

TOTALLY GEODESIC REAL HYPERSURFACES IN COMPLEX QUADRIC

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ABSTRACT. The Complex quadric is a complex hypersurface in complex projective space. It also can be regarded as a kind of real Grassmann manifold of compact type with rank 2. On the other hand Jacobi fields along geodesics of a given Riemannian manifold satisfy a well known differential equation, called Jacobi equation. This equation naturally inspires the so-called Jacobi operator. The purpose of this article is to give a non-existence result for any totally geodesic real hypersurfaces with recurrent normal Jacobi operator $\bar{R}_N$ and also with recurrent structure Jacobi operator R_ξ in Q^m . In the continuation, we obtain classifications of totally geodesic real hypersurfaces in Q^m with recurrent Ricci tensor Ric .

1. INTRODUCTION

A complex quadric $Q^m = SO_{m+2}/SO_m SO_2$ is a Hermitian symmetric space with rank 2 of compact type. It is a complex hypersurface in complex projective space CP^m and admits both a complex conjugation structure A and a Kähler structure J , which anti-commutes with each other, that is, $AJ = -JA$. Let $(\bar{M}, g)$ be a Riemannian manifold. A vector field U along a geodesic in $\bar{M}$ is said to be a *Jacobi field* if it satisfies a differential equation

$$\bar{\nabla}_{\dot{\gamma}}^2 U + \bar{R}(U(t), \dot{\gamma}(t))\dot{\gamma}(t) = 0, \quad (1.1)$$

where $\bar{\nabla}_{\dot{\gamma}} U$ and $\bar{R}$ denote the covariant derivative of the vector field U along the curve γ in $\bar{M}$ and the curvature tensor of the Riemannian manifold $(\bar{M}, g)$, respectively. Then this equation is called the *Jacobi equation*. The *Jacobi operator* $\bar{R}_X$, for any tangent vector field X at $z \in \bar{M}$, is defined by

$$(\bar{R}_X Y)(z) = (\bar{R}(Y, X)X)(z),$$

for any $Y \in T_z \bar{M}$. It becomes a self adjoint endomorphism of the tangent bundle $T\bar{M}$ of $\bar{M}$, where $\bar{R}$ denotes the curvature tensor of $(\bar{M}, g)$. That is, the Jacobi operator satisfies $\bar{R}_X \in \text{End}(T_z \bar{M})$ and is symmetric in the sense of

$$g(\bar{R}_X Y, Z) = g(\bar{R}_X Z, Y),$$

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for any vector fields Y, Z on $\overline{M}$. Let T be a tensor field of type $(1, 1)$ on M . T is said to be *recurrent* if there exists a certain 1-form ω on M such that for any vector fields X, Y tangent to M , [9]

$$(\nabla_X T)(Y) = \omega(X)T(Y).$$

This notion generalizes the fact of T being parallel.

I. Jeong, J.D. Perez and Y.J. Suh have used the concept of recurrent Jacobi operator of real hypersurfaces in complex two-plane Grassmannians in [3]. Suh has investigated real hypersurfaces M in Q^m with parallel normal Jacobi operator in [10]. Also, he has focused on real hypersurfaces M in Q^m with parallel structure Jacobi operator in [11]. He has proved many interesting results in the complex quadric [12, 13]. Motivated by his results, we wish to generalize this notion to recurrent normal Jacobi operator and recurrent structure Jacobi operator.

Our work is outlined as follows: In **Section 2**, we present basic material about the complex quadric Q^m , including its Riemannian curvature tensor and a description of its singular tangent vectors. Further, we define Hopf hypersurfaces and also give some fundamental formulas and facts related to the vector fields $\xi, N, A\xi$ and AN for the complex conjugation A of any real hypersurfaces M in Q^m , which shall be used in the proofs of our main results. **Section 3** deals with a non-existence property of totally geodesic real hypersurfaces M in $Q^m, m \geq 3$ with recurrent structure Jacobi operator. In **Section 4** we prove that there exists no totally geodesic real hypersurfaces M in $Q^m, m \geq 3$ with recurrent normal Jacobi operator. **Section 5** gives complete classifications of totally geodesic real hypersurfaces M in $Q^m, m \geq 3$ with recurrent Ricci tensor.

2. PRELIMINARIES

For more details see [2, 4, 9, 8]. The homogeneous quadratic equation $z_1^2 + \cdots + z_{m+2}^2 = 0$ on C^{m+2} defines a complex hypersurface Q^m in the $(m+1)$ -dimensional complex projective space $CP^{m+1} = SU_{m+2}/S(U_{m+1}U_1)$. The complex hypersurface Q^m is known as the m -dimensional *complex quadric*. The 1-dimensional quadric Q^1 is isometric to the round 2-sphere S^2 . For $m \geq 2$ the triple (Q^m, J, g) is a Hermitian symmetric space of rank two and its maximal sectional curvature is equal to 4. The 2-dimensional quadric Q^2 is isometric to the Riemannian product $S^2 \times S^2$. We will assume $m \geq 3$ for the main results of this paper.

The complex structure J on CP^{m+1} naturally induces a complex structure on Q^m which is also denoted by J . Apart from J there is another distinguished geometric structure on Q^m , namely, a parallel rank two vector bundle $\mathcal{U}$ which contains an S^1 -bundle of real structures, that is, complex conjugation A on the tangent spaces of Q^m . We denote by $A_{\bar{z}}$ the shape operator of Q^m in CP^{m+1} with respect to $\bar{z}$. Then we have $A_{\bar{z}}w = w$ for all $w \in T_z Q^m, z \in Q^m$, that is, $A_{\bar{z}}$ is just complex conjugation restricted to $T_z Q^m$. The shape operator $A_{\bar{z}} = A$ becomes an involution on the complex vector space $T_z Q^m$ and the tangent space $T_z Q^m$ is decomposed as follows:

$$T_z Q^m = V(A_{\bar{z}}) \oplus JV(A_{\bar{z}}),$$

where $V(A_{\bar{z}})$ is the $(+1)$ -eigenspace and $JV(A_{\bar{z}})$ is the (-1) -eigenspace of $A_{\bar{z}}$.

The Gauss equation for the complex hypersurface $Q^m \subset CP^{m+1}$ implies that the Riemannian curvature tensor $\bar{R}$ of Q^m can be expressed in terms of the Riemannian metric g , the complex structure J and a generic real structure A in $\mathcal{U}$:

$$\begin{aligned} \bar{R}(X, Y)Z &= g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ \\ &\quad + g(AY, Z)AX - g(AX, Z)AY + g(JAY, Z)JAX - g(JAX, Z)JAY, \end{aligned} \quad (2.1)$$

for any $X, Y, Z \in T_z Q^m, z \in Q^m$.

A nonzero tangent vector $W \in T_z Q^m$ is called *singular* if it is tangent to more than one maximal flat in Q^m . There are two types of singular tangent vectors for the complex quadric Q^m :

- (1) If there exists a real structure $A \in \mathcal{U}_z$ such that $W \in V(A)$, then W is singular. Such a singular tangent vector is called *$\mathcal{U}$ -principal*.
- (2) If there exist a real structure $A \in \mathcal{U}_z$ and orthonormal vectors $X, Y \in V(A)$ such that $\frac{W}{\|W\|} = \frac{(X+JY)}{\sqrt{2}}$, then W is singular. Such a singular tangent vector is called *$\mathcal{U}$ -isotropic*.

Basic complex linear algebra shows that for every unit tangent vector $W \in T_z Q^m$ there exist a real structure $A \in \mathcal{U}$ and orthonormal vectors $X, Y \in V(A)$ such that

$$W = \cos(t)X + \sin(t)JY,$$

for some $t \in [0, \frac{\pi}{4}]$. The singular tangent vectors correspond to the values $t = 0$ and $t = \frac{\pi}{4}$.

Let M be a real hypersurface in Q^m and denote by the data (ϕ, ξ, η, g) the induced almost contact metric structure on M and by ∇ the induced Riemannian connection on M . Note that $\xi = -JN$, where N is a unit normal vector field of M . The vector field ξ is known as the *Reeb vector field* of M and η is 1-form defined by $\eta(X) = g(\xi, X)$ for any tangent vector field X on M . The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus R\xi$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM . The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and we have $\phi\xi = 0$.

At each point $z \in M$ we define

$$\mathcal{Q}_z = \{X \in T_z M \mid AX \in T_z M \text{ for all } A \in \mathcal{U}_z\},$$

which is the maximal $\mathcal{U}_z$ -invariant subspace of $T_z M$.

If the integral curves of ξ are geodesics in M , the hypersurface M is called a *Hopf hypersurface*. The integral curves of ξ are geodesics in M if and only if ξ is a principal curvature vector of M everywhere. If we assume that M is a Hopf hypersurface, then we have

$$S\xi = \alpha\xi,$$

where S denotes the shape operator of the real hypersurfaces M with the smooth function $\alpha = g(S\xi, \xi)$, which is known as Reeb function on M .

By the Kähler structure J of the complex quadric Q^m , one can transfer any tangent vector field X on M in Q^m as follows:

$$JX = \phi X + \eta(X)N, \quad (2.2)$$

where $\phi X = \tan(JX)$ and N is a unit normal vector field on M in Q^m .

Then it naturally satisfies the following relations: [7]

$$\begin{aligned} \phi^2 X &= -X + \eta(X)\xi, \\ \eta(\xi) &= 1, \\ \phi\xi &= 0. \end{aligned}$$

In the relation between Q^m and its real hypersurface M , the Gauss and Weingarten formulas for M are given as

$$\bar{\nabla}_X Y = \nabla_X Y + \sigma(X, Y),$$

and

$$\bar{\nabla}_X N = -S(X),$$

for any $X, Y \in T_z M$ and $N \in T_z^\perp M, z \in M$.

The curvature tensor $R(X, Y)Z$ for a real hypersurface M in Q^m is given by

$$\begin{aligned} R(X, Y)Z &= g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\ &\quad + g(AY, Z)AX - g(AX, Z)AY + g(JAY, Z)JAX - g(JAX, Z)JAY \\ &\quad + g(SY, Z)SX - g(SX, Z)SY, \end{aligned} \quad (2.3)$$

for any $X, Y, Z \in T_z M$.

Now assume that the normal vector N_z of M is not $\mathcal{U}$ -principal. Then there exists a real structure $A \in \mathcal{U}_z$ such that

$$N_z = \cos(t)Z_2 + \sin(t)JZ_2,$$

for some orthonormal vectors $Z_1, Z_2 \in V(A)$, $t \in [0, \frac{\pi}{4}]$. Note that t is a function on M . Then we have

$$\begin{aligned} N_z &= \cos(t)Z_1 + \sin(t)JZ_2, \\ AN_z &= \cos(t)Z_1 - \sin(t)JZ_2, \\ \xi_z &= \sin(t)Z_2 - \cos(t)JZ_1, \\ A\xi_z &= \sin(t)Z_2 + \cos(t)JZ_1, \end{aligned}$$

and therefore $\mathcal{Q}_z = T_z Q^m \ominus ([Z_1] \oplus [Z_2])$ is strictly contained in $\mathcal{C}_z$.

3. NON-EXISTENCE PROPERTY WITH RECURRENT STRUCTURE JACOBI OPERATOR

Let us study the notion of *recurrent structure Jacobi operator* of a real hypersurface M in Q^m . For this, we need to define the *structure Jacobi operator* as follows:

Compute (2.3) for $Y = Z = \xi$, we get

$$\begin{aligned} R_\xi(X) &= R(X, \xi)\xi \\ &= X - \eta(X)\xi + g(A\xi, \xi)AX - g(AX, \xi)A\xi - g(JAX, \xi)JA\xi \\ &\quad + g(S\xi, \xi)SX - g(SX, \xi)S\xi. \end{aligned} \quad (3.1)$$

When M is a totally geodesic ($S = 0$) real hypersurface, then (3.1) takes the following form:

$$\begin{aligned} R_\xi(X) &= R(X, \xi)\xi \\ &= X - \eta(X)\xi + g(A\xi, \xi)AX - g(AX, \xi)A\xi - g(JAX, \xi)JA\xi. \end{aligned} \quad (3.2)$$

For a real hypersurface in complex projective space CP^m , Perez and Santos [6] introduced a new notion of D -recurrent, which is weaker than the structure Jacobi operator being recurrent. Here, the structure Jacobi operator R_ξ is said to be recurrent if it satisfies

$$(\nabla_X R_\xi)(Y) = \omega(X)R_\xi(Y), \quad (3.3)$$

where ω and D respectively denote an 1-form on M and the orthogonal complement of the Reeb vector field $\xi \in T_z M$ and any vector fields $X \in D, Y \in T_z M$.

3.1. $\mathcal{U}$ -principal normal and $\mathcal{U}$ -isotropic normal. In this section, we assume that M has $\mathcal{U}$ -principal unit normal vector field N , then the unit normal vector field satisfies $AN = N$ for a complex conjugation $A \in \mathcal{U}$ and we have the following:

$$\left\{ \begin{array}{l} g(AN, \xi) = 0, \\ g(A\xi, N) = 0, \\ g(AN, N) = 1, \\ A\xi = -\xi, \\ JAN = -\xi, \\ JA\xi = -N. \end{array} \right. \quad (3.4)$$

On the other hand, we put

$$AY = BY + r(Y)N,$$

where BY denotes the tangential component of AY and

$$r(Y) = g(AY, N) = g(Y, N) = 0.$$

Hence, $AY = BY$ for any vector field Y on M with $\mathcal{U}$ -principal unit normal vector field N in Q^m and it follows that

$$(\nabla_X A)Y = q(X)JAY, \quad (3.5)$$

where we have used the Gauss and Weingarten formulae and the fact that $S = 0$.

Theorem 3.1. *Let M be totally geodesic real hypersurfaces in $Q^m, m \geq 3$ with recurrent structure Jacobi operator. If M has $\mathcal{U}$ -principal unit normal vector field N , then any one of the following can hold:*

- (1) *There do not exist M in Q^m , or*

(2) Q^m is totally η -umbilical.

Proof. Since the unit normal vector field N is $\mathcal{U}$ -principal, then the structure Jacobi operator (3.2) can be rearranged as follows:

$$R_\xi(X) = X - 2\eta(X)\xi - AX. \quad (3.6)$$

Differentiating (3.6) along the direction Y , we have

$$(\nabla_Y R_\xi)(X) = -q(Y)JAX.$$

Using recurrent structure Jacobi operator, equation (3.5) and the fact that $\nabla_Y \xi = \phi SY = 0$, we get

$$\omega(Y)\{X - 2\eta(X)\xi - AX\} = -q(Y)JAX. \quad (3.7)$$

Computing (3.7) for $X = \xi$, it follows that

$$q(Y)N = 0. \quad (3.8)$$

Taking the inner product on both sides of equation (3.8) with the unit normal vector field N , we find that $q(Y) = 0$. If we put it into (3.7), then we get either $\omega(Y) = 0$ or $AX = X - 2\eta(X)\xi$. If $\omega(Y) = 0$, then it affects the given condition. Hence, our assertion (1) follows. From latter case, we get our assertion (2). $\square$

If we consider M has $\mathcal{U}$ -isotropic unit normal vector field N , then the unit normal vector field N satisfies

$$N = \frac{Z_1 + JZ_2}{\sqrt{2}},$$

for any $Z_1, Z_2 \in V(A)$. Then it follows that

$$\left\{ \begin{array}{l} g(\xi, A\xi) = 0, \\ g(\xi, AN) = 0, \\ g(A\xi, N) = 0, \\ g(JAN, \xi) = -g(AN, N), \\ g(AN, N) = 0, \\ g(AN, AN) = 1, \\ g(AN, A\xi) = 0, \\ g(A\xi, A\xi) = 1, \\ g(A\xi, AN) = 0. \end{array} \right. \quad (3.9)$$

Theorem 3.2. *Let M be totally geodesic real hypersurfaces in $Q^m, m \geq 3$ with recurrent structure Jacobi operator. If M has $\mathcal{U}$ -isotropic unit normal vector field N , then any one of the following can hold:*

- (1) *There do not exist M in Q^m , or*
- (2) *(∇A) vanishes identically on M .*

Proof. Since the unit normal vector field N is $\mathcal{U}$ -isotropic, then the structure Jacobi operator (3.2) can be rewritten as follows:

$$R_\xi(X) = X - \eta(X)\xi - g(AX, \xi)A\xi - g(X, AN)AN. \quad (3.10)$$

Differentiating (3.10) along the direction Y , also using the following facts:

$$\nabla_Y(A\xi) = q(Y)JA\xi$$

and

$$\nabla_Y(AN) = q(Y)JAN,$$

we get

$$\begin{aligned} (\nabla_Y R_\xi)(X) &= -q(Y)g(A\xi, X)A\xi - q(Y)g(X, A\xi)JA\xi - q(Y)g(X, AN)AN \\ &\quad - q(Y)g(X, AN)JAN. \end{aligned}$$

By using the assumption of recurrent structure Jacobi operator, we see that

$$\begin{aligned} \omega(Y)\{X - \eta(X)\xi - g(AX, \xi)A\xi - g(X, AN)AN\} &= -q(Y)g(A\xi, X)A\xi \\ &\quad - q(Y)g(X, A\xi)JA\xi \\ &\quad - q(Y)g(X, AN)AN \\ &\quad - q(Y)g(X, AN)JAN. \end{aligned}$$

If $X = AN$, then above equation takes the following form:

$$q(Y)AN + q(Y)JAN = 0, \quad (3.11)$$

and if $X = A\xi$, then same equation becomes

$$q(Y)A\xi + q(Y)JA\xi = 0. \quad (3.12)$$

By combining (3.11), (3.12) and the fact that $JAN = A\xi$, we get either $AN = 0$ or $q(Y) = 0$. If $AN = 0$, then it makes a contradiction. Hence, our assertion (1) follows. From latter case, we get our assertion (2). $\square$

4. NON-EXISTENCE PROPERTY WITH RECURRENT NORMAL JACOBI OPERATOR

We study the notion of *recurrent normal Jacobi operator* of a real hypersurface M in Q^m . For this, we need to define the *normal Jacobi operator* as follows:

Plugging $Y = Z = N$ into (2.1), we get

$$\begin{aligned} \bar{R}_N(X) &= \bar{R}(X, N)N \\ &= X + 3\eta(X)\xi + g(AN, N)AX - g(AX, N)AN - g(AX, \xi)A\xi. \end{aligned} \quad (4.1)$$

Here, the normal Jacobi operator $\bar{R}_N$ is said to be recurrent if it satisfies [9]

$$(\nabla_X \bar{R}_N)(Y) = \omega(X)\bar{R}_N(Y), \quad (4.2)$$

where ω denotes an 1-form on M and any vector fields X, Y tangent to M .

4.1. $\mathcal{U}$ -principal normal and $\mathcal{U}$ -isotropic normal. In this subsection, we assume two cases. First we consider M has $\mathcal{U}$ -principal unit normal vector field N . Then the unit normal vector field satisfies $AN = N$ for a complex conjugation $A \in \mathcal{U}$.

Theorem 4.1. *Let M be totally geodesic real hypersurfaces in $Q^m, m \geq 3$ with recurrent normal Jacobi operator. If M has $\mathcal{U}$ -principal unit normal vector field N , then any one of the following can hold:*

- (1) *There do not exist M in Q^m , or*
- (2) *Q^m is totally η -umbilical.*

Proof. Since the unit normal vector field N is $\mathcal{U}$ -principal, then the normal Jacobi operator (4.1) can be rearranged as follows:

$$\overline{R}_N(Y) = Y + 2\eta(Y)\xi + AY. \quad (4.3)$$

Differentiating (4.3) along the direction X , we have

$$(\nabla_X \overline{R}_N)(Y) = 2(\nabla_X \eta)(Y)\xi + 2\eta(Y)\nabla_X \xi + (\nabla_X A)Y.$$

Using the assumption of recurrent structure Jacobi operator, equation (3.5) and the fact that $\nabla_X \xi = \phi SX = 0$, we get

$$\omega(X)\{Y + 2\eta(Y)\xi + AY\} = q(X)JAY. \quad (4.4)$$

Taking inner product on both sides of equation (4.4) with the unit normal vector field N . It follows that

$$q(X)\eta(Y) = 0.$$

Putting $Y = \xi$ into the last equation and using (3.4), it becomes $q(X) = 0$. If we put it into (4.4), then we get either $\omega(X) = 0$ or $AY = -Y - 2\eta(Y)\xi$. If $\omega(X) = 0$, then it affects the given condition. Hence, our assertion (1) follows. From later case, we get our assertion (2). $\square$

Now we consider M has $\mathcal{U}$ -isotropic unit normal vector field N . Then the unit normal vector field N satisfies

$$N = \frac{Z_1 + JZ_2}{\sqrt{2}}$$

for any $Z_1, Z_2 \in V(A)$.

Theorem 4.2. *Let M be totally geodesic real hypersurfaces in $Q^m, m \geq 3$ with recurrent normal Jacobi operator. If M has $\mathcal{U}$ -isotropic unit normal vector field N , then any one of the following can hold:*

- (1) *There do not exist M in Q^m , or*
- (2) *(∇A) vanishes identically on M .*

Proof. Since the unit normal vector field N is $\mathcal{U}$ -isotropic, then the normal Jacobi operator (4.1) can be rewritten as follows:

$$\overline{R}_N(Y) = Y + 3\eta(Y)\xi - g(AY, N)AN - g(AY, \xi)A\xi. \quad (4.5)$$

Differentiating (4.5) along the direction X , also using the following:

$$\nabla_X(A\xi) = q(X)JA\xi$$

and

$$\nabla_X(AN) = q(X)JAN,$$

we find that

$$(\nabla_X \overline{R}_N)(Y) = -q(X)g(Y, A\xi)AN - q(X)g(AN, Y)A\xi - q(X)\eta(Y)A\xi - q(X)g(A\xi, Y)\xi.$$

By using the assumption of recurrent structure Jacobi operator, we see that

$$\begin{aligned} \omega(X)\{Y + 3\eta(Y)\xi - g(AY, N)AN - g(AY, \xi)A\xi\} &= -q(X)g(Y, A\xi)AN - q(X)g(AN, Y)A\xi \\ &\quad - q(X)\eta(Y)A\xi - q(X)g(A\xi, Y)\xi. \end{aligned}$$

If $Y = A\xi$, then above equation becomes

$$q(X)(AN + \xi) = 0. \quad (4.6)$$

From which we get either $AN = -\xi$ or $q(X) = 0$. If $AN = -\xi$, then it makes a contradiction. Hence, our assertion (1) follows. From latter case, we get our assertion (2). $\square$

5. NON-EXISTENCE PROPERTY WITH RECURRENT RICCI TENSOR

By using Gauss equation, we find Ricci tensor as follows [14, 15]:

$$\begin{aligned} Ric(X) = & (2m - 1)X - 3\eta(X)\xi - g(AN, N)AX + g(AX, N)AN - g(JAN, N)JAX \\ & + g(JAX, N)JAN + (trS)SX - S^2X, \end{aligned} \quad (5.1)$$

for any $X \in T_zM, z \in M$.

Definition 5.1. The Ricci tensor $Ric(X)$ is said to be *recurrent* if it satisfies

$$(\nabla_Y Ric)(X) = \beta(Y)Ric(X),$$

for any $X, Y \in T_zM, z \in M$. Here β is 1-form.

Definition 5.2. A Riemannian manifold (M, g) is called η -Einstein if there exist two real constants a and b such that Ricci curvature tensor field is

$$Ric = ag + b\eta \otimes \eta \quad (5.2)$$

5.1. $\mathcal{U}$ -principal normal and $\mathcal{U}$ -isotropic normal. If we assume that M has $\mathcal{U}$ -principal unit normal vector field N . Then the unit normal vector field satisfies $AN = N$ for a complex conjugation $A \in \mathcal{U}$. Then (5.1) becomes

$$Ric(X) = (2m - 1)X - 2\eta(X)\xi - AX + hSX - S^2X, \quad (5.3)$$

where $h = trS$.

Theorem 5.3. Let M be totally geodesic real hypersurfaces in $Q^m, m \geq 3$ with $\mathcal{U}$ -principal unit normal vector field N . Then any one of the following can hold:

- (1) There do not exist any M with recurrent Ricci tensor and $\beta(Y) \neq 0$ for any $Y \in T_zM$, or
- (2) M is η -Einstein real hypersurface.

Proof. Since M is totally geodesic, then (5.6) takes the following form:

$$Ric(X) = (2m - 1)X - 2\eta(X)\xi - AX. \quad (5.4)$$

Let us assume that $AX = \lambda X$, then we have

$$Ric(X, Y) = (2m - 1 - \lambda)g(X, Y) - 2\eta(X)\eta(Y),$$

which shows that M is η -Einstein real hypersurface.

On the other hand, if we put $X = \xi$ into (5.4), then we get

$$Ric(\xi) = 2(m - 1)\xi.$$

Suppose that the Ricci tensor is recurrent on M . Then we find that [15]

$$\begin{aligned} (\nabla_Y Ric)(X) &= \nabla_Y Ric(X) - Ric(\nabla_Y X) \\ \beta(Y)Ric(X) &= -(\nabla_Y A)X \\ &= -q(Y)JAX, \end{aligned}$$

where we have used (3.5).

For $X = \xi$, the above equation gives

$$\begin{aligned} \beta(Y)Ric(\xi) &= -q(Y)JA\xi \\ 2(m-1)\beta(Y)\xi &= q(Y)N. \end{aligned}$$

Taking inner product with ξ on both sides of last equation, it follows that

$$2(m-1)\beta(Y) = 0. \quad (5.5)$$

From (5.5), we conclude that $m = 1$, which gives a contradiction since $m \geq 3$. Hence, our assertion follows. $\square$

If M has $\mathcal{U}$ -isotropic unit normal vector field N . Then the unit normal vector field N satisfies

$$N = \frac{Z_1 + JZ_2}{\sqrt{2}},$$

for any $Z_1, Z_2 \in V(A)$. Then (5.1) becomes

$$Ric(X) = (2m-1)X - 3\eta(X)\xi + g(AX, N)AN + g(AX, \xi)A\xi + hSX - S^2X. \quad (5.6)$$

Theorem 5.4. *Let M be totally geodesic real hypersurfaces in $Q^m, m \geq 3$ with $\mathcal{U}$ -isotropic unit normal vector field N . Then any one of the following can hold:*

- (1) *There do not exist any M with recurrent Ricci tensor and $\beta(Y) \neq 0$ for any $Y \in T_z M$, or*
- (2) *M is η -Einstein real hypersurface.*

Proof. Since M is totally geodesic, then (5.3) takes the following form:

$$Ric(X) = (2m-1)X - 3\eta(X)\xi + g(AX, N)AN + g(AX, \xi)A\xi. \quad (5.7)$$

We fix $AX = \lambda X$ and $AY = \gamma Y$, then we get

$$Ric(X, Y) = (2m-1)g(X, Y) + (3 - \lambda\gamma)\eta(X)\eta(Y),$$

which shows that M is η -Einstein real hypersurface.

On the other hand, for $X = \xi$, the equation (5.7) becomes

$$Ric(\xi) = 2(m-2)\xi.$$

Suppose that the Ricci tensor is recurrent on M . Then we have [15]

$$\begin{aligned} (\nabla_Y Ric)(X) &= \nabla_Y Ric(X) - Ric(\nabla_Y X) \\ \beta(Y)Ric(X) &= q(Y)g(JAN, X) - g(AX, N)\{q(Y)JAN\}^T + q(Y)g(JA\xi, X)A\xi \\ &\quad + \eta(AX)\{q(Y)JA\xi\}^T. \end{aligned}$$

For $X = \xi$, the above equation gives

$$\begin{aligned}\beta(Y)Ric(\xi) &= q(Y)g(JAN, \xi) + q(Y)g(JA\xi, \xi)A\xi \\ 2(m-2)\beta(Y)\xi &= 0.\end{aligned}$$

Taking inner product with ξ on both sides of last equation, it follows that

$$2(m-2)\beta(Y) = 0. \quad (5.8)$$

From (5.8), we conclude that $m = 2$, which gives a contradiction since $m \geq 3$. Hence, our assertion follows. $\square$

REFERENCES

1. B. Smyth, *Differential geometry of complex hypersurfaces*, Ann. Math. **85**, 246-266 (1967).
2. H. Reckziegel, *On the geometry of the complex quadric*, *Geometry and Topology of Submanifolds VIII*, Brussels/Nordfjordeid, World Sci. Publ., River Edge, NJ 302-315 (1995).
3. I. Jeong, J. D. Perez and Y. J. Suh, *Recurrent Jacobi Operator of real hypersurfaces in complex two-plane Grassmannians*, Bull. Korean Math. Soc. **50**(2), 525-536 (2013).
4. J. Berndt and Y. J. Suh, *Contact hypersurfaces in Kahler manifolds*, Proceedings of the American Math. Soc. **143**, 2637-2649 (2015).
5. J. D. Perez, *Commutativity of Cho and structure Jacobi operators of a real hypersurface in a complex projective space*, Ann. Mat. Pura Appl. **194**, 6 (2015).
6. J. D. Perez and F. G. Santos, *Real hypersurfaces in complex projective space with recurrent structure Jacobi operator*, Diff. Geom. Appl. **26**, 2, 218-223 (2008).
7. K. Yano and M. Kon, *Structures on manifolds*, Worlds Scientific, Singapore (1984).
8. S. Klein, *Totally geodesic submanifolds in the complex quadric*, Differential Geom. Appl. **26**, 79-96 (2008).
9. S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. I, Wiley Classics Library Edition Publ. (1996).
10. Y. J. Suh, *Real hypersurfaces in the complex quadric with parallel normal Jacobi operator*, Math. Nachr. **290**, 203, 442-451 (2017).
11. Y. J. Suh, *hypersurfaces in the complex quadric with parallel structure Jacobi operator*, Diff. Geom. Appl. **51**, 33-48 (2017).
12. Y. J. Suh, *Real hypersurfaces in the complex quadric with Reeb parallel shape operator*, Internat. J. Math. **25** (2014), pp. 17.
13. Y. J. Suh, *Real hypersurfaces in the complex quadric with Reeb invariant shape operator*, Diff. Geom. Appl. **38**, 10-21 (2015).
14. Y. J. Suh, *Real hypersurfaces in the complex quadric with parallel Ricci tensor*, Adv. Math. **281**, 886-905 (2015).
15. Y. J. Suh, *Real hypersurfaces in the complex quadric with commuting and parallel Ricci tensor*, J. Geom. Phys. **106**, 130-142 (2016).

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