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