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ON THE BASE LOCUS OF LINEAR SYSTEMS OF GENERAL DOUBLE POINTS

EDOARDO BALLICO*

ABSTRACT. Fix integers $n \geq 1$, $d \geq 4$ and $x > 0$ such that $(n+1)(x-1) + \binom{n+2}{2} \leq \binom{n+d}{n}$. Take a general $S \subset \mathbb{P}^n$ such that $\#S = x$ and let $\mathcal{B}$ denote the scheme-theoretic base locus of $|\mathcal{I}_{2S}(d)|$, where $2S$ is the union of the double points with S as their reduction. Then $2S$ is the union of the connected components of $\mathcal{B}$ containing at least one point of S. We prove this theorem proving that a general union of $x-1$ double points and one triple point has no higher cohomology in degree d.

1. Introduction and preliminaries

For any smooth point of p of an algebraic variety M and positive integer t let (tp, M) denote the closed subscheme of M with $(\mathcal{I}_{p,M})^t$ as its ideal sheaf. For $t = 2$ (resp. $t = 3$) we call this scheme a double point (resp. a triple point) of M. For any finite $S \subset M$ set $(2S, M) := \cup_{p \in S}(2p, M)$. If $M = \mathbb{P}^n$ we often write tp instead of $(tp, \mathbb{P}^n)$ and $2S$ instead of $(2S, \mathbb{P}^n)$.

By the Terracini lemma the Hilbert function of general unions of a prescribed number of general unions of x double points of $\mathbb{P}^n$ uniquely determine the dimension of the x-secant variety of the Veronese embeddings of $\mathbb{P}^n$ ([1, Cor. 1.10]). J. Alexander and A. Hirschowitz proved a very strong theorem, i.e. that, with a few known examples, the Hilbert function of a general union of x double points in $\mathbb{P}^n$ is the " expected one " ([2, 3, 4]), computing also the dimension of all the secant varieties of the Veronese embedding. A related problem is the number of minimal additive decompositions of degree d forms with a prescribed rank. If the rank is less that the generic one, then uniqueness holds, except known examples, and a key step of the proof is to prove that (except known examples) for all $n \geq 1$, $d \geq 3$ and $(n+1)x < \binom{n+d}{n}$ the base locus $\mathcal{B}$ of $|\mathcal{I}_{2S}(d)|$, S a general subset of $\mathbb{P}^n$ with $\#S = x$, contains no scheme $2p$ for some $p \in \mathbb{P}^n \setminus S$ ([8, Theorem 1.2]).

Theorem 1.1. *Fix integers $n \geq 1$, $d \geq 4$ and $y \geq 0$ such that $(n+1)y + \binom{n+2}{2} \leq \binom{n+d}{n}$. Let $Z \subset \mathbb{P}^n$ be a general union of y double points and one triple point. Then $h^1(\mathcal{I}_Z(d)) = 0$.*

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* Corresponding author.
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Obviously, Theorem 1.1 cannot be extended to the case $d = 3$ (Remark 2.2). For any positive integer x let $S(\mathbb{P}^n, x)$ denote the set of all $S \subset \mathbb{P}^n$ such that $\#S = x$.

Theorem 1.2. *Fix integers $n \ge 1$, $d \ge 4$ and $x > 0$ such that $(n+1)(x-1) + \binom{n+2}{2} \le \binom{n+d}{n}$. Take a general $S \in S(\mathbb{P}^n, x)$ and let $\mathcal{B}$ denote the scheme-theoretic base locus of $|\mathcal{I}_{2S}(d)|$. Then $2S$ is the union of the connected components of $\mathcal{B}$ containing at least one point of S.*

Fix integers $n \ge 1$, $d \ge 3$ and $x > 0$ such that $(n+1)x < \binom{n+d}{n}$. Take a general $S \in S(\mathbb{P}^n, x)$ and let $\mathcal{B}$ denote the scheme-theoretic base locus of $|\mathcal{I}_{2S}(d)|$. By [8] $\mathcal{B}$ contains no double points. If $(n+1)x \ge \binom{n+d}{n} - n + 1$, the obviously $\dim \mathcal{B} > 0$.

Question 1.3. Fix integers $n \ge 1$, $d \ge 3$ and $x > 0$ such that $(n+1)x < \binom{n+d}{n} - n$. Take a general $S \in S(\mathbb{P}^n, x)$ and let $\mathcal{B}$ denote the scheme-theoretic base locus of $|\mathcal{I}_{2S}(d)|$. Is $\mathcal{B} = 2S$? Is $\mathcal{B} = 2S$ at least with stronger assumptions on x, e.g. with the assumptions of Theorem 1.2?

The assumptions $d \ge 3$ and $(n+1)x < \binom{n+d}{n} - n$ give that (n, d, x) is not an exceptional cases of the Alexander-Hirschowitz theorem.

In general, controlling the Hilbert function of at least one triple point is not easy ([7, 6]), although it is known in $\mathbb{P}^2$ ([9]).

2. Preliminaries

Fix a hyperplane $H \subset \mathbb{P}^n$ and let $Z \subset \mathbb{P}^n$ be a zero-dimensional scheme. We often use the residual scheme $\mathrm{Res}_H(Z)$ of Z with respect to H. We have $\mathrm{Res}_H(Z) \subseteq Z$, $\mathrm{Res}_H(Z) = Z$ if $Z \cap H = \emptyset$, $\mathrm{Res}_H(Z) = \emptyset$ if and only if $Z \subset H$ and $\deg(Z) = \deg(Z \cap H) + \deg(\mathrm{Res}_H(Z))$. If $p \in H$, then $\mathrm{Res}_H(tp) = (t-1)p$, with the convention $0p = \emptyset$. We use the existence of the residual exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_H(Z)}(t-1) \to \mathcal{I}_Z(t) \to \mathcal{I}_{Z\cap H,H}(t) \to 0 \tag{2.1}$$

We need the following easy lemma.

Lemma 2.1. *Fix integers $t \ge 0$ and $a \ge 0$, a hyperplane $H \subset \mathbb{P}^n$ and a zero-dimension scheme $W \subset \mathbb{P}^n$ such that $h^1(\mathcal{I}_W(t)) = 0$. Let $A \subset H$ be a general subset such that $\#A = a$. Assume $h^0(\mathcal{I}_{\mathrm{Res}_H(W)}(t-1)) \le \max\{0, h^0(\mathcal{I}_W(t)) - a\}$. Then either $h^1(\mathcal{I}_{W\cup A}(t)) = 0$ or $h^0(\mathcal{I}_{W\cup A}(t)) = 0$.*

Proof. We use induction on the integer a, the case $a = 0$ being trivial. Assume $a > 0$ and take a general $A \subset H$ such that $\#A = a$. We order the elements $p_1, \dots, p_a$ of A and set $E := \{p_1, \dots, p_{a-1}\}$. The inductive assumption gives that either $h^1(\mathcal{I}_{W\cup E}(t)) = 0$ or $h^0(\mathcal{I}_{W\cup E}(t)) = 0$. If $h^0(\mathcal{I}_{W\cup E}(t)) = 0$, then $h^0(\mathcal{I}_{W\cup A}(t)) = 0$. Now assume $h^1(\mathcal{I}_{W\cup E}(t)) = 0$. Since $h^1(\mathcal{I}_W(t)) = 0$, we get $h^0(\mathcal{I}_{W\cup E}(t)) = h^0(\mathcal{I}_W(t) - a + 1 \ge 0$. Recall that A is the union of E and a general point of H. If $h^0(\mathcal{I}_{W\cup A}(t)) < h^0(\mathcal{I}_{W\cup E}(t))$, then $h^1(\mathcal{I}_{W\cup A}(t)) = 0$. Now assume $h^0(\mathcal{I}_{W\cup A}(t)) = h^0(\mathcal{I}_{W\cup E}(t))$. Since p_a is general in H, H is contained in the base locus of $|\mathcal{I}_{W\cup E}(t)|$. Thus $h^0(\mathcal{I}_{\mathrm{Res}_H(W)}(t-1)) = h^0(\mathcal{I}_{W\cup E}(t)) = h^0(\mathcal{I}_W(t) - a + 1 > \max\{0, h^0(\mathcal{I}_W(t)) - a\}$, a contradiction. □

Remark 2.2. Fix an integer $d \geq 3$ and any $p \in \mathbb{P}^n$. Any $T \in |\mathcal{I}_{dp}(d)|$ is a cone with vertex p. Thus $h^1(\mathcal{I}_Z(d)) \geq y$ for a general union $Z \subset \mathbb{P}^n$ of dp and y double points.

3. The proofs

Proof of Theorem 1.1: The cohomology of line bundles on $\mathbb{P}^1$ gives the case $n = 1$. Thus we may assume $n \geq 2$ and use induction on n. By the semicontinuity theorem for cohomology it is sufficient to find a union W of y double points and one triple point such that $\#W_{\mathrm{red}} = y + 1$ and $h^1(\mathcal{I}_W(d)) = 0$. Fix a hyperplane H of $\mathbb{P}^n$ and $p \in H$. Increasing if necessary y we may assume

$$\binom{n+d}{n} - n \leq (n+1)y + \binom{n+2}{2} \leq \binom{n+d}{n} \tag{3.1}$$

(a) Assume $n = 2$.

(a1) Assume d even. Take $E \subset H \setminus \{p\}$ such that $\#E = d/2 - 1$. Take a general $F \subset \mathbb{P}^2 \setminus H$ such that $\#F = y + 1 - d/2$ and set $W := 2E \cup 3p$. We have $h^1(H, \mathcal{I}_{W\cap H}(d)) = 0$, $i = 0, 1$. Thus it is sufficient to prove that $h^1(\mathcal{I}_{\mathrm{Res}_H(W)}(d-1)) = 0$. We have $\mathrm{Res}_H(Z) = 2F \cup 2p \cup E$. We have $\deg(\mathrm{Res}_H(Z)) = 3y + 6 - d - 1 \leq \binom{d+1}{2}$. We first check that $h^1(\mathcal{I}_{2F\cup 2p}(d-1)) = 0$. Since we take a general F after fixing p and any point of $\mathbb{P}^n$ is contained in a hyperplane, $2F \cup 2p$ has the Hilbert function of a general union of $y+2+d/2$ double points. Since $d-1$ is an odd integer ≥ 3 and $\deg(2F \cup \{p\}) \leq \binom{d+1}{2}$, $h^1(\mathcal{I}_{2F\cup 2p}(d-1)) = 0$ by the case $n = 2$ of the Alexander-Hirschowitz theorem ([2, 3, 4]). Since $h^1(\mathcal{I}_{2F\cup 2p}(d-1)) = 0$, $h^0(\mathcal{I}_{2F\cup 2p}(d-1) = \binom{d+1}{2} - 3y - 6 + 3d/2 \geq \#E$. Since E is general in H, $h^0(\mathcal{I}_{2F\cup 2p}(d-1) \geq \#E$ and $\mathrm{Res}_H(\mathrm{Res}_H(W)) = 2F \cup \{p\}$, to get $h^1(\mathcal{I}_W(d)) = 0$ it is sufficient to prove that $h^0(\mathcal{I}_{2F\cup\{p\}}(d-2)) \leq \max\{0, \binom{d+1}{2} - \deg(\mathrm{Res}_H(W)\}$. If $d \geq 8$, then the latter inequality is true by the Alexander-Hirschowitz theorem.

Now assume $d = 6$. We have $y = 7$ and hence $y + 2 - d/2 = 6$. We use that $h^0(\mathcal{I}_A(4)) = 0$ for a general union of 6 double points.

Now assume $d = 4$ and hence $y = 3$ and $\#F = y + 1 - d/2 = 2$. The set $|\mathcal{I}_{2F}(2)|$ is the double line spanned by F and hence $h^0(\mathcal{I}_{2F\cup\{p\}}(d-2)) = 0$ by the generality of F.

(a2) Assume d odd. Hence $d \geq 5$. Take a general $G \subset H \setminus \{p\}$ such that $\#G = (d-1)/2$. Take $o \in G$ and set $E := G \setminus \{o\}$. Take a general $F \subset \mathbb{P}^2 \setminus H$ such that $\#F = y - (d-1)/2$. Set $Z' := 2F$ and $W' := 2F \cup 2E \cup 3p$. Note that $h^i(H, \mathcal{I}_{\{o\}\cup(3p,H)\cup(2E,H),H}(d)) = 0$, $i = 0, 1$. Applying the Differential Horace Lemma ([2, 3, 4, 5]) to prove that $h^1(\mathcal{I}_{W_1}(d)) = 0$ for a general union of W' and 1 double point, it is sufficient to prove that $h^1(\mathcal{I}_{2F\cup E\cup(2o,H)\cup 2p}(d-1)) = 0$. We have $\deg(2F \cup E \cup (2o, H) \cup 2p) = 3y + 6 - d - 1 \leq \binom{d+1}{2}$. We first check that $h^1(\mathcal{I}_{2F\cup 2p}(d-1)) = 0$. The scheme $2F \cup 2p$ has the Hilbert function of a general union of $y - (d-1)/2$ double points. If $d \geq 7$ it is sufficient to apply the Alexander-Hirschowitz theorem.

Now assume $d = 5$ and hence $y = 5$. Since $h^1(\mathcal{I}_B(4)) = 0$ for a general union of 4 double points, $h^0(\mathcal{I}_{2F\cup 2p}(d-1)) = \binom{d+1}{2} - 3y + 3(d-1)/2$. By Lemma 2.1 it is sufficient to prove that the restriction map $H^0(\mathcal{I}_{2F\cup 2p}(d-1)) \to$

$H^0(\mathcal{O}_H(d-1))$ has image of dimension at least $\deg(E\cup(2o,H)) = (d+1)/2$. Since $\mathrm{Res}_H(2F\cup 2p) = 2F\cup\{p\}$, it is sufficient to prove that $h^0(\mathcal{I}_{2F\cup\{p\}}(d-2)) \le \binom{d+1}{2} - 3y + 3(d-1)/2 - (d+1)/2$. Since $2F\cup\{p\}$ has the Hilbert function of a general union of one point and $y-(d-1)/2$ double points, it is sufficient to prove that $h^0(\mathcal{I}_{2F}(d-2)) \le \binom{d+1}{2} - 3y + 3(d-1)/2 - (d+1)/2$. Since $d-2$ is an odd integer ≥ 3, it is sufficient to quote the Alexander-Hirschowitz theorem.

(b) By step (a) we may assume $n\ge 3$ and use induction on n. Set $e := \lfloor(\binom{d+n-1}{n-1} - \binom{n+1}{2})/n\rfloor$ and $f := \binom{d+n-1}{n-1} - \binom{n+1}{2} - ne$. Thus $0\le f\le n-1$. Take a general $A\cup B\subset H\setminus\{p\}$ such that $\#A = e$, $\#B = f$ and $A\cap B = \emptyset$. Let $Z'\subset\mathbb{P}^n$ be a general union of $y-e-f$ double points. The inductive assumption on n and the generality of B give $h^i(H,\mathcal{I}_{(2A,H)\cup B\cup(3p,H)}(d)) = 0$, $i=0,1$. By the Differential Horace Lemma applied to all points of B to prove the theorem for the triple (n,d,y) it is sufficient to prove that $h^1(\mathcal{I}_{Z'\cup A\cup(2B,H)\cup 2p}(d-1)) = 0$.

Claim 1: $h^1(\mathcal{I}_{Z'\cup 2B\cup 2p}(d-1)) = 0$.

Proof of Claim 1: Since $f\le n-1$, any n points of $\mathbb{P}^n$ are contained in a hyperplane of $\mathbb{P}^n$ and we fix a general Z' after fixing p and F, the scheme $Z'\cup 2F\cup 2p$ has the Hilbert function of a general union of $y+1-e$ double points. Since $d-1\ge 3$, the Alexander-Hirschowitz theorem gives $h^1(\mathcal{I}_{Z'\cup 2F\cup 2p}(d-1)) = 0$ if $\deg(Z'\cup 2F\cup 2p)\le\binom{n+d-1}{n}$ and $(n,d)\notin\{(3,5),(4,4),(4,5)\}$. Since $(n+1)y+\binom{n+2}{2}\le\binom{n+d}{n}$ and $\deg((2A,H)\cup B\cup(3p,H)) = \binom{n+d-1}{n-1}$, we have $\deg(Z'\cup 2F\cup 2p)\le\binom{n+d-1}{n}$ if $e\ge f$. Since $f\le n-1$, it is sufficient to have $e\ge n-1$, i.e. $\binom{n+d-1}{n-1} - \binom{n+1}{2}\ge n(n-1)$. Call $f_n(d)$ the difference between the left hand side and the right hand side of the last inequality. Since $f_n(d)$ is (for a fixed n) an increasing function of n, it is sufficient (for a fixed n) to have $\binom{n+3}{4}\ge n(3n-1)/2$, which is true for all $n\ge 3$. Now assume $(n,d) = (3,5)$ and hence $y=6$, $e=3$ and $f=0$ and hence $h^1(\mathcal{I}_{Z'\cup 2F\cup 2p}(4)) = 0$. Now assume $(n,d) = (4,4)$ and hence $y=11$, $e=6$, $f=1$ and hence $y-e+1 = 6 < 7$. We have $h^1(\mathcal{I}_{Z'\cup 2F\cup 2p}(4)) = 0$. Now assume $(n,d) = (4,5)$. We have $y=22$, $e=11$, $f=2$ and hence $22-11+1 < 14$. Thus $h^1(\mathcal{I}_{Z'\cup 2F\cup 2p}(4)) = 0$.

Observation 1: Claim 1 implies $h^1(\mathcal{I}_{Z'\cup(2B,H)\cup 2p}(d-1)) = 0$.

We have $\mathrm{Res}_H(Z'\cup A\cup(2B,H)\cup 2p) = Z'\cup\{p\}$. Since $\binom{n+d-1}{n} - \deg(Z'\cup A\cup(2B,H)\cup 2p) = \binom{n+d}{n} - \binom{n+2}{2} - (n+1)y\ge 0$, to prove that $h^1(\mathcal{I}_{Z'\cup A\cup(2B,H)\cup 2p}(d-1)) = 0$ (and hence to prove the theorem for n and d) it is sufficient to prove that either $h^1(\mathcal{I}_{Z'\cup A\cup(2B,H)\cup 2p}(d-1)) = 0$ or $h^0(\mathcal{I}_{Z'\cup A\cup(2B,H)\cup 2p}(d-1)) = 0$. Since A is general in H, Observation 1 and Lemma 2.1 show that to conclude the proof it is sufficient to prove that $h^0(\mathcal{I}_{Z'\cup\{p\}}(d-2))\le\max\{0,\binom{n+d-1}{n} - (n+1)(y-e-f) - nf - (n+1)\}$. If $d\ge 5$ and $(n,d)\notin\{(3,6),(4,5),(4,6)\}$, then it is sufficient to use the Alexander-Hirschowitz theorem.

Now assume $(n,d) = (3,6)$. Since $\binom{9}{3} = 84$ and $\binom{8}{2} = 28$, we have $y=18$, $e=7$ and $f=1$. Thus Z' is a general union of 10 double points. Thus $h^0(\mathcal{I}_{Z'}(4)) = 0$.

Now assume $(n,d) = (4,5)$. Since $\binom{9}{4} = 126$ and $\binom{8}{3} = 56$, we have $y=22$, $e=11$ and $f=2$. Thus Z' is a general union of 9 double points. Hence $h^0(\mathcal{I}_{Z'}(3)) = 0$.

Now assume $(n,d)=(4,6)$. Since $\binom{10}{4}=210$ and $\binom{9}{3}=84$, we have $y=39$, $e=18$ and $f=2$. Thus Z' is a general union of 19 double points. Thus $h^0(\mathcal{I}_{Z'}(4))=0$.

(c) Assume $d=4$. We have $y=\lfloor((n+4)(n+3(n+2)(n+1)/24-(n+2)(n+1)/2)/(n+1)\rfloor=\lfloor((n+4)(n+3)(n+2)/24-(n+2)/2\rfloor$, $e=\lfloor((n+3)(n+2)(n+1)/24-(n+1)/2\rfloor$ and $f=(n+3)(n+2)(n+1)n/24-(n+1)n/2-ne$. Since we took Z' general after fixing p, we have $h^0(\mathcal{I}_{Z'\cup\{p\}}(2))=0$ if and only if $y-e-f\ge n$. Since A is general in H, we need to prove the following inequality (Lemma 2.1):

$$h^0(\mathcal{I}_{Z'\cup\{p\}}(2))\le\max\{0,\binom{n+3}{3}-(n+1)(y-e-f)-nf-(n+1)\} \quad (3.2)$$

We often write $y(n)$, $e(n)$ and $f(n)$ to stress the dependency of n. We first check all cases with $3\le n\le 11$. We have $y(3)=6$, $e(3)=3$, $f(3)=0$, $y(4)=11$, $e(4)=6$, $f(4)=1$, $y(5)=17$, $e(5)=11$, $f(5)=0$, $y(6)=26$, $e(6)=17$, $f(6)=3$, $y(7)=36$, $e(7)=26$, $f(7)=0$, $y(8)=50$, $e(8)=36$, $f(8)=6$, $y(9)=66$, $e(9)=50$, $f(9)=0$, $y(10)=85$, $e(10)=66$, $f(10)=0$, $y(11)=107$, $e(11)=85$, $f(11)=0$. Thus we may assume $n\ge 12$. We have $y\ge(n+4)(n+3)(n+2)/24-(n+2)/2-1$ and $e+f\le(n+3)(n+2)(n+1)/24-(n+1)/2+n-1$. Thus $y-e-f\ge(n+3)(n+2)/8-n\ge n$ for all $n\ge 12$. □

Proof of Theorem 1.2: Fix $p\in S$ and set $S':=S\setminus\{p\}$ and $Z:=3p\cup 2S'$. Theorem 1.1 gives $h^1(\mathcal{I}_Z(d))=0$. Thus the restriction map $H^0(\mathcal{I}_{2S'}(d))\to H^0(\mathcal{O}_{3p}(d))$ is surjective.This is equivalent to says that the connected component of $\mathcal{B}$ containing p is the scheme $2p$. Since this is proved for all $p\in S$, $2S$ is the union of the connected components of $\mathcal{B}$ containing a point of S. □

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Department of Mathematics, University of Trento, Trento, Italy.
Email address: edoardo.ballico@unitn.it, https://orcid.org/0000-0002-1432-7413