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LI-YAU TYPE ESTIMATES FOR A SEMILINEAR PARABOLIC EQUATION ON AN EVOLVING MANIFOLD

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ABSTRACT. Let $(M, g(t))$ be a complete Riemannian manifold of dimension n, we obtain Li-Yau type gradient estimates on positive bounded solutions to the following semilinear parabolic equation

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + a(x)u^s(t,x) - \lambda u(t,x),$$

where $(t,x) \in ([0,T] \times M)$, $T < \infty$, $s > 1$, $\lambda \in \mathrm{R}$ and $a \in C^2(M)$ on evolving Riemannian metrics $g(t)$ with bounded below Ricci tensor. The application of our gradient estimates yields the classical differential Harnack inequality, which compares a solution at some time with those at previous time.

1. INTRODUCTION

In their celebrated paper [15] Li and Yau derived gradient and Harnack estimates on the positive solutions $(t,x) \mapsto u(t,x)$ to the heat equation

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x)$$

defined on an n-dimensional complete Riemannian (with or without boundary) whose Ricci curvature satisfies some condition. Precisely, they obtained

$$|\nabla w|^2 - \beta w_t \leq \frac{n\beta^2}{2t} + \frac{n\beta^2 K}{2(\beta-1)} \qquad \forall\ t > 0,$$

where $w := \log u$, w_t is the derivative of w with respect to t, the Ricci curvature is bounded below by $-K$, $K \geq 0$, $\beta > 1$ is a constant and $|\nabla w|$ denotes the length of ∇w. This gradient estimate yields parabolic Harnack inequalities, in particular for the case $K = 0$

$$u(t_1,x) \leq u(t_2,y)\Big(\frac{t_2}{t_1}\Big)^{n\beta/2} \exp\Big(\frac{\beta d^2(x,y)}{4(t_2-t_1)}\Big), \qquad t_1 < t_2,$$

where $d(x,y)$ is the Riemann distance between points x and y. Consequently, they derived various optimal bounds on the heat kernel. These results have become very powerful tools in geometric analysis and have been successfully applied to

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the setting of evolving manifolds. We note that the geometric flow versions of the above estimates were first obtained under the Ricci flow in [5] by Băileşteanu, Cao and Pulemotov, see also Băileşteanu [3]. Meanwhile, Hamilton in [11] had earlier shown that Li-Yau estimate is the trace of a full matrix inequality, thereby proving a different version of the estimate. He then observed a similar phenomena for the Ricci flow on a closed manifold with nonnegative Ricci curvature [12] and then for mean curvature flow [13]. For more on geometric flows setiing see [1, 2, 8, 10, 14, 20] and the references therein.

Let (M, g) be an n-dimensional complete Riemannian manifold without boundary and with Ricci curvature bounded below by a nonpositive constant. We consider estimates of Li-Yau type on the following semilinear parabolic equation

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + a(x)u^s(t,x) - \lambda u(t,x) \tag{1.1}$$

where $(t,x) \in ([0,T] \times M)$, $T < \infty$, $s > 1$ and $\lambda \in \mathrm{R}$. Here $a = a(x)$ is a C^2-spatial function on M.

Consideration of equation (1.1) in this paper is motivated by its physical applications and geometric relevance. This equation arises in many different physical applications to model reaction-diffusion phenomena. For instance, given certain values of s and λ, it reduces to Parabolic Allen Cahn equation or Fisher-Komolgrov-Petrovskii-Piskunov equation (see for instance [4, 9] and some references therein). These equations (i.e., with specific values of s and λ) arise in the modelling of travelling waves [19], propagation of evolutionarily advantageous gene density, propagation of electro-chemical waves in organisms, descriptions of branching Brownian motion [16] and certain chemical reactions.

Another motivation is the relationship of (1.1) with some geometric quantities. For instance, the authors in [7] show that the stationary part of (1.1), i.e., the semilinear elliptic equation

$$\Delta u(x) + a(x)u^s(x) - \lambda(x)u(x) = 0, \tag{1.2}$$

is equivalent to Yamabe problem on noncompact Riemannian manifold. Clearly, setting $\widetilde{g} = u^{4/n-2}g$, $u > 0$, then for $\mathcal{R}(x)$ and $K(x) \in C^\infty(M)$, the scalar curvature of g and $\widetilde{g}$ respectively, we have the relation

$$\Delta u + K(x)u^{\frac{n+2}{n-2}} - \frac{n-2}{4(n-1)}\mathcal{R}(x)u = 0, \tag{1.3}$$

which is of the form (1.2). Yamabe problem demands the existence of a positive everywhere defined solution of (1.3). Indeed, the existence and uniqueness of such solution depends on the geometry of the underlying manifold. Thus, g can be pointwise conformally deformed to a complete metric $\widetilde{g}$ of a scalar curvature $K(x)$. For further discussions on existence, uniqueness and a priori estimates of (1.2) (resp. Yamabe-type equation) see [17]. Another motivation is the work of Bidaut-Veron and Veron [6]. They study (1.2) for $\lambda > 0$, $a = 1$ and $s > 1$ on compact manifolds and show that it has only constant solution under the conditions on the Ricci tensor, s and the manifold's dimension as follow

$$Ric \geq \frac{n-1}{n}(s-1)\lambda g, \quad s \leq \frac{n+2}{n-2} \quad \text{and} \quad n \geq 2.$$

Our goal is to derive Li-Yau type gradient and Harnack estimates for the positive solutions of (1.1) on $(M, g(t))$, where the time-dependent Riemannian metrics $g = g(t)$ evolve by the abstract geometric flow

$$\frac{\partial g(t,x)}{\partial t} = 2h(t,x), \tag{1.4}$$

where $(t,x) \in ([0,T]\times M)$, $g(0,x) = g_0(x)$ is the initial metric, h is a general time-dependent symmetric $(0,2)$-tensor and $0 < T < T_\varepsilon$ is taken to be the maximum time of existence for the flow i.e T_ε is the first time where the flow blows-up. We obtain our result on the assumption that the geometry of the manifold remains uniformly bounded throughout the evolution. Our estimates are more general and can be found for specific geometric flows on complete manifolds. We quickly remark that (1.4) encompasses many intrinsic geometric flows. For instance, (1.4) is the Ricci flow, when $h = -Ric$, the Ricci curvature tensor, the Yamabe flow, when $h = 1/2(r - \mathcal{R})g$ (r being the average scalar curvature and $\mathcal{R}$ the scalar curvature) and Mean curvature flow, when $h = -\Pi A_{ij}$, where A_{ij} is the component of the second fundamental form A on M and $\Pi = g^{ij}A_{ij}$ (this is related to an immersion of submanifolds into Euclidean or Lorentzian space). Note that the Yamabe flow and the normalized Ricci flow coincide on Riemann surfaces.

The rest of the paper is organized as follows. In Section 2 we state our main result and its applications. We also describe the methodolgy for achieving the results. Section 3 is devoted to the proof of the main theorem. We first state and prove an important lemma that would be applied to prove the results. We also give a description of the cut-off function needed in the proof. Lastly in this section, we derive a Harnack type inequality by integrating along the space-time path.

2. Statement of results and methodology

Let M be an n-dimensional complete Riemannian manifold without boundary and with Ricci curvature bounded below by a negative constant. Let $u \in C^\infty([0,T]\times M)$ be a positive solution to the following semilinear evolution equation

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + a(x)u^s(t,x) - \lambda u(t,x), \tag{2.1}$$

where $s > 1$ and λ is a constant. Note that a simple calculation shows that if we take $w(t,x) = \log u(t,x)$, then, w solves

$$\frac{\partial w(t,x)}{\partial t} = \Delta w(t,x) + |\nabla w(t,x)|^2 + ae^{(s-1)w} - \lambda, \qquad s > 1 \tag{2.2}$$

which is another nonlinear evolution equations. The above equations are defined on a complete time-dependent Riemannian manifold $(M, g(t))$. Here $g(t)$ is a smooth one parameter family of Riemannian metric evolving by the abstract geometric flow (1.4).

A natural function that will be defined on M is the distance function from a given point, namely, let $y \in M$ and define $d(x,y)$ for all $x \in M$, where $d(\cdot,\cdot)$ is the geodesic distance. Note that d is everywhere continuous except on the cut locus of y and on the point where x and y coincide. It is then easy to see that $|\nabla d| = 1$ on $M \setminus \{\{y\} \cup cut(y)\}$. Let $d(x, y, t)$ be the geodesic distance between x and y with respect to the metric $g(t)$, we shall define a smooth cut-off function $\varphi(x, t)$ with support in the geodesic cube $\mathcal{Q}_{2R,T}$ around a point in M

$$\mathcal{Q}_{2R,T} := \{(x, t) \in (0, T] \times M : d(x, y, t) \leq 2R,\ R > 0\}.$$

In the rest of this paper we shall make use of the following assumptions

2.1. Assumptions.

(I) The curvatures of the metrics $g(t)$ are uniformly bounded on $[0, T] \times M$. In particular, the Ricci curvature of M is $Ric \geq -k_1 g$, where $k_1 > 0$.

(II) h and $|\nabla h|$ are uniformly bounded on $[0, T] \times M$. Precisely, $-k_2 g \leq h \leq k_3 g$ and $|\nabla h| \leq k_4$, where $k_2, k_3, k_4 > 0$.

(III) $a = a(x), |\nabla a|$ and Δa are uniformly bounded on M. Precisely, $|\nabla a| \leq \gamma(M)$ and $\Delta a \leq \theta(M)$ for some positive constants γ and θ.

We now state our main result and its applications.

2.2. Main results.

Firstly, we state some notations that will appear in the results. We will assume that the solution of (1.1) is bounded by $0 < u \leq \mathbb{M}$ and take $\widetilde{\mathbb{M}} = \mathbb{M}^{s-1}$. Then

$$D_2 := D_2(n, k_1, k_2, k_3, R) = \frac{C_1(n-1)\sqrt{k_1}R + \sqrt{C_1\widetilde{K}}R + 2C_1^2 + C_2}{R^2},$$

where C_1, C_2 are absolute constants, $\widetilde{K} = (k_2 + k_3)^2$ and $R > 0$.

$$A_1 := 2(1-\alpha)k_3 + 2\alpha k_1 + \ \|a\|(s-1)(s-\alpha)\widetilde{\mathbb{M}} + \frac{3}{2}k_4 + \frac{2(s-\alpha)}{\varepsilon}\widetilde{\mathbb{M}}$$

$$A_2 := \widetilde{\mathbb{M}}\theta + 2\varepsilon(s-\alpha)\widetilde{\mathbb{M}}\gamma^2 + \ \frac{n}{2}\Big(\frac{q}{\alpha}(k_2 + k_3)^2 + 3k_4\Big)$$

$$A_3 := \frac{np}{2\alpha}\|a\|(s-1)\widetilde{\mathbb{M}} + \sqrt{\frac{np}{2\alpha}\Big(\frac{npA_1^2}{8\alpha(1-\alpha)^2} + A_2\Big)} + \lambda,$$

where $0 < \alpha < 1$, $\varepsilon > 0$, $\|a\| = \|a\|_{L^\infty(\mathcal{Q}_{2R})} = \sup_{\mathcal{Q}_{2R}} |a(x)|$ and $1 < p, q < \infty$ such that $1/p + 1/q = 1$.

Theorem 2.1. *(Local gradient estimate). Let $(M, g(t))$ be a complete solution to the geometric flow (1.4) in some time interval $[0, T]$. Suppose there exist some positive constants $k_1, k_2, k_3,, k_4, \gamma$ and θ such that $Ric(g), h, |\nabla h|, |\nabla a|$ and Δa satisfy the assumptions (I)–(III) in Section 2.1 for all $t \in [0, T]$. Let $u \in C^{1,2}([0, T] \times M)$ be any smooth positive solution to (1.1) with $u \leq \mathbb{M}$ in the geodesic cube $\mathcal{Q}_{2R,T}$.*

Then, the following gradient estimate holds

$$\sup_{x\in\mathcal{Q}_{2R}}\Big\{(\alpha|\nabla w|^2+ae^{(s-1)w}-w_t-\lambda)\Big\}\leq\frac{np}{2\alpha t}$$
$$+\frac{np}{2\alpha}\Big\{D_2+\|a\|(s-1)\widetilde{\mathbb{M}}+\frac{C_1^2np}{4R^2\alpha(1-\alpha)}\Big\}+\sqrt{\frac{np}{2\alpha}\Big(\frac{npA_1^2}{8\alpha(1-\alpha)^2}+A_2\Big)} \tag{2.3}$$

for all $(t,x)\in\mathcal{Q}_{2R,T}$, $t>0$. *Here* $w=\log u$, α *is any constant satisfying* $0<\alpha<1$, $1<p,q<\infty$ *such that* $1/p+1/q=1$ *and* $D_2,A_1,A_2,\widetilde{\mathbb{M}}$ *are as defined above.*

Letting $R\to\infty$, we obtain the following global Li-Yau gradient estimates for the solutions of (1.1).

Corollary 2.2. *(Global gradient estimates). Let* $(M,g(t))$ *be a complete solution to the geometric flow (1.4) in some time interval* $[0,T]$. *Assume the conditions of Theorem 2.1. Then, the following global gradient estimate holds*

$$\sup_{x\in\mathcal{Q}_{2R}}\Big\{(\alpha|\nabla w|^2+ae^{(s-1)w}-w_t-\lambda)\Big\}\leq\frac{np}{2\alpha t}+\frac{np}{2\alpha}\|a\|(s-1)\widetilde{\mathbb{M}}$$
$$+\sqrt{\frac{np}{2\alpha}\Big(\frac{npA_1^2}{8\alpha(1-\alpha)^2}+A_2\Big)} \tag{2.4}$$

for all $(t,x)\in\mathcal{Q}_{2R,T}$, $t>0$, *where* A_1,A_2 *and* $\widetilde{\mathbb{M}}$ *are as defined above.*

As an application of the above theorem we derive the following Li-Yau Harnack inequality. This is a comparison of values of positive solution at different points in space-time.

Corollary 2.3. *(Harnack inequality). Suppose the assumptions of Theorem 2.1 hold. If in addition* $u\leq\mathbb{M}$ *for all* $(t,x)\in[0,T]\times M$. *Then for any* $x_1,x_2\in M$ *and* $0<t_1<t_2\leq T$, *then the following Harnack inequality holds*

$$u(t_1,x_1,)\leq u(t_2,x_2,)\Big(\frac{t_2}{t_1}\Big)^{\frac{np}{2\alpha}}\exp\Big\{\frac{1}{4\alpha}\Gamma(t_1,x_1;t_2,x_2)+A_4(t_2-t_1)\Big\}, \tag{2.5}$$

where $\Gamma(t_1,x_1;t_2,x_2)=\inf_\gamma\int_{t_1}^{t_2}|\frac{d}{dt}\gamma(t)|^2dt$ *with the infimum taken over the set of all smooth paths* $\gamma:[t_1,t_2]\to M$ *connecting points* x_1 *and* x_2 *and* $A_4=A_3+\|a\|\widetilde{\mathbb{M}}$, $A_3,\widetilde{\mathbb{M}}$ *are as defined above.*

2.3. **Methodology.** The proof of our Li-Yau type estimate involves defining a space-time function

$$G(t,x)=t(\alpha|\nabla w|^2+ae^{(s-1)w}-w_t-\lambda)$$

called Harnack quantity and applying the heat operator $(\frac{\partial}{\partial t}-\Delta)$ to it. By this, one obtains a nonlinear parabolic equation of the form

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)G+2\nabla w\nabla G+\frac{G}{t}\geq A,$$

where A depends on t, dimension of M, bounds of $Ric, h, |\nabla h|, a, |\nabla a|$ and Δa. For a smooth cut-off function $\varphi(t,x)$ (as will be difined later) with compact support in the geodesic cube $\mathcal{Q}_{2R,T}$, let (φG) attains its maximum value at point (t_0, x_0) in $\mathcal{Q}_{2R,T}$, for $t_0 > 0$. We then assume without loss of generality that there exists $(\varphi G)(t_0, x_0) > 0$ (otherwise $G \leq 0$ and the proof is trivial). At this stage we can apply the Maximum principle on φG with the assumption that φ is everywhere differentiable and obtain local gradient estimate. Letting $R \to \infty$, we then obtain global gradient estimate. Integrating the Harnack quantity along space-time path yields the classical Harnack inequality.

3. Gradient estimates and Harnack inequality

3.1. An important lemma.

We first prove the following technical lemma which was originally proved for heat equation on static metric by Li and Yau [15]. This is very crucial to the derivation of our gradient estimate of Li-Yau type.

Lemma 3.1. *Let $(M, g(t))$ be a complete solution to the geometric flow (1.4) in some time interval $[0,T]$. Suppose there exist some positive constants k_1, k_2, k_3, and k_4 such that $Ric(g) \geq -k_1 g$, $-k_2 g \leq h \leq k_3 g$ and $|\nabla h| \leq k_4$ for all $t \in [0,T]$. For any smooth positive solution $u \in C^\infty([0,T]\times M)$ to equation (1.1) in the cube $\mathcal{Q}_{2R,T}$, it holds that*

$$\begin{aligned}\Big(\frac{\partial}{\partial t} - \Delta\Big)G \geq& -2\nabla w \nabla G + t\Big\{\frac{2\alpha}{np}(\Delta w)^2 + \Big[2(\alpha-1)k_3 - 2\alpha k_1 \\ &+ a(s-1)(s-\alpha)e^{(s-1)w} - \frac{3}{2}k_4\Big]|\nabla w|^2 - \frac{n}{2}\Big(\frac{q}{\alpha}(k_2+k_3)^2 + 3k_4\Big) \\ &+ e^{(s-1)w}\Delta a + 2(s-\alpha)e^{(s-1)w}\nabla w \nabla a\Big\} \\ &- a(s-1)e^{(s-1)w}G - \frac{G}{t},\end{aligned} \tag{3.1}$$

where

$$w = \log u,$$
$$G = t(\alpha|\nabla w|^2 + ae^{(s-1)w} - w_t - \lambda),$$

and α is any constant satisfying $0 < \alpha < 1$ and $1 < p, q < \infty$ such that $1/p + 1/q = 1$.

Proof. Define

$$G = t(\alpha|\nabla w|^2 + ae^{(s-1)w} - w_t - \lambda)$$

then

$$\Delta G = t(\alpha\Delta|\nabla w|^2 + a\Delta(e^{(s-1)w}) - \Delta w_t). \tag{3.2}$$

By the Bochner-Weitzenböck's formula and the assumption on the Ricci curvature tensor we have

$$\Delta|\nabla w|^2 \geq 2|\nabla^2 w|^2 + 2\nabla w \nabla(\Delta w) - 2k_1|\nabla w|^2. \tag{3.3}$$

Recall the following evolutions under the flow (1.4) from [2, Lemma 2.1]

$$(|\nabla w|^2)_t = -2h\nabla w \nabla w + 2\nabla w \nabla w_t \tag{3.4}$$

and

$$\Delta w_t = (\Delta w)_t + 2h\nabla\nabla w + 2\Big(div\ h - \frac{1}{2}\nabla\mathcal{H}\Big)\nabla w, \tag{3.5}$$

where div is the divergence operator i.e., $(div\ h)_k = g^{ij}\nabla_i h_{jk}$ and $\mathcal{H} = g^{ij}h_{ij}$, the metric trace of $(0,2)$-tensor h. .

Note that

$$\begin{aligned}\Delta w &= -|\nabla w|^2 - ae^{(s-1)w} + w_t + \lambda \\ &= \Big(\frac{1}{\alpha} - 1\Big)\Big(ae^{(s-1)w} - w_t - \lambda\Big) - \frac{G}{\alpha t} \\ &= (1-\alpha)(-|\nabla w|^2) - \frac{G}{t}.\end{aligned} \tag{3.6}$$

We now compute using (3.3) and (3.6)

$$\begin{aligned}\alpha\Delta|\nabla w|^2 &\geq 2\alpha|\nabla^2 w|^2 + 2\alpha\nabla w\nabla(\Delta w) - 2\alpha k_1|\nabla w|^2 \\ &= 2\alpha|\nabla^2 w|^2 + 2\alpha\nabla w\nabla\Big[\Big(\frac{1}{\alpha} - 1\Big)\Big(ae^{(s-1)w} - w_t - \lambda\Big) - \frac{G}{\alpha t}\Big] \\ &\quad - 2\alpha k_1|\nabla w|^2 \\ &= 2\alpha|\nabla^2 w|^2 + 2(1-\alpha)e^{(s-1)w}\nabla w\nabla a + 2a(1-\alpha)(s-1)e^{(s-1)w}|\nabla w|^2 \\ &\quad - 2(1-\alpha)\nabla w\nabla w_t - \frac{2}{t}\nabla w\nabla G - 2\alpha k_1|\nabla w|^2.\end{aligned}$$

By direct computation using (3.6)

$$\begin{aligned}\Delta\Big(ae^{(s-1)w}\Big) &= e^{(s-1)w}\Delta a + 2(s-1)e^{(s-1)w}\nabla w\nabla a + a(s-1)^2e^{(s-1)w}|\nabla w|^2 \\ &\quad + a(s-1)e^{(s-1)w}\Delta w \\ &= e^{(s-1)w}\Delta a + 2(s-1)e^{(s-1)w}\nabla w\nabla a + a(s-1)^2e^{(s-1)w}|\nabla w|^2 \\ &\quad + a(s-1)e^{(s-1)w}\Big[(1-\alpha)(-|\nabla w|^2) - \frac{G}{t}\Big]\end{aligned}$$

and

$$\begin{aligned}\Delta w_t &= (\Delta w)_t + 2h\nabla\nabla w + 2\Big(div\ h - \frac{1}{2}\nabla\mathcal{H}\Big)\nabla w \\ &= (-|\nabla w|^2 - ae^{(s-1)w} + \lambda + w_t)_t + 2h\nabla\nabla w + 2\Big(div\ h - \frac{1}{2}\nabla\mathcal{H}\Big)\nabla w \\ &= -(\alpha|\nabla w|^2 + ae^{(s-1)w} - w_t)_t - (1-\alpha)(|\nabla w|^2)_t + 2h\nabla\nabla w \\ &\quad + 2\Big(div\ h - \frac{1}{2}\nabla\mathcal{H}\Big)\nabla w \\ &= -\frac{G_t}{t} + \frac{G}{t^2} + 2(1-\alpha)h(\nabla w, \nabla w) - 2(1-\alpha)\nabla w\nabla w_t + 2h\nabla\nabla w \\ &\quad + 2\Big(div\ h - \frac{1}{2}\nabla\mathcal{H}\Big)\nabla w,\end{aligned}$$

where we have used the identities

$$G_t = \frac{G}{t} + t(\alpha|\nabla w|^2 + ae^{(s-1)w} - w_t)_t$$

and

$$(1-\alpha)(|\nabla w|^2)_t = -2(1-\alpha)h(\nabla w, \nabla w) + 2(1-\alpha)\nabla w \nabla w_t.$$

Putting all of the above into (3.2) we obtain

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)G \geq -2\nabla w \nabla G + t\Big\{2\alpha|\nabla^2 w|^2 - 2h\nabla\nabla w - 2(1-\alpha)h(\nabla w, \nabla w) + a(s-1)(s-\alpha)e^{(s-1)w}|\nabla w|^2 + 2(s-\alpha)e^{(s-1)w}\nabla w \nabla a + e^{(s-1)w}\Delta a - 2\alpha k_1|\nabla w|^2 - 2\Big(div\ h - \frac{1}{2}\nabla\mathcal{H}\Big)\nabla w\Big\} - a(s-1)e^{(s-1)}G - \frac{G}{t}.$$

Note that

$$|\nabla^2 w|^2 \geq \frac{1}{n}(\Delta w)^2 \tag{3.7}$$

by Cauchy-Schwarz inequality. The boundedness assumption on h can also be written as

$$-(k_2+k_3)g \leq |h| \leq (k_2+k_3)g$$

so that

$$\sup_M |h|^2 \leq n(k_2+k_3)^2. \tag{3.8}$$

Hence, choosing any two real numbers $1 < p, q < \infty$ such that 1/p+1/q=1, we have

$$2\alpha|\nabla^2 w|^2 - 2h\nabla\nabla w = \frac{2\alpha}{p}|\nabla^2 w|^2 + 2\Big(\frac{\alpha}{q}|\nabla^2 w|^2 - h\nabla\nabla w\Big) \geq \frac{2\alpha}{np}(\Delta w) - \frac{nq}{2\alpha}(k_2+k_3)^2$$

where we have used identities (3.7), (3.8) and an inequality of form $Ax^2 - Bx \geq -B^2/4A$.

On the other hand

$$2\Big(div\ h - \frac{1}{2}\nabla\mathcal{H}\Big)\nabla w = 2\Big(g^{kl}\nabla_k h_{li} - \frac{1}{2}g^{kl}\nabla_i h_{kl}\Big)\nabla_j w \leq 2\Big(\frac{3}{2}|g||\nabla h|\Big)|\nabla w| \leq 3n^{\frac{1}{2}}k_4|\nabla w| \leq 3k_4\Big(\frac{n}{2} + \frac{|\nabla^2 w|^2}{2}\Big).$$

The last inequality follows from Young's inequality.

Therefore

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)G \geq -2\nabla w \nabla G + t\Big\{\frac{2\alpha}{np}(\Delta w)^2 + 2(\alpha-1)k_3|\nabla w|^2 - 2\alpha k_1|\nabla w|^2 + a(s-1)(s-\alpha)e^{(s-1)w}|\nabla w|^2 + 2(s-\alpha)e^{(s-1)w}\nabla w \nabla a + e^{(s-1)w}\Delta a - \frac{nq}{2\alpha}(k_2+k_3)^2 - \frac{3}{2}nk_4 - \frac{3}{2}k_4|\nabla w|^2\Big\} - a(s-1)e^{(s-1)w}G - \frac{G}{t}.$$

Rearranging, we arrive at the desired result. Our calculation is valid in the cube $\mathcal{Q}_{2R,T}$.

□

3.2. **The cut-off function.** Define a smooth cut-off function $\varphi(x,t)$ with support in the geodesic cube

$$\mathcal{Q}_{2R,T} := \{(x,t) \in (0,T] \times M : d(x,y,t) \leq 2R, R > 0\}.$$

Let $\psi(r)$ on $[0,+\infty)$ be a C^2 function defined on $[0,\infty)$, such that $\psi(r) = 1$ for $0 \leq r \leq 1$, $\psi(r) = 0$ for $2 \leq r \leq +\infty$ and $0 \leq \psi(r) \leq 1$. Furthermore, let $\psi(r)$ satisfies the following estimates

$$-C_1 \leq \frac{\psi'(r)}{\psi^{\frac{1}{2}}(r)} \leq 0, \qquad \frac{|\psi'(r)|^2}{\psi} \leq C_1 \quad \text{and} \quad \psi''(r) \geq -C_2$$

for some absolute constants C_1, C_2. Fix a point $p \in M$, denote by $d(p,x)$ the geodesic distance between points p and x in M. Let $R \geq 1$ and define a smooth function

$$\varphi(x,t) = \psi\Big(\frac{d(x,p,t)}{R}\Big) \quad \textit{and} \quad \varphi\Big|_{\mathcal{Q}_{2R,T}} = 1.$$

Using Calabi's trick (see the argument of Li-Yau in [15]) we assume without loss of generality that the function $\varphi(x,t)$ is everywhere smooth with support in $\mathcal{Q}_{2R,T}(p)$ since $\psi(r)$ is in general Lipschitz. Then by direct calculation we have on $\mathcal{Q}_{2R,T}$

$$\frac{|\nabla\varphi|^2}{\varphi} = \frac{|\psi'|^2 \cdot |\nabla d|^2}{R^2\psi} \leq \frac{C_1^2}{R^2}$$

and by the Laplacian comparison theorem [18] (since $Ric \geq -k_1 g$) we have

$$\Delta d \leq (n-1)\sqrt{k_1}\coth(\sqrt{k_1}R)$$

and

$$\Delta\varphi = \frac{\psi'\Delta d}{R} + \frac{\psi''|\nabla d|^2}{R^2} \geq -\frac{C_1(n-1)\sqrt{k_1}R + C_2}{R^2}.$$

Next is to estimate time derivative of φ: consider a fixed smooth path $\gamma : [a,b] \to M$ whose length at time t is given by $d(\gamma) = \int_a^b |\gamma'(\sigma)|_{g(t)} d\sigma$, where σ is the arc length along the path and $d\sigma$ is its element. Differentiating we get

$$\frac{\partial}{\partial t}(d(\gamma)) = \frac{1}{2}\int_a^b \Big|\gamma'(\sigma)\Big|_{g(t)}^{-1} \frac{\partial g}{\partial t}\Big(\gamma'(\sigma), \gamma'(\sigma)\Big) d\sigma = \sup_M \int_\gamma |h|(X,X) d\sigma,$$

where X is the unit tangent vector to the path γ. Now

$$\begin{aligned} \frac{\partial}{\partial t}\varphi = \frac{\psi'}{R}\frac{d}{dt}(d(t)) &= \frac{|\psi'|}{R}\sup_M \int_\gamma |h|(X,X) d\sigma \\ &\leq \frac{\sqrt{C_1}\psi^{\frac{1}{2}}}{R}(k_2 + k_3)\int_\gamma d\sigma \leq \frac{\sqrt{C_1}}{R}(k_2 + k_3) \end{aligned}$$

by choosing $0 \leq \sigma \leq 1$, $R \geq 1$ and fixing the path to be of length not more than unit so that it always stays inside the geodesic cube $\mathcal{Q}_{2R,T}$. Hence we denote

$$\begin{aligned}\Big(\Delta - \frac{\partial}{\partial t}\Big)\varphi &\geq -\frac{C_1(n-1)\sqrt{k_1}R + C_2}{R^2} - \frac{\sqrt{C_1}}{R}(k_2+k_3)\\ &=: D_1(n,k_1,k_2,k_3,R) = D_1.\end{aligned}$$

which will be used in the proof of our result.

3.3. **Gradient estimates.**

Proof. **of Theorem 2.1**. Using the same notations as in the last lemma. For a fixed $\tau \in (0,T]$ and a smooth cut-off function $\varphi(t,x)$ (chosen as before). Suppose (φG) attains its maximum value at $(t_0,x_0) \in ([0,T]\times M)$, for $t_0 > 0$. If $(\varphi G)(t_0,x_0) \leq 0$ for any $R \geq 1$, then the result holds trivially in $[0,T]\times M$ and we are done. Hence we may assume without loss of generality that there exists $(\varphi G)(t_0,x_0) > 0$. Then since $(\varphi G)(0,x) = 0$ for all $x \in M$, we have by the maximum principle that

$$\nabla(\varphi G)(t_0,x_0) = 0, \quad \frac{\partial}{\partial t}(\varphi G)(t_0,x_0) \geq 0, \quad \Delta(\varphi G)(t_0,x_0) \leq 0, \tag{3.9}$$

where the function (φG) is being considered with support on $\mathcal{Q}_{2R,T}$ and we have assumed that $(\varphi G)(t_0,x_0) > 0$ for $t_0 > 0$.

It then follows that

$$0 \geq \Big(\Delta - \frac{\partial}{\partial t}\Big)(\varphi G) \geq 2\nabla\varphi\nabla G + D_1 G + \varphi\Big(\Delta - \frac{\partial}{\partial t}\Big)G \tag{3.10}$$

The above inequality holds in the part of $\mathcal{Q}_{2R,T}$ where $\varphi(x,t)$ is strictly positive ($0 < \varphi(t,x) \leq 1$). Notice that since $\nabla(\varphi G) = 0$, the product rule tells us that we can always replace $-G\nabla\varphi$ with $\varphi\nabla G$ at the maximum point (t_0,x_0). Indeed, the following identity holds

$$2\nabla\varphi\nabla G = 2\varphi\frac{\nabla\varphi}{\varphi}\nabla G = 2\frac{\nabla\varphi}{\varphi}(-G\nabla\varphi) \geq -2G\frac{C_1^2}{R^2},$$

so by (3.10) we obtain

$$0 \geq -D_2 G + \varphi\Big(\Delta - \frac{\partial}{\partial t}\Big)G \tag{3.11}$$

where

$$D_2 = D_2(n,k_1,k_2,k_3,R) := \frac{C_1(n-1)\sqrt{k_1}R + \sqrt{C_1\widetilde{K}}R + 2C_1^2 + C_2}{R^2}.$$

and $\widetilde{K} = (k_2+k_3)^2$.

Using Lemma 3.1 in (3.11) at the maximum point (t_0, x_0) we have

$$0 \geq -D_2 G + \varphi\Big\{ -\frac{G}{t_0} - 2\nabla w \nabla G - a(s-1)e^{(s-1)w}G + \frac{2\alpha t_0}{np}(\Delta w)^2$$
$$+ t_0\Big[2(\alpha-1)k_3 - 2\alpha k_1 + a(s-1)(s-\alpha)e^{(s-1)w} - \frac{3}{2}k_4\Big]|\nabla w|^2 \tag{3.12}$$
$$+ t_0 e^{(s-1)w}\Delta a + 2t_0(s-\alpha)e^{(s-1)w}\nabla w \nabla a - \frac{nt_0}{2}\Big(\frac{q}{\alpha}\widetilde{K} + 3k_4\Big)\Big\}.$$

Using the identity

$$-2\varphi\nabla w\nabla G = 2G\nabla\varphi\nabla w = 2G|\nabla w|\varphi\frac{|\nabla\varphi|}{\varphi} \geq -2\frac{C_1}{R}|\nabla w|\varphi^{\frac{1}{2}}G$$

the condition $u \leq \mathbb{M}$ with $\widetilde{\mathbb{M}} = \mathbb{M}^{s-1}$ and the boundedness assumptions on a, $|\nabla a|$ and Δa, (i.e., $\|a\| = \|a\|_{L^\infty(\mathcal{Q}_{2R})}$, $|\nabla a| \leq \gamma$ and $\Delta a \leq \theta$) we obtain

$$0 \geq -D_2 G - \frac{\varphi G}{t_0} - \frac{2C_1}{R}\varphi^{\frac{1}{2}}|\nabla w|G - \|a\|\varphi(s-1)\widetilde{\mathbb{M}}G + \frac{2\alpha t_0}{np}\varphi(\Delta w)^2$$
$$- t_0\varphi\Big[2(1-\alpha)k_3 + 2\alpha k_1 + \|a\|(s-1)(s-\alpha)\widetilde{\mathbb{M}} + \frac{3}{2}k_4\Big]|\nabla w|^2$$
$$- t_0\varphi\Big[\widetilde{\mathbb{M}}\theta + 2(s-\alpha)\widetilde{\mathbb{M}}|\nabla w|\gamma + \frac{n}{2}\Big(\frac{q}{\alpha}\widetilde{K} + 3k_4\Big)\Big]$$
$$\geq -D_2 G - \frac{\varphi G}{t_0} - \frac{2C_1}{R}\varphi^{\frac{1}{2}}|\nabla w|G - \|a\|\varphi(s-1)\widetilde{\mathbb{M}}G + \frac{2\alpha t_0}{np}\varphi(\Delta w)^2$$
$$- t_0\varphi\Big[2(1-\alpha)k_3 + 2\alpha k_1 + \|a\|(s-1)(s-\alpha)\widetilde{\mathbb{M}} + \frac{3}{2}k_4$$
$$+ \frac{2(s-\alpha)}{\varepsilon}\widetilde{\mathbb{M}}\Big]|\nabla w|^2 - t_0\varphi\Big[\widetilde{\mathbb{M}}\theta + 2\varepsilon(s-\alpha)\widetilde{\mathbb{M}}\gamma^2 + \frac{n}{2}\Big(\frac{q}{\alpha}\widetilde{K} + 3k_4\Big)\Big].$$

where we have used the following inequality (for $\varepsilon > 0$)

$$-2(s-\alpha)\widetilde{\mathbb{M}}|\nabla w|\gamma \geq -\frac{2(s-\alpha)}{\varepsilon}\widetilde{\mathbb{M}}|\nabla w|^2 - 2\varepsilon(s-\alpha)\widetilde{\mathbb{M}}\gamma^2$$

Multiplying the last inequality through by $(t_0\varphi)$, we get

$$0 \geq -t_0\varphi D_2 G - \varphi^2 G - \frac{2C_1}{R}t_0\varphi^{\frac{3}{2}}|\nabla w|G - \|a\|t_0\varphi^2(s-1)\widetilde{\mathbb{M}}G$$
$$+ \frac{2\alpha t_0^2}{np}\varphi^2(\Delta w)^2 - t_0^2\varphi^2 A_1|\nabla w|^2 - t_0^2\varphi^2 A_2 \tag{3.13}$$

where

$$A_1 = 2(1-\alpha)k_3 + 2\alpha k_1 + \|a\|(s-1)(s-\alpha)\widetilde{\mathbb{M}} + \frac{3}{2}k_4 + \frac{2(s-\alpha)}{\varepsilon}\widetilde{\mathbb{M}}$$

and

$$A_2 = \widetilde{\mathbb{M}}\theta + 2\varepsilon(s-\alpha)\widetilde{\mathbb{M}}\gamma^2 + \frac{n}{2}\Big(\frac{q}{\alpha}\widetilde{K} + 3k_4\Big).$$

Using the identity

$$(\Delta w)^2 = (|\nabla w|^2 + ae^{(s-1)w} - w_t - \lambda)^2$$

and a technique as in Li-Yau paper [15], when $t_0 > 0$, we denote by

$$y = \varphi|\nabla w|^2 \quad \text{and} \quad z = \varphi(w_t - ae^{(s-1)w} + \lambda)$$

to obtain

$$\varphi^2|\nabla w|^2 = \varphi y \le y, \quad y^{\frac{1}{2}}(\alpha y - z) = \frac{1}{t_0}|\nabla w|\varphi^{\frac{3}{2}}G \quad \text{and} \quad \varphi G = t_0(\alpha y - z).$$

Then (3.13) implies

$$\begin{aligned} 0 \ge \frac{2t_0^2}{n}&\left(\frac{\alpha}{p}(y-z)^2 - \frac{A_1}{2}ny - \frac{nC_1}{R}y^{\frac{1}{2}}(\alpha y - z)\right) \\ &+\Big(-D_2t_0 - 1 - \|a\|t_0(s-1)\widetilde{\mathbb{M}}\Big)(\varphi G) - t_0^2A_2. \end{aligned} \tag{3.14}$$

Notice by direct calculation that

$$\begin{aligned} (y-z)^2 &= \Big(\alpha(y - \frac{z}{\alpha}) + (1-\alpha)y\Big)^2 \\ &= (\alpha y - z)^2 + (1-\alpha)^2y^2 + 2(1-\alpha)y(\alpha y - z). \end{aligned}$$

Then, the first term in the right hand side of inequality (3.14) can be simplified as follows:

$$\begin{aligned} &\frac{2t_0^2}{n}\Big\{\frac{\alpha}{p}\Big[(y-z)^2 - \frac{A_1np}{2\alpha}y - \frac{np}{\alpha}\frac{C_1}{R}y(\alpha y - z)\Big]\Big\} \\ &\quad = \frac{2t_0^2}{n}\Big\{\frac{\alpha}{p}\Big[(\alpha y - z)^2 + \Big((1-\alpha)^2y^2 - \frac{A_1np}{2\alpha}y\Big) \\ &\qquad + \Big(2(1-\alpha)y - \frac{np}{\alpha}\frac{C_1}{R}y^{\frac{1}{2}}\Big)(\alpha y - z)\Big]\Big\} \\ &\quad \ge \frac{2t_0^2}{n}\Big\{\frac{\alpha}{p}(\alpha y - z)^2 - \frac{n^2pA_1^2}{16\alpha(1-\alpha)^2} - \frac{n^2pC_1^2}{8R^2\alpha(1-\alpha)}(\alpha y - z)\Big\} \\ &\quad = \frac{2\alpha}{np}(\varphi G)^2 - \frac{npA_1^2}{8\alpha(1-\alpha)^2}t_0^2 - \frac{npC_1^2}{4R^2\alpha(1-\alpha)}t_0(\varphi G). \end{aligned}$$

We have used the inequality of the form $ax^2 - bx \ge -\frac{b^2}{4a}$, $(a, b > 0)$, to compute

$$(1-\alpha)^2y^2 - \frac{A_1np}{2\alpha}y \ge -\frac{n^2p^2A_1^2}{16\alpha^2(1-\alpha)^2} \tag{3.15}$$

and

$$2(1-\alpha)y - \frac{np}{\alpha}\frac{C_1}{R}y^{\frac{1}{2}} \ge -\frac{n^2p^2C_1^2}{8R^2\alpha^2(1-\alpha)}. \tag{3.16}$$

Therefore putting all these together into (3.14), we get a quadratic polynomial in (φF)

$$0 \geq \frac{2\alpha}{np}(\varphi G)^2 + \Big(- D_2 t_0 - 1 - \|a\| t_0 (s-1)\widetilde{\mathbb{M}} - \frac{npC_1^2}{4R^2\alpha(1-\alpha)} t_0 \Big)(\varphi G) - \Big(\frac{npA_1^2}{8\alpha(1-\alpha)^2} + A_2 \Big) t_0^2.$$

Then we develope a formula for quadratic inequality of the form $ax^2 + bx + c \leq 0$, for $x \in \mathbb{R}$. Note that when $a > 0$ and $c < 0$, then $b^2 - 4ac > 0$ and we have an upper bound

$$x \leq \frac{-b + \sqrt{b^2 - 4ac}}{2a} \leq \frac{1}{a}\Big\{ - b + \sqrt{-ac} \Big\}. \tag{3.17}$$

Hence, we have by applying (3.17)

$$\varphi G \leq \frac{np}{2\alpha} + \frac{np}{2\alpha}\Big\{ D_2 t_0 + |a| t_0 (s-1)\widetilde{\mathbb{M}} + \frac{C_1^2 np}{4R^2\alpha(1-\alpha)} t_0 \Big\} + \sqrt{\frac{np}{2\alpha}\Big(\frac{npA_1^2}{8\alpha(1-\alpha)^2} + A_2 \Big)} t_0.$$

To obtain the required bound on $G(\tau, x)$ for an appropriate range of $x \in M$, we take $\varphi(\tau, x) \equiv 1$ whenever $d(x, x_0, \tau) < 2R$ and since (t_0, x_0) is the maximum point for (φG) in $\mathcal{Q}_{2R,T}$, we have

$$\begin{aligned} G(\tau, x) &= (\varphi G)(\tau, x) \leq (\varphi G)(t_0, x_0) \\ &\leq \frac{np}{2\alpha} + \frac{np}{2\alpha}\Big\{ D_2 t_0 + \|a\| t_0 (s-1)\widetilde{\mathbb{M}} + \frac{C_1^2 np}{4R^2\alpha(1-\alpha)} t_0 \Big\} \\ &\quad + \sqrt{\frac{np}{2\alpha}\Big(\frac{npA_1^2}{8\alpha(1-\alpha)^2} + A_2 \Big)} t_0 \end{aligned}$$

for all $x \in M$, such that $d(x, x_0, \tau) < R$ and $\tau \in (0, T]$ was arbitrarily chosen. This ends the proof of Theorem 2.1. □

Remark 3.2. Global estimate follows by letting $R \to \infty$ for all $t > 0$. Clearly, if R goes to infinity, we have the estimate

$$\alpha \frac{|\nabla u|^2}{u^2} + a u^{(s-1)} - \frac{u_t}{u} \leq \frac{np}{2\alpha t} + A_3 \tag{3.18}$$

where

$$A_3 = \frac{np}{2\alpha}\|a\|(s-1)\widetilde{\mathbb{M}} + \sqrt{\frac{np}{2\alpha}\Big(\frac{nA_1^2 p}{8\alpha(1-\alpha)^2} + A_2 \Big)} + \lambda.$$

3.4. **Harnack inequality.**

Proof. **of Corollary 2.3.** Let $\gamma : [t_1, t_2] \to M$ be a smooth path connecting points $x_1 = \gamma(t_1)$ and $x_2 = \gamma(t_2)$ in M. Note that equation (2.4) implies

$$w_t \geq \alpha |\nabla w|^2 + a e^{(s-1)w} - \frac{np}{2\alpha t} - A_3 \tag{3.19}$$

where A_3 is as defined in Remark 3.2.

A straightforward computation yields

$$\begin{aligned} w(t_2, x_2) - w(t_1, x_1) &= \int_{t_1}^{t_2} \frac{d}{dt}\Big(w(t, \gamma(t)\Big)dt \\ &= \int_{t_1}^{t_2} \Big(w_t + \langle \nabla w, \frac{d}{dt}\gamma(t)\rangle\Big)dt \\ &\geq \int_{t_1}^{t_2} \Big(\alpha|\nabla w|^2 + ae^{(s-1)w} - \frac{np}{2\alpha t} - A_3 + |\nabla w|\Big|\frac{d}{dt}\gamma(t)\Big|\Big)dt \\ &\geq -\int_{t_1}^{t_2} \Big(\frac{1}{4\alpha}\Big|\frac{d}{dt}\gamma(t)\Big|^2 + \|a\|e^{(s-1)w} + \frac{np}{2\alpha t} + A_3\Big)dt. \end{aligned}$$

In the computation above we have used (3.19) to arrive at the inequality in the third line and we applied an inequality of the form $Ay^2 + By \geq -B^2/4A$ to obtain the inequality in the last line. Precisely, we took $|\nabla w|$ as a variable and the integrand as a quadratic polynomial in $|\nabla w|$.

Notice that the condition $u \leq \mathbb{M}$ implies $e^{(s-1)w} \leq \widetilde{\mathbb{M}}$ and $-|a|e^{(s-1)w} \geq -|a|\widetilde{\mathbb{M}}$. Define

$$\Gamma(t_1, x_1; t_2, x_2) = \inf_{\gamma} \int_{t_1}^{t_2} |\frac{d}{dt}\gamma(t)|^2 dt$$

with the infimum taken over all the smooth paths γ. Then

$$w(t_2, x_2) - w(t_1, x_1) \geq -\frac{1}{4\alpha}\Gamma(t_1, x_1; t_2, x_2) - (A_3 + \|a\|\widetilde{\mathbb{M}})(t_2 - t_1) - \frac{np}{2\alpha}\log\Big(\frac{t_2}{t_1}\Big).$$

Notice that the expression in the left hand side of the last inequality is the same as

$$w(t_2, x_2) - w(t_1, x_1) = \log\frac{u(t_2, x_2)}{u(t_1, x_1)},$$

then the assertion of the corollary follows by exponentiation and that concludes its proof.

□

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