

TENSION SPLINE APPROACH TO SINGULARLY PERTURBED DELAY DIFFERENTIAL EQUATIONS INVOLVING LARGE DELAY

DANY JOY^{1*} AND AKHILA MARIYA REGAL²

ABSTRACT. To solve the singularly perturbed delay differential equations of convection type, spline in tension technique is used. Taylor series approximation is used to tackle the first order derivative term involved in the problem and finally a tridiagonal scheme is obtained, which is then solved by Thomas algorithm. An example is illustrated to show the method's applicability and it is solved numerically for different values of perturbation parameter. The maximum absolute errors and rate of convergence are given in tables and it is seen that our method is of first order convergent.

1. INTRODUCTION

In the biosciences and control theory, delay differential equations are crucial for the mathematical modelling of a variety of real-world phenomena. There will almost always be time delays in any system that uses a feedback control. These occur because the process of sensing information and responding to it needs short time. Retarded delay differential equations have the delay argument missing from the highest order derivative term. If we restrict this category to those in which the largest derivative term is multiplied by a small parameter, then such a class of problems is known as a singularly perturbed retarded delay differential equation. These issues are dependent on a small positive parameter (perturbation parameter) in a way that causes the solution to vary quickly in some regions and slowly in others. As a result, the solution typically behaves predictably and varies slowly outside of thin transition layers where it changes quickly or jumps. Due to their applications in numerous scientific and technical fields, such as micro scale heat transfer [3], control theory [7], hydrodynamics of liquid helium [4], the first exit-time problem [6], describing the human pupil-light reflex [8] and models for different physiological processes or diseases [9] etc. In recent years, there has been a rise in the level of interest around the numerical investigation of singularly perturbed delay differential equations (SPDDEs).

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* Corresponding author.

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Tang and Geng [13] investigated a fitted reproducing kernel method to solve singularly perturbed delay differential equations. Selvi and Ramanujam [11] suggested an iterative method in which the solution of delay term and non- delay term is computed by different methods. Kudu and et al [5] developed implicit difference rules for the differential component and for the integral part, there are composite numerical quadrature rules. Sahu and Mohapatra [10] developed an upwind scheme on the fitted Shishkin mesh. The authors in [1] worked on non-smooth delay differential equations and the authors in [2] used cubic B-spline method for solving singularly perturbed delay problems. Delay differential equations(DDEs) of third order is handled by N Senu et al [12] .

2. DESCRIPTION OF THE PROBLEM

We consider the following SPDDE of convection diffusion type with shift in space.

$$-\epsilon x''(z) - b(z)x'(z) + c(z)x(z) + d(z)x(z-1) = f(z), \quad z \in \Omega = (0, 2) \quad (2.1)$$

with

$$\begin{aligned} x(z) &= \phi'(z), \quad z \in [-1, 0] \\ x(2) &= 0 \end{aligned} \quad (2.2)$$

where $0 < \epsilon \ll 1$. Here, ϵ is a small quantity known as perturbation parameter. All the functions in (2.1) and (2.2) are such that $b(z) \geq b^* > 0$, $d(z) \geq 0$. Let $\Omega_1 = (0, 1)$ and $\Omega_2 = (1, 2)$. (2.1) and (2.2) can be written as

$$Lx \equiv l(z), \quad z \in \Omega$$

where

$$Lx = \begin{cases} L_1x(z) &= -\epsilon x''(z) - b(z)x'(z) + c(z)x(z), & z \in \Omega_1 \\ L_2x(z) &= -\epsilon x''(z) - b(z)x'(z) + c(z)x(z) + d(z)x(z-1), & z \in \Omega_2 \end{cases}$$

$$l(z) = \begin{cases} f(z) - d(z)x(z-1), & z \in \Omega_1 \\ f(z), & z \in \Omega_2 \end{cases}$$

Let $Kx(2) = x(2)$, $\Omega^* = \Omega_1 \cup \Omega_2$ and $X = C^0(\Omega) \cup C^1(\Omega) \cup C^2(\Omega)$.

3. STABILITY ANALYSIS

Lemma 3.1. Let $\psi(z)$ be any function in X such that $\psi(0) \geq 0$, $K\psi(2) \geq 0$, $L_1\psi(2) \geq 0$ for all $z \in \Omega_1$, $L_2\psi(2) \geq 0$ for all $z \in \Omega_2$ and $[\psi'](1) \leq 0$, then $\psi(z) \geq 0$ for all $z \in \Omega$.

Proof: Define the function $t(u)$ as:

$$t(z) = \begin{cases} \frac{1}{8} + \frac{z}{2}, & z \in [0, 1] \\ \frac{3}{8} + \frac{z}{4}, & z \in [1, 2] \end{cases}$$

Then $t(z)$ is positive for all $z \in \bar{\Omega}$.

Also $Lt(z) > 0$ for all $z \in \Omega_1 \cup \Omega_2$, $t(0) > 0$, $Kt(2) > 0$ and $[t]'(1) < 0$. Let

$$\bar{\mu} = \max \left\{ \frac{-\xi(z)}{t(z)} : z \in \bar{\Omega} \right\}.$$

Then $\exists z_0 \in \bar{\Omega}$ satisfying $\xi(z_0) + \bar{\mu}t(z_0) = 0$ and $\xi(z) + \bar{\mu}t(z) \geq 0$, for all $u \in \bar{\Omega}$. Hence $\xi + \bar{\mu}t(z_0)$ gives minimum value.

Now, suppose that the theorem is not true. Then $\bar{\mu} > 0$.

Case 1: $z_0 = 0$.

Then

$$0 < (\xi + \bar{\mu}t)(0) = \xi(0) + \bar{\mu}t(0) = 0,$$

which is a contradiction.

Case 2: $z_0 \in \Omega_1$.

Then

$$\begin{aligned} 0 < L(\xi + \bar{\mu}t)(z_0) &= -\epsilon(\xi + \bar{\mu}t)''(z_0) \\ &\quad + a(z_0)(\xi + \bar{\mu}t)'(z_0) + b(z_0)(\xi + \bar{\mu}t)(z_0) \leq 0, \end{aligned}$$

a contradiction.

Case 3: $z_0 = 1$.

Then

$$0 \leq [(\xi + \bar{\mu}t)'](1) = [\xi'](1) + \bar{\mu}[t'](1) < 0,$$

it is a contradiction.

Case 4: $z_0 \in \Omega_2$.

Then

$$\begin{aligned} 0 < L(\xi + \bar{\mu}t)(z_0) &= -\epsilon(\xi + \bar{\mu}t)''(z_0) + a(z_0)(\xi + \bar{\mu}t)'(z_0) \\ &\quad + b(z_0)(\xi + \bar{\mu}t)(z_0) + c(z_0)(\xi + \bar{\mu}t)(z_0 - 1) \leq 0, \end{aligned}$$

again a contradiction.

Case 5: $z_0 = 2$.

Then

$$0 \leq K(\xi + \bar{\mu}t)'(2) = (\xi + \bar{\mu}t)'(2) - \epsilon \int_0^2 g(z)(\xi + \bar{\mu}t)(z_0) du \leq 0,$$

which is a contradiction. Hence the proof.

Lemma 3.2. The solution $x(z)$ for (2.1)- (2.2) satisfies

$$|x(z)| \leq \bar{P} \max\{|x(0)|, |Kx(2)|, \sup_{z \in \Omega^*} |Lx(z)|\}, z \in \bar{\Omega}$$

Proof:

This is proved using the above lemma with the functions:

$$\theta^\pm(z) = \bar{C} \cdot \bar{M} \cdot t(z) \pm x(z), z \in \bar{\Omega},$$

where $\bar{M} = \max\{|x(0)|, |Kx(2)|, \sup_{z \in \Omega^*} |Lx(u)|\}$ and $t(x)$ are the test functions in the above lemma.

Lemma 3.3. Let $x(z)$ be the solution for the problem (2.1)- (2.2). Then

$$\|x^{(k)}(z)\|_{\Omega^*} \leq P\epsilon^{-k}, \text{ for } k = 1, 2, 3$$

4. METHOD

Let $[0, 2]$ be discretized as $z_i = ih$, $h = 2/N$, $z_0 = 0$ and $z_{2N} = 2$. A function $w(z, t)$ satisfying the following in $[z_i, z_{i+1}]$, where $w(z_i) = x_i$ and $t > 0$ is cubic spline in tension.

$$w''(z) - tw(z) = [w''(z_{i+1}) - tw(z_{i+1})]\frac{(z - z_i)}{h} + [w''(z_i) - tw(z_i)]\frac{(z_{i+1} - z)}{h} \quad (4.1)$$

Solving the equation (4.1) and apply $w(z_{i+1}) = x_{i+1}$, $w(z_i) = x_i$. Also let $t' = t^{1/2}h$ and $S_i = w''(z_i)$ to obtain

$$w(z) = \frac{\sinh(t'(z - z_i)/h)}{\sinh t'} \frac{h^2}{t'^2} S_{i+1} + \frac{\sinh(t'(z_{i+1} - z)/h)}{\sinh t'} \frac{h^2}{t'^2} S_i - \frac{(z - z_i)h}{t'^2} S_{i+1} + \frac{(z - z_i)}{h} x_{i+1} - \frac{(z_{i+1} - z)h}{t'^2} S_i + \frac{(z_{i+1} - z)}{h} x_i \quad (4.2)$$

Differentiate equation (4.2) and let $z \rightarrow z_i$ to obtain $w'(z_i^+)$. Apply the same method in (z_{i-1}, z_i) to obtain $w'(z_i^-)$ and equate them to obtain the following system.

$$(t_1 S_{i-1} + 2t_2 S_i + t_1 S_{i+1}) = \frac{x_{i+1} - 2x_i + x_{i-1}}{h^2}, \quad i = 1(1)2N - 1 \quad (4.3)$$

where t_1 and t_2 are given by $(\frac{1}{t^2} - \frac{1}{t^2 \sinh t})$ and $(\frac{\coth t}{t} - \frac{1}{t^2})$ respectively and $w''(z_i) = S_i$. If $t_1 + t_2 = \frac{1}{2}$, the system is consistent.

At $z = z_i$,

$$\epsilon S_i = c(z_i)x_i + d(z_i)x(z_{i-1}) - b(z_i)x'_i - f(z_i) \quad (4.4)$$

with $x_i = \phi_i$, for $-N \leq i \leq 0$ and $z_{2N} = 0$. We use equation (4.4) and Taylor series approximation for x'_{i-1}, x'_i, x'_{i+1} in the system (4.3) to get the scheme.

$$\begin{aligned} & \left(\epsilon - \frac{3t_1 b_{i-1} h}{2} - h^2 t_1 c_{i-1} - t_2 b_i h + \frac{t_1 b_{i+1} h}{2} \right) x_{i-1} \\ & + (-2\epsilon + 2t_1 b_{i-1} h - 2t_2 c_i h^2 - 2t_1 b_{i+1} h) x_i \\ & + \left(\epsilon - \frac{t_1 b_{i-1} h}{2} - h^2 t_1 c_{i+1} - t_2 b_i h + \frac{3t_1 b_{i+1} h}{2} \right) x_{i+1} \\ & = h^2 [t_1 (d_{i-1} x(z_{i-1-N}) - f_{i-1}) + 2t_2 (d_i x(z_{i-N}) - f_i)] \\ & + h^2 [t_1 (d_{i+1} x(z_{i+1-N}) - f_{i+1})], \quad i = 1(1)2N - 1 \end{aligned} \quad (4.5)$$

5. NUMERICAL ALGORITHM

We obtain the following initial value problem by considering the reduced problem of (2.1) and we use Runge-Kutta method to get the solution at $z = 1$, say γ' [ie $x_0(1) = \gamma'$].

$$z'_0 = \frac{1}{b(z)} [c(z)x(z) + d(z)x(z-1) - f(z)] \text{ with } x_0(0) = \phi'(0).$$

We rewrite the scheme in (4.5) with the following fitting factor.

$$\sigma_{\rho'} = -b(1)\rho'(t_1 + t_2) \coth \left(\frac{-b(1)\rho'}{2} \right)$$

where $h = \epsilon\rho'$.

The scheme in Ω_1 is as follows:

$$P_i x_{i-1} + Q_i x_i + R_i x_{i+1} = S_i, \quad 1 < i < N-1 \quad (5.1)$$

where

$$\begin{aligned} P_i &= \epsilon\sigma_{\rho'} - \frac{3t_1 b_{i-1} h}{2} - h^2 t_1 c_{i-1} - t_2 b_i h + \frac{t_1 b_{i+1} h}{2} \\ Q_i &= -2\epsilon\sigma_{\rho'} + 2t_1 b_{i-1} h - 2t_2 c_i h^2 - 2t_1 b_{i+1} h \\ R_i &= \epsilon\sigma_{\rho'} - \frac{t_1 b_{i-1} h}{2} - h^2 t_1 c_{i+1} + t_2 b_i h + \frac{3t_1 b_{i+1} h}{2} \\ S_i &= h^2 [t_1 (d_{i-1} \phi'(z_{i-1-N}) - f_{i-1}) + 2t_2 (d_i \phi'(z_{i-N}) - f_i) + t_1 (d_{i+1} \phi'(z_{i+1-N}) - f_{i+1})] \end{aligned}$$

We solve the tridiagonal system (5.1) using Thomas algorithm with the conditions $x_0 = \phi'(0)$ and $x_N = \gamma'$.

The scheme with fitting factor in Ω_2 is:

$$P_i x_{i-1} + Q_i x_i + R_i x_{i+1} = S_i, \quad N+1 < i < 2N-1 \quad (5.2)$$

where

$$\begin{aligned} P_i &= \epsilon\sigma_{\rho'} - \frac{3t_1 b_{i-1} h}{2} - h^2 t_1 c_{i-1} - t_2 b_i h + \frac{t_1 b_{i+1} h}{2} \\ Q_i &= -2\epsilon\sigma_{\rho'} + 2t_1 b_{i-1} h - 2t_2 c_i h^2 - 2t_1 b_{i+1} h \\ R_i &= \epsilon\sigma_{\rho'} - \frac{t_1 b_{i-1} h}{2} - h^2 t_1 c_{i+1} + t_2 b_i h + \frac{3t_1 b_{i+1} h}{2} \\ S_i &= h^2 [t_1 (d_{i-1} x(z_{i-1-N}) - f_{i-1}) + 2t_2 (d_i x(z_{i-N}) - f_i) + t_1 (d_{i+1} x(z_{i+1-N}) - f_{i+1})] \end{aligned}$$

The tridiagonal system (5.2) can again be solved by Thomas algorithm with the conditions $x_N = \gamma'$ and $x_{2N} = 0$.

6. NUMERICAL EXPERIMENTS

To illustrate the method, we took an example for $t_1 = 1/12$ and $t_2 = 5/12$. The computed maximum absolute errors are estimated using double mesh principle and it is presented in table.

$$E_\epsilon^N = \max_i | z_i^N - z_{2i}^{2N} | \quad (6.1)$$

The ϵ -uniform maximum absolute error for N is calculated by

$$E^N = \max_{\epsilon} E_{\epsilon}^N \quad (6.2)$$

Rate of Convergence. The numerical rate of convergence ρ is defined as follows:

$$\rho = \frac{\log(E_h) - \log(E_{h/2})}{\log 2}$$

TABLE 1. The maximum absolute error of the Example 1

	N						
	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
E^N	4.12e-02	2.12e-02	1.08e-02	5.43e-03	2.72e-03	1.36e-03	6.83e-04
Result in [15]							
E^N	1.27e-01	6.52e-02	3.31e-02	1.67e-02	8.36e-03	4.19e-03	2.10e-03

Example:1. Consider

$$-\epsilon x''(z) - (2+z)x'(z) + (3+z)x(z) - d(z)x(z-1) = 3, \quad z \in (0, 2)$$

with

$$\begin{aligned} x(z) &= z^2, \quad z \in [-1, 0] \\ x(2) &= 0 \end{aligned}$$

where

$$d(z) = \begin{cases} 1-z & z < 1 \\ (2 + \sin 4\pi z) & z \geq 1 \end{cases}$$

Maximum absolute errors for this example is given in Table 1 for different values of ϵ . Rate of convergence table is obtained as one and the computation is given in Table 2.

Example:2. Consider

$$-\epsilon x''(z) - (3-z)x'(z) + (2+z^2)x(z) - 2x(z-1) = 2, \quad z \in (0, 2)$$

with

$$\begin{aligned} x(z) &= z^2, \quad z \in [-1, 0] \\ x(2) &= 1 \end{aligned}$$

Maximum absolute errors for this example is given in Table 3 for different values of ϵ .

TABLE 2. Rate of convergence ρ of the example for $\epsilon = 2^{-14}$

	h	$\frac{h}{2}$	E_h	$\frac{h}{4}$	$E_{\frac{h}{2}}$	ρ
	1/32	1/64	2.1242e-02	1/128	1.0776e-02	0.9791
	1/64	1/128	1.0776e-02	1/256	5.4268e-03	0.9897
	1/128	1/256	5.4268e-03	1/512	2.7231e-03	0.9949
	1/256	1/512	2.7231e-03	1/1024	1.3640e-03	0.9974

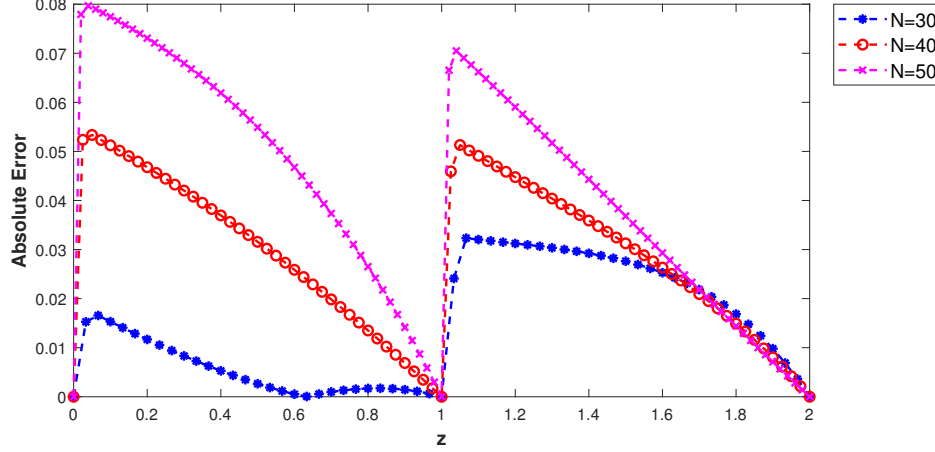


FIGURE 1. The numerical solution of Example 1 with

7. CONCLUSION

In this work, we approached singularly perturbed delay differential equations involving large delay with spline method. To demonstrate the method, we have given an example with maximum absolute error and rate of convergence tables. We obtained rate of convergence as one. From the findings, we can observe that

TABLE 3. The maximum absolute error of the Example 2 for different values of ϵ

	N						
	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
10^{-1}	1.18e-01	5.80e-02	2.82e-02	1.38e-02	6.84e-03	3.40e-03	1.69e-03
10^{-2}	1.20e-01	1.18e-01	1.48e-01	1.14e-01	6.35e-02	3.22e-02	1.60e-02
10^{-3}	1.09e-01	5.68e-02	2.90e-02	1.89e-02	5.20e-02	1.20e-01	1.26e-01
10^{-4}	1.09e-01	5.68e-02	2.89e-02	1.46e-02	7.34e-03	3.68e-03	3.15e-03
10^{-5}	1.09e-01	5.68e-02	2.89e-02	1.46e-02	7.34e-03	3.68e-03	1.84e-03
10^{-6}	1.09e-01	5.68e-02	2.89e-02	1.46e-02	7.34e-03	3.68e-03	1.84e-03
10^{-7}	1.09e-01	5.68e-02	2.89e-02	1.46e-02	7.34e-03	3.68e-03	1.84e-03
10^{-8}	1.09e-01	5.68e-02	2.89e-02	1.46e-02	7.34e-03	3.68e-03	1.84e-03
10^{-9}	1.09e-01	5.68e-02	2.89e-02	1.46e-02	7.34e-03	3.68e-03	1.84e-03
E^N	1.09e-01	5.68e-02	2.89e-02	1.46e-02	7.34e-03	3.68e-03	1.84e-03

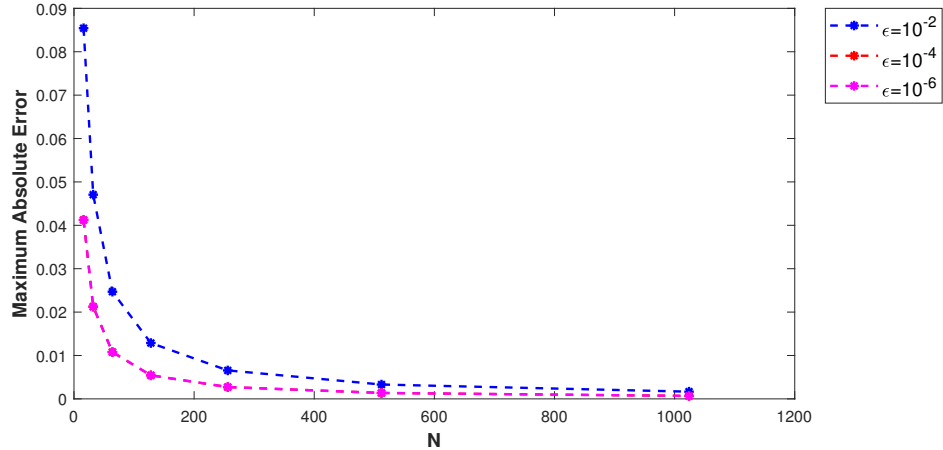


FIGURE 2. The numerical solution of Example 1 with

our errors are uniform and are getting smaller with large N . Hence our method gives consistent and convergent numerical results.

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Conflict of interest The authors declare that they have no conflict of interest, relevant to the content of this article.

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¹ DEPARTMENT OF MATHEMATICS, NIRMALA COLLEGE, MUVATTUPUZHA, INDIA
Email address: danyjoy4@gmail.com

² DEPARTMENT OF MATHEMATICS, ST.JOSEPH'S COLLEGE, DEVAGIRI, INDIA
Email address: akhilamariyard@gmail.com