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KUMMER EXTENSIONS AND PÒLYA FIELDS

GÉRARD KIENTEGA[1], THÉODORE TAPSOBA[2], CHARLES WEND-WAOGA TOUGMA[3*]

Abstract. A number field is called a Pólya field if the module of integer-valued polynomials over its ring of integers has a regular basis. Let $a \in \mathbb{Z}$, $a \neq 0$ and $p \geq 3$ be a prime number. In this paper, we characterize the number fields of the form $\mathbb{Q}(\zeta_p, \sqrt[p]{a})$ which are Pólya fields. We hence generalize results of [9]. New bounds for the number of ramified prime numbers in these fields are derived.

1. Introduction

1.1. **Notations.** Throughout this paper, if K is a number field, $\mathbb{Z}_K$ is its ring of integers, U_K its units group and $\mathcal{I}_K$ its group of fractional ideals. We denote by h_K the class number of K. Let q be a prime number. $\Pi_{q^{\mathfrak{f}}}(K)$ is the product of all maximal (prime) ideals of K with norm $q^{\mathfrak{f}}$, $\mathfrak{f} \in \mathbb{N}$. If $q^{\mathfrak{f}}$ is not a norm of an ideal, then $\Pi_{q^{\mathfrak{f}}}(K) = \mathbb{Z}_K$. Finally, by (r, s) we mean the gcd of r and s with $r, s \in \mathbb{Z}$. The same notation will be used for ideals. We note also $r \mid s$ when r divides s.

1.2. **A result of Amandine Leriche.** In the 1910's, Pòlya [11] and Ostrowski [10] had introduced the ring of integers valued polynomials on the ring $\mathbb{Z}_K$ of integers of the number field K:

$$\mathrm{Int}(\mathbb{Z}_K) = \{P \in K[X],\ P(\mathbb{Z}_K) \subset \mathbb{Z}_K\}$$

Pòlya was interested by number fields K whose $\mathbb{Z}_K$-module $\mathrm{Int}(\mathbb{Z}_K)$ has a regular basis $(f_n)_n$ which means that $(f_n)_n$ is a basis of $\mathrm{Int}(\mathbb{Z}_K)$ such that $\deg(f_n) = n$, for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, the leading coefficients of the polynomials in $\mathrm{Int}(\mathbb{Z}_K)$ of degree n together with 0 form a fractional ideal denoted by $\mathfrak{J}_n(K)$. Pòlya proved that K has the above propriety if and only if $\mathfrak{J}_n(K)$ is a principal ideal for each $n \in \mathbb{N}$. Ostrowski showed just after Pòlya that this propriety is satisfied if and only if the ideals $\Pi_{q^{\mathfrak{f}}}$, $\mathfrak{f} \in \mathbb{N}$ are principal.
It is Zantema that in 1982, called these fields Pòlya fields. In his 1982's work [15], he gave a cohomological interpretation of Pòlya fields that are Galois extension of $\mathbb{Q}$ and showed that all cyclotomic fields are Pòlya fields.
After Zantema's results on cyclotomic fields, Kummer extensions seem like a

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* Corresponding author.
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natural choice. In [9] and [8], Leriche gave a description of Pòlya fields of the form $\mathbb{Q}(j, \sqrt[3]{a})$ in terms of the subfield $\mathbb{Q}(\sqrt[3]{a})$:

Theorem 1.1 ([9], Theorem 6.2 or [8], Prop 3.7.)**.** *Let $a = bc^2$ where a is an integer ≥ 2, b and c square free and coprime. Let $M = \mathbb{Q}(j, \sqrt[3]{a})$, $L = \mathbb{Q}(j)$ and $K = \mathbb{Q}(\sqrt[3]{a})$. The field M is a Pòlya field if and only if:*

- *when $b^2 \not\equiv c^2 \mod 9$, for each prime q dividing $3a$, there is an element $\omega \in K$ such that $N_{K/\mathbb{Q}}(\omega) = \pm q$*
- *when $b^2 \equiv c^2 \mod 9$, for each prime q dividing a, there is an element $\omega \in K$ such that $N_{K/\mathbb{Q}}(\omega) = \pm q$.*

Corollary 1.2 ([9], Corollary 6.6 or [8], Cr. 3.10.)**.** *Let q be a prime number. The field $\mathbb{Q}(j, \sqrt[3]{q})$ is a Pòlya field if and only if:*

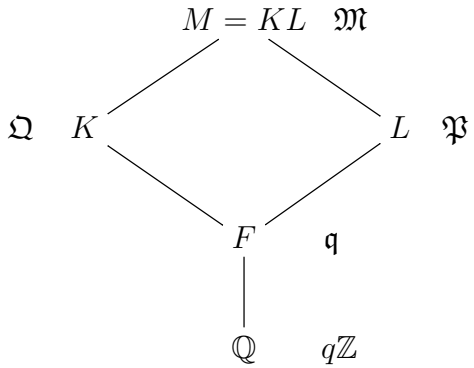
- *either $q^2 \equiv 1 \mod 9$*
- *or there is an integer of $\mathbb{Q}(\sqrt[3]{q})$ with norm ± 3.*

In this paper, we propose to generalize theses results of Leriche to the fields $\mathbb{Q}(\zeta_p, \sqrt[p]{a})$, $p \geq 3$, $a \in \mathbb{Z}$. We also improve the bounds of the ramified numbers in these fields and we apply them to refine Theorem 1.1.

2. Preliminaries

Let K/F be an extension of number fields. If $\mathfrak{q}$ is a maximal ideal of $\mathbb{Z}_F$ and $\mathfrak{Q}$ is a maximal ideal of K lying over $\mathfrak{q}$, we denote by $\mathfrak{e}(\mathfrak{Q}/\mathfrak{q})$ and $\mathfrak{f}(\mathfrak{Q}/\mathfrak{q})$ the ramification index and the residue degree of $\mathfrak{Q}$ over $\mathfrak{q}$. If $I \in \mathcal{I}_K \cap \mathbb{Z}_K$, $\mathrm{Rac}(I)$ is the product of all prime ideals of K dividing I.

Suppose now that F, K, L and M are number fields such that $F = K \cap L$ and $M = KL$. Let $\mathfrak{q}$, $\mathfrak{Q}$, $\mathfrak{P}$ and $\mathfrak{M}$ be respectively prime ideals of F, K, L and M such that $\mathfrak{M} \cap K = \mathfrak{Q}$, $\mathfrak{M} \cap L = \mathfrak{P}$, $\mathfrak{M} \cap F = \mathfrak{q}$ and $\mathfrak{M} \cap \mathbb{Q} = q\mathbb{Z}$.



(2.1)

Recall that $\mathcal{N}_{M/K}(\mathfrak{M}) = \mathfrak{Q}^{\mathfrak{f}(\mathfrak{M}/\mathfrak{Q})}$ is the relative norm of $\mathfrak{M}$ over K. If $I \in \mathcal{I}_M$ then

$$I = \prod_i \mathfrak{M}_i^{e_i}$$

with $e_i \in \mathbb{Z}$ and $\mathfrak{M}_i$ prime ideals of M. As $\mathcal{I}_M$ is a multiplicative group generated by the prime ideals of M, if $\mathfrak{Q}_\mathfrak{i} = \mathfrak{M}_i \cap K$ and $\mathfrak{f}_i = \mathfrak{f}(\mathfrak{M}_i/\mathfrak{Q}_i)$ then

$$\mathcal{N}_{M/K}(I) = \prod_i \mathfrak{Q}_i^{e_i \mathfrak{f}_i}$$

Lemma 2.1. *With the above notations,*

1) $\mathfrak{e}(\mathfrak{M}/\mathfrak{q})$ *is divisible by the least common multiple of* $\mathfrak{e}(\mathfrak{Q}/\mathfrak{q})$ *and* $\mathfrak{e}(\mathfrak{P}/\mathfrak{q})$
2) *If* $K/\mathbb{Q}$ *or* $L/\mathbb{Q}$ *is galoisian, then* $\mathfrak{e}(\mathfrak{M}/\mathfrak{q})$ *divides* $\mathfrak{e}(\mathfrak{Q}/\mathfrak{q})\mathfrak{e}(\mathfrak{P}/\mathfrak{q})$
3) *If* $K/\mathbb{Q}$ *or* $L/\mathbb{Q}$ *is galoisian and if* $(\mathfrak{e}(\mathfrak{Q}/\mathfrak{q}), \mathfrak{e}(\mathfrak{P}/\mathfrak{q})) = 1$ *then* $\mathfrak{e}(\mathfrak{M}/\mathfrak{q}) = \mathfrak{e}(\mathfrak{Q}/\mathfrak{q})\mathfrak{e}(\mathfrak{P}/\mathfrak{q})$

Proof. See [8], Lemma 2.19, p.41. □

Lemma 2.2. *If* $M/\mathbb{Q}$ *is galoisian, then the following assertions are equivalent:*

1) M *is a Pòlya field*
2) *For each prime number* q *and for each* $\mathfrak{f} \in \mathbb{N}$, $\Pi_{q^{\mathfrak{f}}}(M)$ *is a principal ideal.*
3) *For each ramified prime number* q *and for each* $\mathfrak{f} \in \mathbb{N}$, $\Pi_{q^{\mathfrak{f}}}(M)$ *is a principal ideal.*
4) *For each prime number* q, $\mathrm{Rac}(q\mathbb{Z}_M)$ *is a principal ideal.*
5) *For each ramified prime number* q, $\mathrm{Rac}(q\mathbb{Z}_M)$ *is a principal ideal.*

Proof. 1) $\Leftrightarrow$ 2) is Ostrowski's result[1]. Other equivalences are Ostrowski's results in the galoisian case. Indeed, in a galoisian extension, all prime ideals lying over q have the same residue degree $\mathfrak{f} = \mathfrak{f}(q)$. Hence $\mathrm{Rac}(q\mathbb{Z}_M) = \Pi_{q^{\mathfrak{f}}}(M)$. This shows that assertions 2) and 4) are equivalent. If q is not ramified in M, then $\mathrm{Rac}(q\mathbb{Z}_M) = \Pi_{q^{\mathfrak{f}}}(M) = q\mathbb{Z}_M$ and hence $\mathrm{Rac}(q\mathbb{Z}_M)$ is a principal ideal showing that assertions 2), 3) and 5) are equivalent. □

Remark 2.3. If $M/\mathbb{Q}$ is not galoisian, Lemma 2.2 is not necessary true.

From now on, throughout the paper $p \geq 3$ and $q \geq 2$ will be prime numbers; $a > 1$ a p-th power free integer, ζ_p a p-th primitive root of unit, $L = \mathbb{Q}(\zeta_p)$. The polynomial $X^p - a$ has a unique real root. We denote this root by $\sqrt[p]{a}$. We set $K = \mathbb{Q}(\sqrt[p]{a})$ and $M = KL = \mathbb{Q}(\zeta_p, \sqrt[p]{a})$ as in the following diagram

$$\begin{array}{ccccc} & & M = \mathbb{Q}(\zeta_p, \sqrt[p]{a}) & & \\ & \swarrow & & \searrow & \\ K = \mathbb{Q}(\sqrt[p]{a}) & & & & L = \mathbb{Q}(\zeta_p) \\ & \searrow & & \swarrow & \\ & & \mathbb{Q} & & \end{array} \tag{2.2}$$

Lemma 2.4. p *is totally ramified in* L*:*

$$p\mathbb{Z}_L = (1 - \zeta_p)^{p-1}\mathbb{Z}_L.$$

Proof. See [1], Lemma 3, p.87 □

It is well known that p is the only ramified prime number in L (see [12], p.90). The ramified prime numbers in K are those that divide pa. Westlund's following Theorem [14] gives their decomposition in K.

[1] See Introduction

Theorem 2.5 (Westlund)**.** *The integer a may be expressed in one way only in the form $a = a_1 a_2^2 \dots a_{p-1}^{p-1}$ where $a_1, a_2, \dots a_{p-1}$ are relatively prime and are not divisible by the square of a prime number. Let $b = a_1 a_2^2 \dots a_{p-2}^{p-2} = \dfrac{a}{a_{p-1}^{p-1}}$ and $d = b^{p-1} - a_{p-1}^{p-1}$. Then*

1) *$q\mathbb{Z}_K = \mathfrak{Q}^p$ if q is a prime number such that $q \mid a$*
2) *If $(p, a) = 1$ then*
   - *$p\mathbb{Z}_K = \mathfrak{Q}^p$ if $d \not\equiv 0 \mod p^2$*
   - *$p\mathbb{Z}_K = \mathfrak{Q}_1^{p-1}\mathfrak{Q}_2$ if $d \equiv 0 \mod p^2$.*

Proof. [14], p.389-391. □

Lemma 2.6. *With the notations of the previous Theorem, the following assertions are equivalent.*

1) $d \equiv 0 \mod p^2$
2) $a^{p-1} \equiv 1 \mod p^2$

Proof.
1) $\Rightarrow$ 2) : $d \equiv 0 \mod p^2 \Rightarrow b^{p-1} - a_{p-1}^{p-1} \equiv 0 \mod p^2$. It is clear that $a_{p-1}^{p-1} \equiv 0 \mod p^2 \Leftrightarrow p \mid a_{p-1}$ since p is an odd prime. Thus if we suppose that $a_{p-1}^{p-1} \equiv 0 \mod p^2$ then $p \nmid b$ since $a_1, a_2, \dots a_{p-1}$ are relatively primes. ($a_{p-1}^{p-1} \equiv 0 \mod p^2$ and $p \nmid b$) $\Rightarrow d = b^{p-1} - a_{p-1}^{p-1} \equiv b^{p-1} \not\equiv 0 \mod p^2$ and this contradicts our hypothesis. So, we may assume that $a_{p-1}^{p-1} \not\equiv 0 \mod p^2$.
$d \equiv 0 \mod p^2 \Leftrightarrow \dfrac{a^{p-1}}{a_{p-1}^{p-1}} - a_{p-1}^{p-1} \equiv 0 \mod p^2 \Leftrightarrow a^{p-1} - a_{p-1}^{(p-1)^2} a_{p-1}^{p-1} \equiv 0 \mod p^2$ ie $a^{p-1} - a_{p-1}^{p(p-1)} \equiv 0 \mod p^2$. Hence $a^{p-1} - 1 \equiv 0 \mod p^2$ since the multiplicative group of $\mathbb{Z}/p^2\mathbb{Z}$ has order $p(p-1)$.
2) $\Rightarrow$ 1) : We have $a_{p-1}^{p-1} \not\equiv 0 \mod p^2$ (this is equivalent to $p \nmid a_{p-1}$). In fact if $a_{p-1}^{p-1} \equiv 0 \mod p^2$ then $a \equiv 0 \mod p^2$ because a_{p-1}^{p-1} divides a: which is absurd since by hypothesis $a^{p-1} \equiv 1 \mod p^2$. First, one has $d = b^{p-1} - a_{p-1}^{p-1} = \dfrac{a^{p-1}}{a_{p-1}^{(p-1)^2}} - a_{p-1}^{p-1}$. So $d a_{p-1}^{(p-1)^2} = a^{p-1} - a_{p-1}^{p(p-1)}$. On the other hand, we have $a_{p-1}^{p(p-1)} \equiv 1 \mod p^2$ since we have already shown that $p \nmid a_{p-1}$. Thus $a^{p-1} - 1 \equiv 0 \mod p^2$ implying $d a_{p-1}^{(p-1)^2} \equiv a^{p-1} - a_{p-1}^{p(p-1)} \equiv 0 \mod p^2$. This last congruence implies that $d \equiv 0 \mod p^2$ since $p \nmid a_{p-1}^{(p-1)^2}$. □

Corollary 2.7. *Under the hypothesis of Theorem 1.1, the following assertions are equivalent:*

(1) $b^2 \equiv c^2 \mod 9$
(2) $a^2 \equiv 1 \mod 9$

Westlund[14] also found an integral basis of $K = \mathbb{Q}(\sqrt[p]{a})$.

Theorem 2.8 (Westlund). *Let $\alpha_i = \sqrt[p]{\beta_i}$ where β_i is the p-th power free part of a^i. Let b and d be as in the Theorem 2.5 and $\gamma = \dfrac{\alpha_1^{p-1} + \alpha_1^{p-2}b + \cdots + \alpha_1 b^{p-2} + 1}{p}$. An integral basis of K is*

1) $(1, \alpha_1, \cdots, \alpha_{p-1})$ *if* $d \not\equiv 0 \mod p^2$
2) $(\gamma, \alpha_1, \cdots, \alpha_{p-1})$ *if* $d \equiv 0 \mod p^2$

Theorem 2.9 (Hilbert). *Let $\mu \in L$ such that $X^p - \mu$ is irreducible over L. Assume that $\mathfrak{P}$ is a prime ideal of L such that $\mathfrak{P} \mid p\mathbb{Z}_L$ and $(\mu, \mathfrak{P}) = 1$. Then in M,*

1) $\mathfrak{P}$ *splits completely into p factors,* $\mathfrak{P}\mathbb{Z}_M = \mathfrak{M}_1 \cdots \mathfrak{M}_p$ *iff* $\mu \equiv \xi^p \mod \mathfrak{P}^{p+1}$
2) $\mathfrak{P}$ *remains prime:* $\mathfrak{P}\mathbb{Z}_M = \mathfrak{M}$ *iff* $\mu \not\equiv \xi^p \mod \mathfrak{P}^{p+1}$ *and* $\mu \equiv \xi^p \mod \mathfrak{P}^p$
3) $\mathfrak{P}$ *becomes the p-th power of a prime ideal:* $\mathfrak{P}\mathbb{Z}_M = \mathfrak{M}^p$ *iff* $\mu \not\equiv \xi^p \mod \mathfrak{P}^p$

Proof. [13], Theorem 1.1, p.589. □

Theorem 2.10. *If $q\mathbb{Z}_K = \mathfrak{Q}^p$ then the following assertions are equivalent:*

1) $\mathfrak{Q} = \mathrm{Rac}(q\mathbb{Z}_K)$ *is a principal ideal*
2) $\mathrm{Rac}(q\mathbb{Z}_M)$ *is a principal ideal.*

Proof. 1) $\Longrightarrow$ 2) : *i)* $q \neq p$: q is not ramified in $\mathbb{Z}_L$ since p is the only ramified prime number in L. Hence by Lemma 2.1, $\mathfrak{Q}$ is not ramified in $\mathbb{Z}_M$. So we have $\mathrm{Rac}(q\mathbb{Z}_M) = \mathrm{Rac}(\mathfrak{Q}\mathbb{Z}_M) = \mathfrak{Q}\mathbb{Z}_M$. Thus if $\mathfrak{Q}$ is a principal ideal then $\mathrm{Rac}(q\mathbb{Z}_M)$ is a principal ideal.

ii) $q = p$: By Lemma 2.4, $p\mathbb{Z}_L = (1 - \zeta_p)^{p-1}\mathbb{Z}_L$. Lemma 2.1, shows that there exists a prime ideal $\mathfrak{M}$ of $\mathbb{Z}_M$ such that $p\mathbb{Z}_M = \mathfrak{M}^{p(p-1)}$. Hence we have $\mathrm{Rac}(p\mathbb{Z}_M) = \mathfrak{M}$. It follows that $\mathfrak{Q}\mathbb{Z}_M = \mathfrak{M}^{p-1}$ and $(1 - \zeta_p)\mathbb{Z}_M = \mathfrak{M}^p$. So $\mathfrak{M}^{p-1}$ and $\mathfrak{M}^p$ are principal ideals. As $\mathfrak{M} = \mathfrak{M}^p \left(\mathfrak{M}^{p-1}\right)^{-1}$ then $\mathfrak{M}$ is a principal ideal.

2) $\Longrightarrow$ 1): M/K is galoisian since $M/\mathbb{Q}$ is galoisian. Thus $\mathfrak{Q}\mathbb{Z}_M = (\mathfrak{M}_1 \ldots \mathfrak{M}_r)^e$ with $ref = [M : K] = (p - 1)$. Hence: $\mathfrak{Q}^{p-1} = \mathfrak{Q}^{ref} = \mathcal{N}_{M/K}(\mathfrak{Q}\mathbb{Z}_M) = \mathcal{N}_{M/K}\left((\mathrm{Rac}(q\mathbb{Z}_M))^e\right) = \mathcal{N}_{M/K}\left((\mathrm{Rac}(q\mathbb{Z}_M))^e \mathbb{Z}_K\right.$. Thus $\mathfrak{Q}^{p-1}$ is a principal ideal because the realative norm of a principal ideal is a principal ideal. Since by hypothesis, $\mathfrak{Q}^p$ is a principal ideal, then $\mathfrak{Q} = \mathfrak{Q}^p \left(\mathfrak{Q}^{p-1}\right)^{-1}$ is a principal ideal. □

3. Pòlya fields of the form $\mathbb{Q}(\zeta_p, \sqrt[p]{a})$

With the notations of the previous sections, the following result generalizes Theorem 1.1. Recall that $M = \mathbb{Q}(\zeta_p, \sqrt[p]{a})$ and $K = \mathbb{Q}(\sqrt[p]{a})$.

Theorem 3.1. *M is a Pòlya field if and only if:*

- *when $a^{p-1} \not\equiv 1 \mod p^2$, for each prime q dividing pa, there is an algebraic integer $\omega \in K$ such that $N_{K/\mathbb{Q}}(\omega) = q$.*
- *when $a^{p-1} \equiv 1 \mod p^2$, for each prime q dividing a, there is an algebraic integer $\omega \in K$ such that $N_{K/\mathbb{Q}}(\omega) = q$.*

Proof. We look at diagram 2.2: the ramified prime numbers in M are those dividing pa. Indeed by Lemma 2.1, q is ramified in M iff it is ramified at least in

one of the extensions L or K, thus q is ramified in M if and only if it is ramified in K since p is ramified in K and is the only ramified prime number in L.

By Lemma 2.2, M is a Pòlya field iff for each ramified prime number q, $\mathrm{Rac}(q\mathbb{Z}_M)$ is a principal ideal. By Theorem 2.5 and Lemma 2.6,

i) if $a^{p-1} \not\equiv 1 \mod p^2$ and $q \mid pa$ then $q\mathbb{Z}_K = \mathfrak{Q}^p$
ii) if $a^{p-1} \equiv 1 \mod p^2$ then
$q\mathbb{Z}_K = \mathfrak{Q}^p$ if $q \mid a$, $q \neq p$
$p\mathbb{Z}_K = \mathfrak{Q}_1^{p-1}\mathfrak{Q}_2$.

In case i), Theorem 2.10 implies that $\mathfrak{Q} = \mathrm{Rac}(q\mathbb{Z}_K)$ is a principal ideal if $q \mid pa$. Hence there exists $\omega \in \mathbb{Z}_K$ such that $\mathfrak{Q} = \omega\mathbb{Z}_K$. Taking norm, it is equivalent to $N_{K/\mathbb{Q}}(\omega) = \pm q$. If $N_{K/\mathbb{Q}}(\omega) = -q$, one takes $\omega' = -\omega$ and we then have $N_{K/\mathbb{Q}}(\omega') = q$ since p is odd.

In case ii), the previous approach is valid for all prime numbers q dividing a. Now, look what happens to p. By Theorem 2.5, $p\mathbb{Z}_K = \mathfrak{Q}_1^{p-1}\mathfrak{Q}_2$. On the other hand, $p\mathbb{Z}_L = (1-\zeta_p)^{p-1}\mathbb{Z}_L$ by Lemma 2.4. Only the decomposition 1) of Theorem 2.9 must hold. Indeed, since $p\mathbb{Z}_L = (1-\zeta_p)^{p-1}\mathbb{Z}_L$, it suffices to factorize $(1-\zeta_p)\mathbb{Z}_L$ in M. We have $a^{p-1} \equiv 1 \mod p^2$ hence $(p, a) = 1$. Thus $((1-\zeta_p)\mathbb{Z}_L, a\mathbb{Z}_L) = 1$ since $(1-\zeta_p)\mathbb{Z}_L$ divides $p\mathbb{Z}_L$. So we can apply Theorem 2.9. In decomposition 2) and 3) of this Theorem, there is only one prime ideal of M lying over $(1-\zeta_p)\mathbb{Z}_L$ thus only one prime ideal of M lying over p: this is impossible since by Theorem 2.5, there are already two primes ideals lying over p in $K \subset M$. Hence it remains only the decomposition 1) of Theorem 2.9: $(1-\zeta_p)\mathbb{Z}_L$ splits completely in M and thus $\mathrm{Rac}(p\mathbb{Z}_M)) = \mathrm{Rac}\left((1-\zeta_p)^{p-1}\mathbb{Z}_M\right) = \mathrm{Rac}\left((1-\zeta_p)\mathbb{Z}_M\right) = (1-\zeta_p)\mathbb{Z}_M$. □

Remark 3.2. Leriche's result (Theorem 1.1) deals with $\omega \in K$ but her proof shows clearly that ω must be an algebraic integer. We obtain this Theorem by taking $p = 3$ and $a = bc^2$. It follows from Corollary 2.7, that the assertions ($a^2 \equiv 1 \mod 9$) and ($b^2 \equiv c^2 \mod 9$) are equivalent.

It is well known that the norm form equation $N_{K/\mathbb{Q}}(\omega) = q$ in our previous Theorem is difficult to solve in general. Our following result gives simple necessary conditions for M to be a Pòlya field based only on congruence relations. Before, define the radical of a to be

$$r(a) = \prod_{p|a} p$$

ie, the product of all the prime numbers dividing a, taken with multiplicity 1 (see [7], p.196).

Corollary 3.3. *If $M = \mathbb{Q}(\zeta_p, \sqrt[p]{a})$ is a Pòlya field then:*

- *when $a^{p-1} \not\equiv 1 \mod p^2$, for each prime number q dividing pa, we have $q \equiv u^p \mod r(a)$ with $u \in \mathbb{Z}$*
- *when $a^{p-1} \equiv 1 \mod p^2$, for each prime number q dividing a, we have $q \equiv u^p \mod r(a)$ with $u \in \mathbb{Z}$*

Proof. Let $\omega \in \mathbb{Z}_K$. By Theorem 2.8, we have:

- $\omega = x_0 + x_1\alpha_1 + \cdots + x_{p-1}\alpha_{p-1}$ where $x_i \in \mathbb{Z}$ if $a^{p-1} \not\equiv 1 \mod p^2$ ($\Leftrightarrow d \not\equiv 0 \mod p^2$ by Lemma 2.6)

- $\omega = x_0\gamma + x_1\alpha_1 + \cdots + x_{p-1}\alpha_{p-1}$ where $x_i \in \mathbb{Z}$ if $a^{p-1} \equiv 1 \mod p^2$.

Thus:

- $\omega - x_0 = x_1\alpha_1 + \cdots + x_{p-1}\alpha_{p-1}$ if $a^{p-1} \not\equiv 1 \mod p^2$
- $p\omega - x_0 = x_0(\alpha_1^{p-1} + b\alpha_1^{p-2} + \cdots + b^{p-2}\alpha_1) + p(x_1\alpha_1 + \cdots + x_{p-1}\alpha_{p-1})$ if $a^{p-1} \equiv 1 \mod p^2$

If σ is an embedding of K in $\mathbb{C}$ then $\sigma(\alpha_i) = \sigma(\sqrt[p]{\beta_i}) = \zeta_p^j \sqrt[p]{\beta_i}$ ($K = \mathbb{Q}(\sqrt[p]{a}) = \mathbb{Q}(\sqrt[p]{\beta_i})$). Thus let:

- $\theta = \omega$ if $a^{p-1} \not\equiv 1 \mod p^2$
- $\theta = p\omega$ if $a^{p-1} \equiv 1 \mod p^2$

Then:

$$N_{K/\mathbb{Q}}(\theta) = x_0^p + r(a)z \text{ with } z \in \mathbb{Z}$$

Indeed, we have:

$$N_{K/\mathbb{Q}}(\theta) - x_0^p \in \langle \alpha_1, \cdots, \alpha_{p-1} \rangle \mathbb{Z}_M = \left\langle \sqrt[p]{\beta_1}, \cdots, \sqrt[p]{\beta_{p-1}} \right\rangle \mathbb{Z}_M$$

As $N_{K/\mathbb{Q}}(\theta) - x_0^p \in \mathbb{Z}$ and $\left(\sqrt[p]{\beta_i}\right)\mathbb{Z}_M \cap \mathbb{Z} = \beta_i\mathbb{Z}$ then

$$N_{K/\mathbb{Q}}(\theta) - x_0^p \in \langle \beta_1, \cdots, \beta_{p-1} \rangle \mathbb{Z}$$

By definition of β_i, it follows that $r(a) \mid \beta_i$ for all i. Hence we have:

$$N_{K/\mathbb{Q}}(\theta) - x_0^p \in r(a)\mathbb{Z}$$

- $N_{K/\mathbb{Q}}(\theta) = N_{K/\mathbb{Q}}(\omega)$ if $a^{p-1} \not\equiv 1 \mod p^2$
- $N_{K/\mathbb{Q}}(\theta) = p^p N_{K/\mathbb{Q}}(\omega)$ if $a^{p-1} \equiv 1 \mod p^2$

If $a^{p-1} \equiv 1 \mod p^2$ then $p \nmid a$, so p^p is invertible modulo $r(a)$. Thus

$$N_{K/\mathbb{Q}}(\omega) - u^p \mod r(a) \text{ with } u \in \mathbb{Z}$$

The corollary now follows from Theorem 3.1. □

Remark 3.4. The necessary conditions of Corollary 3.3 holds if K is a Pòlya field since if K is a Pòlya field then M is also a Pòlya field. Indeed, it follows from Theorem 2.5 that $\Pi_q(K) = \mathrm{Rac}(q\mathbb{Z}_K)$ if $q \mid pa$. Thus by Ostrowski's result, $\mathrm{Rac}(q\mathbb{Z}_K)$ must be a principal ideal if $q \mid pa$ since K is a Pòlya field. The proof of the Theorem 3.1 shows that if ($a^{p-1} \not\equiv 1 \mod p^2$ and $q \mid pa$) or ($a^{p-1} \equiv 1 \mod p^2$ and $q \mid a$) then the assertions ($\mathrm{Rac}(q\mathbb{Z}_K)$ is a principal ideal) and $\left(\text{there is } \omega \in \mathbb{Z}_K \text{ such that } N_{K/\mathbb{Q}}(\omega) = q\right)$ are equivalent.
The converse is false, i.e M must be a Pòlya field whithout K: let $K = \mathbb{Q}(\sqrt[3]{11})$ and $M = \mathbb{Q}(j, \sqrt[3]{11})$. We have $11^2 \not\equiv 1 \mod 9$, $N_{K/\mathbb{Q}}(\sqrt[3]{11}) = 11$ and $N_{K/\mathbb{Q}}(-2 + \sqrt[3]{11}) = 3$. Thus, by Theorem 3.1, M is a Pòlya field. According to Zantema, K is a Pòlya field iff $h_K = 1$ since K is a S_3-field (see [15], Theorem 1.1). But table from [6] shows that $h_K = 2$.

Corollary 3.5. *If $\mathbb{Q}(\zeta_p, \sqrt[p]{q})$ is a Pòlya field then:*

- *either $q^{p-1} \equiv 1 \mod p^2$*
- *or $p \equiv u^p \mod q$ with $u \in \mathbb{Z}$*

Proof. A direct application of the previous Corollary on $\mathbb{Q}(\zeta_p, \sqrt[p]{q})$. □

We now give some sufficient conditions for M to be a Pòlya field.

Corollary 3.6. *If $p \nmid h_K$ then M is Pòlya field.*

Proof. For each case of Theorem 3.1, we have $q\mathbb{Z}_K = \mathfrak{Q}^p$ by Theorem 2.6 and hence $N_{K/\mathbb{Q}}(\omega) = q \Leftrightarrow \omega\mathbb{Z}_K = \mathfrak{Q}$. So it is enough to prove that $\mathfrak{Q}$ is a principal ideal. $\mathfrak{Q}^p$ is a principal ideal since $\mathfrak{Q}^p = q\mathbb{Z}_K$. On the other hand $\mathfrak{Q}^{h_K}$ is a principal ideal. Since $p \nmid h_K$ then by Bezout's Theorem, there exits $u, v \in \mathbb{Z}$ such that $up + vh_K = 1$. Thus $\mathfrak{Q} = (\mathfrak{Q}^p)^u \times \left(\mathfrak{Q}^{h_K}\right)^v$ is a principal ideal. □

Corollary 3.7. *Let $K = \mathbb{Q}(\sqrt[p]{q})$. Then $M = \mathbb{Q}(\zeta_p, \sqrt[p]{q})$ is a Pòlya field if and only if:*

- *either $q^{p-1} \equiv 1 \mod p^2$*
- *or there exists $\omega \in \mathbb{Z}_K$ such that $N_{K/\mathbb{Q}}(\omega) = p$.*

In particular, $\mathbb{Q}(\zeta_p, \sqrt[p]{p})$ is a Pòlya field since $N_{K/\mathbb{Q}}(\sqrt[p]{p}) = p$.

Proof. Recall that $p \geq 3$ and $q \geq 2$ are prime numbers. Take $a = q$. Then q is the only prime dividing a. By Theorem 3.1, $M = \mathbb{Q}(\zeta_p, \sqrt[p]{q})$ is a Pòlya field iff one of the following assertion is satisfied:

- $q^{p-1} \equiv 1 \mod p^2$ and there exists $\omega \in \mathbb{Z}_K$ such that $N_{K/\mathbb{Q}}(\omega) = q$.
- $q^{p-1} \not\equiv 1 \mod p^2$ and there exists $\omega, \omega' \in \mathbb{Z}_K$ such that $N_{K/\mathbb{Q}}(\omega) = q$ and $N_{K/\mathbb{Q}}(\omega') = p$.

Since $\sqrt[p]{q} \in K$ and $N_{K/\mathbb{Q}}(\sqrt[p]{q}) = q$, the two above assertions are equivalent to those of the Corollary 3.7. □

Example 3.8. Let n be a positive integer and p a prime number such that $p \mid n$. If $n^2 + 1$ is a prime number then $\mathbb{Q}(\zeta_p, \sqrt[p]{n^2+1})$ is a Pòlya field. Indeed we have $q = n^2 + 1 \equiv 1 \mod n^2 \equiv 1 \equiv 1 \mod p^2$ so $q^{p-1} \equiv 1 \mod p^2$.
Recall that p is a Wieferich's prime number if and only if

$$2^{p-1} \equiv 1 \mod p^2$$

If p is a Wieferich's prime number then $\mathbb{Q}(\zeta_p, \sqrt[p]{2})$ is a Pòlya field. It is enough to take $q = 2$ in Corollary 3.7.

4. Ramification in the Pòlya fields $K = \mathbb{Q}(\sqrt[p]{a})$ and $M = \mathbb{Q}(\zeta_p, \sqrt[p]{a})$

Recall that the ramified prime numbers in M are those dividing pa. With the notations of the previous section, we have the following result.

Theorem 4.1. *If M is a Pòlya field, let n_M be the number of ramified prime numbers in M. Then $n_M \leq \dfrac{p-1}{2} + 2$. Precisely let n_a be the number of prime divisors of a. Then*

- $n_a \leq \dfrac{p-1}{2} + 1$ *if* $a^{p-1} \equiv 1 \mod p^2$ *or* $p \mid a$
- $n_a \leq \dfrac{p-1}{2}$ *if* $a^{p-1} \not\equiv 1 \mod p^2$ *and* $p \nmid a$

Remark 4.2. By Remark 3.4, necessary conditions of the previous Theorem holds if we replace M by K.
General upperbounds exist for n_M in [8] and [15]. By Corollary 2.43 of [8], we have $n_M \leq p-1+\dfrac{p(p-1)}{2}+c_p$ with $c_p \geq 0$. It follows from Proposition 5.5 of [15] that $n_M \leq p$. Therefore Theorem 4.1 is sharper than these bounds.

To prove Theorem 4.1, we need some lemmas.

Lemma 4.3. *The following are equivalent:*

(1) $\beta \notin \mathbb{Q}^p$
(2) $X^p-\beta$ *is irreductible in* $\mathbb{Q}$.
(3) $X^p-\beta$ *has no roots in* $\mathbb{Q}$.

Proof. See [7], Theorem 9.1, p.297. □

Lemma 4.4. *Let* $\mathrm{Tor}(K^*/\mathbb{Q}^*)=\{x\mathbb{Q}^*,\ \exists n\in\mathbb{N}^*,\ x^n\in\mathbb{Q}\}$. *Then*

$$\mathrm{Tor}(K^*/\mathbb{Q}^*)=\{1\mathbb{Q}^*,\sqrt[p]{a}\mathbb{Q}^*,\cdots,(\sqrt[p]{a})^{p-1}\mathbb{Q}^*\}$$

Proof. Lemma 1.3, p.259 of [4] □

Corollary 4.5. *Let* $b,c\in\mathbb{Q}$. *Then* $\mathbb{Q}(\sqrt[p]{b})=\mathbb{Q}(\sqrt[p]{c})$ *if and only if* $c=b^\alpha u^p$ *with* $u\in\mathbb{Q}$ *and* $0\leq\alpha\leq p-1$.

Proof. $\mathbb{Q}(\sqrt[p]{b})=\mathbb{Q}(\sqrt[p]{c})\Rightarrow \mathrm{Tor}\left(\mathbb{Q}(\sqrt[p]{b})^*/\mathbb{Q}^*\right)=\mathrm{Tor}\left(\mathbb{Q}(\sqrt[p]{c})^*/\mathbb{Q}^*\right)$. Hence by Lemma 4.4, $\sqrt[p]{c}=\left(\sqrt[p]{b}\right)^\alpha\times u$ with $0\leq\alpha\leq p-1$ and $u\in\mathbb{Q}$ ie $c=b^\alpha u^p$. The converse is obvious. □

Lemma 4.6. *Let* q *be a prime number. If* $q\mathbb{Z}_K=\mathfrak{Q}^p$ *then there is an equivalence between the following assertions:*

1) *There is* $\omega\in\mathbb{Z}_K$ *such that* $N_{K/\mathbb{Q}}(\omega)=q$
2) *There is* $\varepsilon\in U_K$ *such that the equation* $\omega^p=\varepsilon N_{K/\mathbb{Q}}(\omega)$ *has a solution* $\omega\in\mathbb{Z}_K$ *with* $N_{K/\mathbb{Q}}(\omega)=q$

Proof. 1) ⇒ 2): $N_{K/\mathbb{Q}}(\omega\mathbb{Z}_K)=q$. Hence $\omega\mathbb{Z}_K$ is a prime ideal lying over q. Thus $\omega\mathbb{Z}_K=\mathfrak{Q}$ since $\mathfrak{Q}$ is the only prime ideal over q in K. We then have $\omega^p\mathbb{Z}_K=q\mathbb{Z}_K$ and hence there is $\varepsilon\in U_K$ such that $\omega^p=\varepsilon q$.
2) ⇒ 1) is trivial. □

Proof of Theorem 4.1. Let r be the number of fundamentals units in K. By Dirichlet's units Theorem, $U_K=\{\pm1\}\times V_K$ where V_K is a free $\mathbb{Z}$-module of rank r since the only roots of units in K are ±1. Suppose M is a Pòlya. Let s be the number of totally ramified prime numbers q_i in K ie ie $q_i\mathbb{Z}_K=\mathfrak{Q}_i^p$. We will show that $s\leq r+1$. Suppose $s\geq r+2$. It follows from Theorem 3.1 and Lemma 4.6

that there are (at least) $r+2$ elements $\omega_i \in \mathbb{Z}_K$ and $\varepsilon_i \in V_K$[2] such that

$$\begin{array}{l} \omega_1^p = \varepsilon_1 q_1 \\ \quad \vdots \qquad \vdots \\ \omega_r^p = \varepsilon_r q_r \\ \omega_{r+1}^p = \varepsilon_{r+1} q_{r+1} \\ \omega_{r+2}^p = \varepsilon_{r+2} q_{r+2} \end{array} \tag{4.1}$$

Let us first consider r and $r+1$. Since the rank of the $\mathbb{Z}$-module V_K is r then there exist $k_1, \ldots, k_r, k_{r+1} \in \mathbb{Z}$ such that

$$\varepsilon_1^{k_1} \times \cdots \times \varepsilon_r^{k_r} \times \varepsilon_{r+1}^{k_{r+1}} = 1 \tag{4.2}$$

There exists $\ell \in \{1, \ldots, r, r+1\}$ such that $p \nmid k_\ell$ even if we need to replace k_i, $i \in \{1, \ldots, r, r+1\}$ by $k_i' = \dfrac{k_i}{p^\alpha}$ with $\alpha = v_p\left(\gcd\left(k_1, \ldots, k_r, k_{r+1}\right)\right)$. Indeed $k_i' \in \mathbb{Z}$ and $p \nmid \gcd\left(k_1', \ldots, k_r', k_{r+1}'\right)$ since p^α is the most power of p that divides $\gcd\left(k_1, \ldots, k_r, k_{r+1}\right)$. From (4.2), we have $\varepsilon_1^{k_1'} \times \cdots \times \varepsilon_r^{k_r'} \times \varepsilon_{r+1}^{k_{r+1}'} = \left(\varepsilon_1^{k_1} \times \cdots \times \varepsilon_r^{k_r} \times \varepsilon_{r+1}^{k_{r+1}}\right)^{\frac{1}{p^\alpha}} = \sqrt[p^\alpha]{1}$. But 1 is the only p^α-root of unit in K since p is odd. So $\varepsilon_1^{k_1'} \times \cdots \times \varepsilon_r^{k_r'} \times \varepsilon_{r+1}^{k_{r+1}'} = 1$ and hence the relation (4.2) does not also change. We can assume that $\ell = r+1$ ie $p \nmid k_{r+1}$ even if we must reorder $\{k_1, \ldots, k_r, k_{r+1}\}$.
It follows from (4.1)) and (4.2) that

$$\left(\omega_1^{k_1} \times \cdots \times \omega_r^{k_r} \times \omega_{r+1}^{k_{r+1}}\right)^p = q_1^{k_1} \times \cdots \times q_r^{k_r} \times q_{r+1}^{k_{r+1}} \tag{4.3}$$

As $p \nmid k_{r+1}$ and the q_i are relatively prime then $q_1^{k_1} \times \cdots \times q_r^{k_r} \times q_{r+1}^{k_{r+1}} \notin \mathbb{Q}^p$. Hence by (4.3) and Lemma 4.3, we have

$$\mathbb{Q}\left(\sqrt[p]{q_1^{k_1} \cdots q_r^{k_r} q_{r+1}^{k_{r+1}}}\right) = \mathbb{Q}\left(\omega_1^{k_1} \cdots \omega_r^{k_r} \omega_{r+1}^{k_{r+1}}\right) \ , \ \left[\mathbb{Q}\left(\sqrt[p]{q_1^{k_1} \cdots q_r^{k_r} q_{r+1}^{k_{r+1}}}\right) : \mathbb{Q}\right] = p$$

So

$$\mathbb{Q}\left(\sqrt[p]{a}\right) = K = \mathbb{Q}\left(\sqrt[p]{q_1^{k_1} \cdots q_r^{k_r} q_{r+1}^{k_{r+1}}}\right)$$

since $\omega_i \in K$ and $[K : \mathbb{Q}] = p$. Thus, by Corollary 4.5, we have

$$q_1^{k_1} \cdots q_r^{k_r} q_{r+1}^{k_{r+1}} = a^\alpha u^p \text{ with } 1 \leq \alpha \leq p-1 \text{ and } u \in \mathbb{Q}$$

It follows that the set of prime divisors of a is contained in $\{q_1, \ldots, q_r, q_{r+1}\}$. Indeed let π be a prime divisor of a. Then $v_\pi\left(q_1^{k_1} \cdots q_r^{k_r} q_{r+1}^{k_{r+1}}\right) = v_\pi\left(a^\alpha u^p\right) = v_\pi\left(a^\alpha\right) + v_\pi\left(u^p\right)$; if $v_\pi\left(q_1^{k_1} \cdots q_r^{k_r} q_{r+1}^{k_{r+1}}\right) = 0$ then $v_\pi\left(a^\alpha\right) = -v_\pi\left(u^p\right)$, which is absurd since $p \mid v_\pi\left(u^p\right)$ and[3] $p \nmid v_\pi\left(a^\alpha\right)$. Morever $q_{r+1} \mid a$. Indeed $k_{r+1} = v_{q_{r+1}}\left(q_1^{k_1} \cdots q_r^{k_r} q_{r+1}^{k_{r+1}}\right) = v_{q_{r+1}}\left(a^\alpha u^p\right) = v_{q_{r+1}}\left(a^\alpha\right) + v_{q_{r+1}}\left(u^p\right)$; hence $v_{q_{r+1}}\left(a^\alpha\right) \neq 0$ since $p \nmid k_{r+1}$ and $p \mid v_{q_{r+1}}\left(u^p\right)$, so $v_{q_{r+1}}\left(a\right) \neq 0$.

[2] $\omega^p = q\varepsilon \Leftrightarrow (-\omega)^p = q(-\varepsilon)$. Since $U_K = \pm V_K$, it's enough to take ε in V_K instead of in U_K.
[3] u^p is a p-th power and a^α is p-th power free since a is p-th power free and $0 \leq \alpha \leq p-1$.

With the same idea, we can show that the set of prime divisors of a is contained in $\{q_1, \ldots, q_r, q_{r+2}\}$. Thus we obtain a contradiction since $q_{r+1} \mid a$ and $q_{r+1} \notin \{q_1, \ldots, q_r, q_{r+2}\}$.
It follows from the previous that if M is a Pòlya field then the number s of totally ramified prime numbers in K satisfied $s \leq r+1$. Since r is the number of fundamental units in K and the signature of K is $\left(1, \frac{p-1}{2}\right)$ then $r = \frac{p-1}{2}$.
Now let n_a be the number of prime divisors of a. Then by Theorem 2.5 and Lemma 2.6,

$$s = \begin{cases} n_a & \text{if } a^{p-1} \equiv 1 \mod p^2 \text{ or } p \mid a \\ n_a + 1 & \text{if } a^{p-1} \not\equiv 1 \mod p^2 \text{ and } p \nmid a \end{cases}$$

Hence we have

$$\begin{aligned} n_a &\leq \frac{p-1}{2} + 1 \qquad \text{if } a^{p-1} \equiv 1 \mod p^2 \text{ or } p \mid a \\ n_a &\leq \frac{p-1}{2} \qquad\quad \text{if } a^{p-1} \not\equiv 1 \mod p^2 \text{ and } p \nmid a \end{aligned}$$

since $s \leq r+1$ and $r = \frac{p-1}{2}$. □

Remark 4.7. We will give an application of the previous Theorem on Theorem 1.1. Suppose a cubefree. Let n_a be the number of prime divisors of a. Suppose that $K = \mathbb{Q}(\sqrt[3]{a})$ or $M = \mathbb{Q}(j, \sqrt[3]{a})$ is a Pòlya field. By Theorem 4.1 and Remark 4.2, we have:

- $n_a \leq 2$ if $a^2 \equiv 1 \mod 9$ or $3 \mid a$
- $n_a \leq 1$ if $a^2 \not\equiv 1 \mod 9$ and $3 \nmid a$

Since $\mathbb{Q}(\sqrt[3]{a}) = \mathbb{Q}(\sqrt[3]{a^2})$, we can also suppose that a is not a square. More generally, for $b, c \in \mathbb{N}$, we have $\mathbb{Q}(\sqrt[3]{bc^2}) = \mathbb{Q}(\sqrt[3]{b^2c})$. Morever if $3 \mid a$ then $a^2 \not\equiv 1 \mod 9$. Hence

- if $a^2 \equiv 1 \mod 9$ then $a = q$ or $a = q_1q_2$ or $a = q_1^2q_2$
- if $a^2 \not\equiv 1 \mod 9$ then $a = q$ or $a = 3q$ or $a = 3q^2$

where q, q_1, q_2 are prime numbers. Thus a Pòlya field of the form $\mathbb{Q}(\sqrt[3]{a})$ or $\mathbb{Q}(j, \sqrt[3]{a})$ must be in one of the sets of the following Definition except when $a = 3$.

Definition 4.8. Let

- E_1 be the set of fields $K = \mathbb{Q}(\sqrt[3]{a})$ and $M = \mathbb{Q}(j, \sqrt[3]{a})$ such that a is a prime number and $a^2 \equiv 1 \mod 9$.
- E_2 be the set of fields $K = \mathbb{Q}(\sqrt[3]{a})$ and $M = \mathbb{Q}(j, \sqrt[3]{a})$ such that $a^2 \equiv 1 \mod 9$ and there exists two prime numbers q_1, q_2 such that $a = q_1q_2$ or $a = q_1^2q_2$.
- E_3 be the set of fields $K = \mathbb{Q}(\sqrt[3]{a})$ and $M = \mathbb{Q}(j, \sqrt[3]{a})$ such that $a^2 \not\equiv 1 \mod 9$ and there exists a prime number q, $q \neq 3$ such that $a = q$ or $a = 3q$ or $a = 3q^2$.

The following Theorem is a refinement of Theorem 1.1.

Theorem 4.9. *$M = \mathbb{Q}(j, \sqrt[3]{a})$ is a Pòlya field if and only if one of the following is satisfied*

1) $M = \mathbb{Q}(j, \sqrt[3]{3})$
2) $M \in E_1$
3) $M \in E_2$ *and there is an algebraic integer* $\omega \in K$ *and a prime* $q \mid a$ *such that* $N_{K/\mathbb{Q}}(\omega) = q$.
4) $M \in E_3$ *and there is an algebraic integer* $\omega \in K$ *such that* $N_{K/\mathbb{Q}}(\omega) = 3$.

Proof. The necessity follows naturally from Remark 4.7 and Theorem 3.1.
By Corollary 3.7, 1) and 2) are sufficient. Let us now look at the sufficiency of 3) and 4). Suppose $M \in E_2$ and $a = q_1 q_2$ where q_1 and q_2 are prime numbers. Suppose that there exsits $\omega \in \mathbb{Z}_K$ such that $N_{K/\mathbb{Q}}(\omega) = q_1$. We will show that there exists $\omega' \in \mathbb{Z}_K$ such that $N_{K/\mathbb{Q}}(\omega') = q_2$ and by Theorem 4.1, we can conclude that M is a Pòlya field. $a\mathbb{Z}_K = q_1 q_2 \mathbb{Z}_K = \omega^3 q_2 \mathbb{Z}_K$. Hence $q_2\mathbb{Z}_K = \left(\omega^{-1}\sqrt[3]{a}\right)^3 \mathbb{Z}_K$. So $\omega^{-1}\sqrt[3]{a} \in \mathbb{Z}_K$ and $N_{K/\mathbb{Q}}\left(\omega^{-1}\sqrt[3]{a}\right) = q_2$. Thus $\omega' = \omega^{-1}\sqrt[3]{a}$. The condition 3) is then sufficient when $a = q_1 q_2$. With the same idea we get the sufficiency of the cases ($M \in E_2$ and $a = q_1^2 q_2$), ($M \in E_3$ and $a = 3q$ or $3q^2$). Finaly let $M \in E_3$ such that a is a prime number and there exists $\omega \in \mathbb{Z}_K$ such that $N_{K/\mathbb{Q}}(\omega) = 3$. By Theorem 3.1, M is a Pòlya field since the only prime divisors of $3a$ are 3 and a, $a^2 \not\equiv 1 \mod 9$, $N_{K/\mathbb{Q}}(\omega) = 3$ and $N_{K/\mathbb{Q}}(\sqrt[3]{a}) = a$. Therefore we can conclude that all conditions of the Theorem are sufficient. □

Remark 4.10. Let us show why Theorem 4.9 is a refinement of Theorem 1.1. By Remark 3.2, the condition ($b^2 \equiv c^2 \mod 9$) of Theorem 1.1 is equivalent to ($a^2 \equiv 1 \mod 9$). By Theorem 1.1, to know a Pòlya field $\mathbb{Q}(j, \sqrt[3]{a})$, we must solve the norm form equations $N_{K/\mathbb{Q}}(\omega) = q$, $\omega \in \mathbb{Z}_K$ for all prime divisors of a or $3a$ (depending on wheter we have $a^2 \equiv 1 \mod 9$ or not). Each of these equations is difficult de solve in general. Theorem 4.9 shows that this resolution is not necessary if the number n_a of prime divisors of a satisfied

- $n_a > 2$ and ($a^2 \equiv 1 \mod 9$ or $3 \mid a$)
- $n_a > 1$ and ($a^2 \not\equiv 1 \mod 9$ and $3 \nmid a$)

since in this case Theorem 4.9 shows clearly that M is not a Pòlya field. For example $\mathbb{Q}(j, \sqrt[3]{30})$ is not a Pòlya field since $n_{30} = 3 > 2$. Similary $\mathbb{Q}(j, \sqrt[3]{77})$ is not since $n_{77} = 2 > 1$, $77^2 \equiv (-4)^2 \equiv -2 \not\equiv 1 \mod 9$ and $3 \nmid 77$.
Morever even if we need to solve norm form equation $N_{K/\mathbb{Q}}(\omega) = q$, only one is necessary in the Theorem 4.9 whereas in Theorem 1.1 it must be solved for all prime divisors of a. For example, suppose $3 \nmid a$ and a cubefree. If the Thue equation $x^3 + ay^3 = 3$ has a solution $x, y \in \mathbb{Z}$ then the field $M = \mathbb{Q}(j, \sqrt[3]{a}) = K(j)$ is a Pòlya field if and only if a is a prime number or a square of a prime. Indeed $x^3, y^3 \equiv \{-1, 0, 1\} \mod 9$. Hence $a \equiv \pm\{4, 3, 2\} \mod 9$, so $a^2 \not\equiv 1 \mod 9$. Thus $M \notin E_1$ and $M \notin E_2$. Hence, by Theorem 4.9, if M is a Pòlya field field then $M \in E_3$. Since $3 \nmid a$ and $\mathbb{Q}(\sqrt[3]{a}) = \mathbb{Q}(\sqrt[3]{a^2})$, a must be a prime number or a square of a prime number. The conclusion follows since $N_{K/\mathbb{Q}}(x + y\sqrt[3]{a}) = x^3 + ay^3 = 3$ give us one solution of norm form equation. In particular let $x \in \mathbb{Z}$ such that $3 \nmid x$ and $3 + x^3$ cubefree. Then $\mathbb{Q}(j, \sqrt[3]{3 + x^3})$ is a Pòlya field if and only if $3 + x^3$ is a prime number or $a = 4$. Indeed it follows from [5] that $3 + x^3$ is a square of an integer if and only if $x = 1$.

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[1] Laboratoire de Théorie des Nombres, Algèbre, Géométrie Algégrique et Topologie Algébrique (TNAGATA), UFR. Sciences Exactes et Appliquées, 03 BP 7021 Ouaga 03, Ouagadougou, Burkina Faso

Email address: kientdia@yahoo.fr

[2] Laboratoire d'Algèbre, de Mathématiques Discrètes et d'Informatique (LAMDI), Université Nazi Boni, 01 BP 1091 Bobo Dioulasso 01, Burkina Faso.

Email address: tmytapsoba@yahoo.fr

[3] Laboratoire de Théorie des Nombres, Algèbre, Géométrie Algégrique et Topologie Algébrique (TNAGATA), UFR. Sciences Exactes et Appliquées, 03 BP 7021 Ouaga 03, Ouagadougou, Burkina Faso

Email address: tougmacharles@yahoo.fr