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GLOBAL EXISTENCE OF A PAIR OF COUPLED CAHN-HILLIARD EQUATIONS WITH NONDEGENERATE MOBILITY AND LOGARITHMIC POTENTIAL

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Abstract. We conducted a mathematical investigation on a system of interconnected Cahn-Hilliard equations featuring a logarithmic potential, nondegenerate mobility, and homogeneous Neumann boundary conditions. This system emerges from a model depicting the phase separation of a binary liquid mixture in a thin film. Assuming certain conditions on the initial data, we successfully established the existence, uniqueness, and stability estimates for the weak solution. Our approach involved initially replacing the logarithmic potential with a smooth counterpart, resulting in the regularization of the original problem (Q) into a regularized problem (Q_ϵ). Utilizing the Faedo-Galerkin method and compactness arguments, we demonstrated the existence and uniqueness of a solution for (Q_ϵ). Subsequently, by letting ϵ approach zero, we attained the existence of a solution for the original problem (Q). Additionally, we addressed higher regularity aspects of the weak solutions for both (Q) and (Q_ϵ). Employing the standard regularity theory for elliptic problems and introducing additional assumptions regarding the domain's boundary and the initial data, we established that the weak solutions belong to higher-order Sobolev spaces.

1. Introduction

Recently, various models of the Cahn-Hilliard equations have garnered significant attention, owing to their widespread application across diverse domains such as the modeling of alloys, glasses, and polymers (refer to [20, 34] for examples). The original Cahn-Hilliard model was introduced by Cahn and Hilliard [10] to elucidate the mechanics of binary mixture separation into distinct phases. This classical model has proven effective in adequately capturing spinodal decomposition or phase separation phenomena. For qualitative explorations on this subject, see, for instance, [11, 28, 29].

There are many applications of the Cahn-Hilliard equation with a logarithmic free energy [10, 11, 28, 29]. Firstly, the equation can be used to study the evolution of concentration fields in binary alloys, helping to understand and predict the microstructure of materials during phase separation. Also, in polymer blends,

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the Cahn-Hilliard equation with a logarithmic free energy can be applied to study the evolution of domain morphology during phase separation, providing insights into the structure of polymer blends. Moreover, the equation can be adapted to model pattern formation in biological systems, such as cell sorting and tissue development, where phase separation occurs. In addition, the Cahn-Hilliard equation can be applied to model the dynamics of phase separation in emulsions and foams, where the logarithmic free energy captures the tendency of immiscible fluids to separate. Furthermore, the equation can be used to study the evolution of surface morphology in thin film coatings, helping to predict and control the development of patterns and structures.

The classical Cahn-Hilliard equation is characterized by a fourth-order, time-dependent, and nonlinear partial differential equation, which can be expressed in the form:

$$\partial_t\phi - \nabla(M(\phi)\nabla w) = 0, \qquad \text{in} \quad \mathfrak{R}_T = \mathfrak{R}\times(0,T), T>0, \tag{1.1}$$

$$w = -\gamma\Delta\phi + \Psi'(\phi), \qquad \text{in} \quad \mathfrak{R}_T. \tag{1.2}$$

The mobility function $M \in C([-1;1])$ is presupposed to meet the following condition:

$$M(1) = M(-1) = 0, \quad m_0 \le M(s) \le M_0, \quad \forall s \in (1-,1), \quad m_0, M_0 > 0. \tag{1.3}$$

The initial condition and the boundary conditions are, respectively,

$$\phi(x,0) = \phi^0(x), \qquad \text{in} \quad \mathfrak{R}, \tag{1.4}$$

$$\frac{\partial\phi}{\partial\nu} = \frac{\partial w}{\partial\nu} = 0, \qquad \text{in} \quad \partial\mathfrak{R}\times(0,T), \tag{1.5}$$

In this context, $\mathfrak{R}$ represents a bounded domain in $\mathbb{R}^d, d = 1,2,3$, featuring a Lipschitz boundary $\partial\mathfrak{R}$. The outward unit normal to $\mathfrak{R}$ is denoted as ν. The variable ϕ signifies the concentration of the two components, while w represents the chemical potential, defined as the variational derivative of the Ginzburg-Landau free energy functional in the form

$$\Lambda(\phi) = \int_{\mathfrak{R}} [\frac{\gamma}{2}|\nabla\phi|^2 + \Psi(\phi)]dx. \tag{1.6}$$

Cahn and Hilliard incorporated the gradient term, $|\nabla\phi|^2$, into the free energy functional Λ to represent the surface energy that distinguishes between phases, where γ denotes a positive constant associated with surface tension.

The function Ψ in equation (1.6) signifies the homogeneous potential, typically adopting a symmetric double well-form. To streamline mathematical computations, Ψ is commonly expressed as a quartic polynomial in the subsequent manner:

$$\Psi(\phi) = a\phi^4 - b\phi^2 + c \quad a,b>0, c\in\mathbb{R}. \tag{1.7}$$

When the quenching temperature, θ, closely approaches a critical temperature ω, the quartic polynomial potential can be interpreted as an approximation of the thermodynamic logarithmic potential given by:

$$\Psi(\phi) = \frac{\theta}{2}[(1+\phi)\ln(1+\phi) + (1-\phi)\ln(1-\phi)] + \frac{\omega}{2}(1-\phi^2) \quad -1\le\phi\le 1. \tag{1.8}$$

The quartic Taylor polynomial for this logarithmic potential is expressed as:

$$\Psi(\phi) \approx \frac{\theta}{12}\phi^4 - \frac{(\omega-\theta)}{2}\phi^2 + \frac{\omega}{2}.$$

This aligns with the structure outlined in equation (1.7). The logarithmic shape of the potential was proposed by Cahn and Hilliard, as documented in [10]. It is worth noting that in this logarithmic expression, Ψ exhibits the necessary double well-form, featuring minima at α and $-\alpha$, i.e. α is the positive root of,$\Psi'(\alpha) = 0$,

$$\ln(\frac{1+\alpha}{1-\alpha}) = \frac{2\alpha\omega}{\theta}.$$

In the scenario where θ approaches 0, the parameter α converges to 1. In this particular case, the logarithmic potential can be substituted with the following obstacle potential:

$$\Psi(\phi) = \begin{cases} \frac{\omega}{2}(1-\phi^2) & \text{if } |\phi| \leq 1, \\ \infty & \text{if } |\phi| > 1. \end{cases} \tag{1.9}$$

The proposed potential form was initially introduced in [30]. For detailed mathematical and numerical investigations concerning the classical Cahn-Hilliard equation with various formulations of the free energy, please consult [1, 6, 15, 17] and the associated references.

In this article, our focus revolves around a pair of interconnected Cahn-Hilliard equations featuring a logarithmic potential, nondegenerate mobility, and homogeneous Neumann boundary conditions. The specific form of these equations is as follows [11]:
Find $\{\phi_1(x,t), \phi_2(x,t)\} \in \mathbb{R}\times\mathbb{R}$ such that

$$\partial_t\phi_1 - \nabla(M(\phi_1)\nabla w_1) = 0, \qquad \text{in} \quad \mathfrak{R}_T, \tag{1.10}$$

$$\partial_t\phi_2 - \nabla(M(\phi_2)\nabla w_1) = 0, \qquad \text{in} \quad \mathfrak{R}_T, \tag{1.11}$$

where

$$w_1 = \frac{\delta\Lambda(\phi_1,\phi_2)}{\delta\phi_1}, \qquad \text{in} \quad \mathfrak{R}_T, \tag{1.12}$$

$$w_2 = \frac{\delta\Lambda(\phi_1,\phi_2)}{\delta\phi_2}, \qquad \text{in} \quad \mathfrak{R}_T, \tag{1.13}$$

where

$$\Lambda(\phi_1,\phi_2) = \int_{\mathfrak{R}} \Psi_1(\phi_1) + \frac{\gamma_1}{2}|\nabla\phi_1|^2 + \Psi_2(\phi_2) + \frac{\gamma_2}{2}|\nabla\phi_2|^2 + D(\phi_2+\alpha_1)^2(\phi_2+\alpha_2)^2, \tag{1.14}$$

with initial conditions

$$\phi_1(x,0) = \phi_1^0(x), \qquad \phi_2(x,0) = \phi_2^0(x) \qquad \text{in} \quad \mathfrak{R}, \tag{1.15}$$

and boundary conditions

$$\frac{\partial\phi_1}{\partial\nu} = M(\phi_1)\frac{\partial w_1}{\partial\nu} = \frac{\partial\phi_2}{\partial\nu} = M(\phi_2)\frac{\partial w_2}{\partial\nu} = 0, \qquad \text{on} \quad \partial\mathfrak{R}\times(0,T), \tag{1.16}$$

In the given context, the symbol $\frac{\delta\Lambda(\phi_1,\phi_2)}{\delta\phi_i}$,,, $i = 1,2$, represents the variational derivative of the free energy functional Λ with respect to the field variable ϕ_i. The variable ϕ_1 conveys information about the local concentration of either component A or B, while ϕ_2 signifies the presence of a liquid or vapor phase. The positive constants $\gamma_i,\ i=1,2$, correspond to the surface tension of ϕ_i, and the coupling constant D is a prescribed positive constant. Additionally, $\alpha_i,\ i=1,2$, represents a positive constant associated with the minimum of a double well potential Ψ_i. In scenarios where Ψ_i transforms into an obstacle double well potential and the D-term is substituted with a bilinear term, the model encompasses three-phase systems, as discussed in prior studies. In the absence of the D-coupling term ($D=0$) in the free energy functional Λ, the model simplifies to two classical single Cahn-Hilliard equations, extensively explored in mathematical literature, as indicated in references such as [7, 9].

Now, when we take into account the logarithmic potential (1.9) and, for simplicity, assume equal surface tensions, i.e., $\gamma=\gamma_1=\gamma_2$, the ensuing problem becomes the focal point of our investigation in this article:
(Q) Find $\{\phi_1(x,t),\phi_1(x,t)\}\in\mathbb{R}\times\mathbb{R}$ such that

$$\partial_t\phi_1-\nabla(M(\phi_1)\nabla w_1)=0, \qquad \text{in} \quad \mathfrak{R}_T, \tag{1.17}$$

$$\partial_t\phi_2-\nabla(M(\phi_2)\nabla w_1)=0, \qquad \text{in} \quad \mathfrak{R}_T, \tag{1.18}$$

where

$$w_1=-\gamma\Delta\phi_1+\Psi_1'(\phi_1)+f_D^{(1)}(\phi_1,\phi_2), \qquad \text{in} \quad \mathfrak{R}_T, \tag{1.19}$$

$$w_2=-\gamma\Delta\phi_2+\Psi_2'(\phi_2)+f_D^{(2)}(\phi_1,\phi_2), \qquad \text{in} \quad \mathfrak{R}_T, \tag{1.20}$$

subject to the initial conditions

$$\phi_1(x,0)=\phi_1^0(x), \quad , \ \phi_2(x,0)=\phi_2^0(x) \qquad \text{in} \quad \mathfrak{R}, \tag{1.21}$$

and boundary conditions

$$\frac{\partial\phi_1}{\partial\nu}=M(\phi_1)\frac{\partial w_1}{\partial\nu}=\frac{\partial\phi_2}{\partial\nu}=M(\phi_2)\frac{\partial w_2}{\partial\nu}=0, \qquad \text{on} \quad \partial\mathfrak{R}\times(0,T), \tag{1.22}$$

where

$$\psi_i(\mathfrak{s})=\psi(\mathfrak{s})+\frac{\theta_i}{2}(1-\mathfrak{s}^2) \quad i=1,2,-1\le\mathfrak{s}\le1,\ \ 0\le\theta<\theta_i, \tag{1.23}$$

$$\psi(\mathfrak{s})=\frac{\theta}{2}[(1+\mathfrak{s})\ln(1+\mathfrak{s})+(1-\mathfrak{s})\ln(1-\mathfrak{s})], \tag{1.24}$$

$$f_D(\mathfrak{s}_1,\mathfrak{s}_2)=D(\mathfrak{s}_1+\sigma_1)^2(\mathfrak{s}_2+\sigma_2)^2, \tag{1.25}$$

$$f_D^{(i)}(\mathfrak{s}_1,\mathfrak{s}_2)=\frac{\partial f_D(\mathfrak{s}_1,\mathfrak{s}_2)}{\partial\mathfrak{s}_i}=2D(\mathfrak{s}_i+\sigma_i)(\mathfrak{s}_j+\sigma_j)^2 \quad i,j=1,2,i\ne j. \tag{1.26}$$

where, $\gamma,\ D,\ \theta,\ \theta_i$ and σ_i are positive constants with $\theta<\theta_i$ and $\Psi_i'(\sigma_i)=0$.
Note that (i) since Ψ_i takes its minimum at $\pm\alpha_i$, we have $0<\alpha_i<1$,
(ii) Ψ_i is defined at $\mathfrak{s}=\pm1$ as

$$\Psi_i(\pm1)=\lim_{\mathfrak{s}\to\pm1}\Psi_i(\mathfrak{s})=\theta\ln 2.$$

Upon introducing a function $\Phi\in C[0,\infty)$ such that

$$\Phi(\mathfrak{s})=\frac{\theta}{2}\mathfrak{s}\ln\mathfrak{s}, \tag{1.27}$$

then, ψ can be reformulated as

$$\psi(\mathfrak{s}) = \Phi(1+\mathfrak{s}) + \Phi(1-\mathfrak{s}) \tag{1.28}$$

For analytical purposes, we define the monotone function $\varphi : (-1,1) \to \mathbb{R}$ as

$$\varphi(\mathfrak{s}) = \psi'(\mathfrak{s}) = \Phi'(1+\mathfrak{s}) - \Phi'(1-\mathfrak{s}) = \frac{\theta}{2}[\ln(1+\mathfrak{s}) - \ln(1-\mathfrak{s})]. \tag{1.29}$$

To formulate a weak form of problem, we introduce a test function $\Pi \in H^1(\mathfrak{R})$ and leverage Green's theorem. Furthermore, a solution is considered weak for the system (1.10)-(1.26) if there exists a set ϕ_1, ϕ_2, w_1, w_2 such that $\phi_1, \phi_2 \in L^\infty(0,T; H^1(\mathfrak{R})) \cap H^1(0,T; (H^1(\mathfrak{R}))')$ and $w_1, w_2 \in L^2(0,T; H^1(\mathfrak{R}))$. These entities satisfy the weak formulation:
(Q) Find $\{\phi_1, \phi_2, w_1, w_2\} \in [H^1(\mathfrak{R})]^4$ such that for a.e. $t \in (0,T)$, $\forall i = 1,2$ and for all $\Pi \in H^1(\mathfrak{R})$

$$< \partial_t \phi_i, \Pi > + (M(\phi_i)\nabla w_i, \nabla \Pi) = 0, \tag{1.30}$$

$$\gamma(\nabla \phi_i, \nabla \Pi) + (\Psi_i'(\phi_i), \Pi) + (f_D^{(i)}(\phi_1, \phi_2), \Pi) = (w_i, \Pi), \tag{1.31}$$

$$\phi_i(.,0) = \phi_i^0, \tag{1.32}$$

While the Cahn-Hilliard equation has been extensively explored, there has been limited attention to mathematical studies involving a diffusional mobility, denoted as $M(\phi)$, that depends on the difference in mass density, represented by ϕ (where ϕ signifies the mass density difference of the two alloy components). The original formulation of the Cahn-Hilliard equation incorporated a concentration-dependent mobility, as seen in [11]. Consequently, a thermodynamically valid choice for $M(\phi)$ is $1-\phi^2$, as acknowledged in [12, 22, 35]. The degeneracy of M introduces mathematical challenges when solving the Cahn-Hilliard equation with such mobility. However, there is optimism that solutions initially confined to the interval $[-1,1]$ will persist within this range for all positive timesa property not applicable to fourth-order parabolic equations lacking degeneracy. It is essential to highlight that only values within the interval $[-1,1]$ hold physical significance.

Existence studies for the Cahn-Hilliard equation with degenerate mobility in a one-dimensional context have been explored in [23]. In [18], existence conclusions are presented for arbitrary space dimensions, utilizing a weak form distinct from that in [23]. Additionally, [18] allows for singularities in the bulk energy when M degenerates. For investigations into fourth-order degenerate parabolic equations in one spatial dimension, refer to the work in [8]. Given the significance of this topic, recent studies have delved into the existence and uniqueness of differential equations, as evidenced in [3, 4, 5, 16, 21, 25, 27, 31, 32, 33, 37].

In [2], mathematical analysis has been undertaken for a pair of coupled Cahn-Hilliard equations with a logarithmic potential. Global existence of a weak solution to the problem is proved. Moreover, higher regularity results of the weak solution are established under further regular requirements on the initial data. In this article, we consider the same system of [2], but with a diffusional mobility $M(\phi_i)$ depending on ϕ_i, $i = 1,2$. This type of dependent mobility was suggested in [11]. Studying coupled Cahn-Hilliard equation with diffusional mobility is a major mathematical challenge and of great importance.

The work presented in [2] involves a comprehensive mathematical analysis of a pair of interconnected Cahn-Hilliard equations featuring a logarithmic potential. The study established the global existence of a weak solution to the problem and further demonstrates higher regularity results for the weak solution, contingent upon additional regularity requirements imposed on the initial data. In the current manuscript, we explore the identical system as investigated in [2], but with a diffusional mobility $M(\phi_i)$ that is depend on the phase field variable ϕ_i, where $i = 1, 2$. This form of mobility, dependent on the phase variable, was initially proposed in [11]. The examination of a coupled Cahn-Hilliard equation with such a diffusional mobility presents a significant mathematical challenge and holds substantial importance.

In this investigation, we employ two sets of assumptions regarding the initial data, denoted as ϕ_1^0, ϕ_2^0, for a thorough mathematical analysis of the problem denoted as (Q). The examination of (Q) involves the introduction of a regularized form, denoted as (Q_ϵ), followed by the exploration of the limit as ϵ approaches 0. This approach is adopted due to the singularity of the potential Ψ_i, where $i = 1, 2$. The methodology employed here aligns with the one utilized by Elliott and Luckhaus [19], who applied a similar technique to investigate a singular Cahn-Hilliard equation. Subsequently, this method found application in the analysis of multiple Cahn-Hilliard equations with non-smooth free energy in mathematical literature, as exemplified in [15, 9].

Here is a quick summary of each section of this article. Section 2 provides a comprehensive discussion of the fundamental notation employed in this paper, focusing on Sobolev spaces. Additionally, various auxiliary results are recalled and demonstrated. In Section 3, a regularized version of problem (Q_ϵ) is introduced. The section also presents essential findings that facilitate the treatment of non-linear terms, offering equivalent weak formulations for both the original problem (Q) and its regularized counterpart, (Q_ϵ). Section 4 employs the Faedo-Galerkin technique and compactness arguments to establish the existence of solutions for (Q) and (Q_ϵ) under a set of assumptions (A1) imposed on the initial conditions. Moving on to Section 5, additional regularity criteria are introduced on the initial conditions and domain boundary, denoted as assumptions (A2). This section aims to demonstrate further regularity results for the weak solutions established in Section 4. The article concludes with a summary of the key findings and implications of the study.

2. Notation and auxiliary results

The $L^2(\mathfrak{R})$ inner product over $\mathfrak{R}$, equipped with the norm $|\cdot|_0$, is represented as $(\cdot,\cdot)$. Furthermore, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $(H^1(\mathfrak{R}))'$ and $H^1(\mathfrak{R})$, where $(H^1(\mathfrak{R}))'$ is the dual space of $H^1(\mathfrak{R})$. A norm on $(H^1(\mathfrak{R}))'$ is expressed as:

$$\|\Pi\|_{(H^1(\mathfrak{R}))'} := \sup_{\Pi\neq 0} \frac{|\langle \Pi, \Pi\rangle|}{\|\Pi\|_1} \equiv \sup_{\|\Pi\|_1=1} |\langle \Pi, \Pi\rangle|. \tag{2.1}$$

Furthermore, we designate the function spaces that depend on both time and space as $L^\sigma(0, T; Y)$ $(1 \leq \sigma \leq \infty)$, where Y is a Banach space. These spaces encompass all functions Π such that, for almost every $s \in (0,T)$, $\Pi \in Y$, and the ensuing norm is finite:

$$\|\Pi(s)\|_{L^\sigma(0,T;Y)} = \left(\int_0^T \|\Pi(s)\|_Y^\sigma ds\right)^{\frac{1}{\sigma}},$$

$$\|\Pi(s)\|_{L^\infty(0,T;Y)} = ess \sup_{s\in(0,T)} \|\Pi(s)\|_Y,$$

$$\langle \eta, \Pi\rangle = (\eta,\Pi), \quad \forall \eta \in L^2(\mathfrak{R}) \quad \text{and} \quad \Pi \in H^1(\mathfrak{R}). \tag{2.2}$$

Additionally, we introduce the spaces $L^\sigma(\mathfrak{R}_T) = L^\sigma(0,T;L^\sigma(\mathfrak{R}))$, where $\sigma \in [1,\infty]$. Furthermore, we define $C([0,T];Y)$ as the space of continuous functions from $[0,T]$ into Y, comprising functions $\Pi(s):[0,T]\to Y$ such that $\Pi(s)\to\Pi(s0)$ in Y as $s\to s_0$. The associated norm for the space $C([0,T];Y)$ can be defined as per [36]:

$$\|\Pi(s)\|_{C([0,T];Y)} = \sup_{s\in[0,T]} \|\Pi(s)\|_Y.$$

In addition, the mean integral can be define as follows:

$$⨍ \zeta = \frac{1}{|\mathfrak{R}|}(\zeta,1), \quad \forall \zeta \in L^1(\mathfrak{R}). \tag{2.3}$$

It is convenient to define "the inverse Laplacian Green's operator" $\mathcal{G}:\mathcal{F}_0\to V_0$ in such a way that

$$(\nabla\mathcal{G}\eta,\nabla\Pi) = \langle\eta,\Pi\rangle, \quad \forall \Pi\in H^1(\mathfrak{R}), \tag{2.4}$$

where $\mathcal{F}_0 = \{\eta\in (H^1(\mathfrak{R}))' : \langle\eta,1\rangle = 0\}$ and $V_0 = \{\ell \in H^1(\mathfrak{R}) : (\ell,1)=0\}$.
The Lax-Milgram theorem and the subsequent Poincaré inequality may be used to determine the well-posedness of $\mathcal{G}$. The Poincaré inequality is defined in the form [38]:

$$|\ell|_0 \le \mathfrak{C}_P(|\ell|_1 + |(\ell,1)|), \quad \forall \ell\in H^1(\mathfrak{R}). \tag{2.5}$$

The norm defined in (2.1) on $(H^1(\mathfrak{R}))'$ serves as a norm on $\mathcal{F}_0$. An equivalent norm on $\mathcal{F}_0$ can be constructed, as illustrated in the proof of Lemma 2.1.1 in [2], in the form

$$\|\ell\|_{-1} = |\mathcal{G}\ell|_1 \equiv \langle \ell, \mathcal{G}\ell\rangle^{\frac{1}{2}}, \quad \forall \ell\in\mathcal{F}_0. \tag{2.6}$$

It follows from (2.2) and (2.5) for any $\ell\in L^2(\mathfrak{R})\cap\mathcal{F}_0$ that

$$\|\ell\|_{-1}^2 = \langle\ell,\mathcal{G}\ell\rangle = (\ell,\mathcal{G}\ell) \le \|\ell\|_0\|\mathcal{G}\ell\|_0 \le \mathfrak{C}_P\|\ell\|_0|\mathcal{G}\ell|_1 = \mathfrak{C}_P\|\ell\|_0\|\ell\|_{-1}, \tag{2.7}$$

and this leads to the following result:

$$\|\ell\|_{-1} \le \mathfrak{C}_P\|\ell\|_0. \tag{2.8}$$

Moreover, we find that

$$\|\eta\|_{(H^1(\mathfrak{R}))'} = \sup_{\|\ell\|_1=1} |\langle\eta,\ell\rangle| = \sup_{\|\ell\|_1=1} |(\nabla\mathcal{G}\eta,\nabla\ell)| \le \sup_{\|\ell\|_1=1} \|\eta\|_{-1}|\ell|_1 \le \|\eta\|_{-1}. \tag{2.9}$$

Additionally, we are recalled of the following well-known Sobolev results [13, 14]:

$$H^1(\mathfrak{R}) \overset{c}{\hookrightarrow} L^\rho(\mathfrak{R}) \hookrightarrow (H^1(\mathfrak{R}))' \text{ holds for } \rho\in \begin{cases} [1,\infty] & \text{if } d=1,\\ [1,\infty) & \text{if } d=2,\\ [1,6] & \text{if } d=3,\end{cases} \tag{2.10}$$

where $\hookrightarrow$ and $\overset{c}{\hookrightarrow}$ are the continuous embedding and compact embedding, respectively. The following Hölders inequality is also required frequently: for $1 \leq r_1, r_2 \leq \infty$ such that $\frac{1}{r_1} + \frac{1}{r_2} = 1$ if $\Pi_1 \in L^{r_1}(\mathfrak{R})$ and $\Pi_2 \in L^{r_2}(\mathfrak{R})$ then $\Pi_1\Pi_2 \in L^1(\mathfrak{R})$ and

$$\|\Pi_1\Pi_2\|_{0,1} = \int_{\mathfrak{R}} |\Pi_1\Pi_2| dx \leq \left(\int_{\mathfrak{R}} |\Pi_1|^{r_1} dx \right)^{\frac{1}{r_1}} \left(\int_{\mathfrak{R}} |\Pi_2|^{r_2} dx \right)^{\frac{1}{r_2}} = \|\Pi_1\|_{0,r_1} \|\Pi_2\|_{0,r_2}. \tag{2.11}$$

By utilizing the aforementioned inequality twice, it can be extended in the following manner:

$$\|\Pi_1\Pi_2\Pi_3\|_{0,1} = \int_{\mathfrak{R}} |\Pi_1\Pi_2\Pi_3| dx$$

$$\leq \left(\int_{\mathfrak{R}} |\Pi_1|^{r_1} dx \right)^{\frac{1}{r_1}} \left(\int_{\mathfrak{R}} |\Pi_2|^{r_2} dx \right)^{\frac{1}{r_2}} \left(\int_{\mathfrak{R}} |\Pi_3|^{r_3} dx \right)^{\frac{1}{r_3}} = \|\Pi_1\|_{0,r_1} \|\Pi_2\|_{0,r_2} \|\Pi_3\|_{0,r_3}, \tag{2.12}$$

for $1 \leq r_1, r_2, r_3 \leq \infty$ such that $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1$.

Let $\beta \in [1, \infty], \varpi \geq 1$ and $\ell \in W^{\varpi,\beta}(\mathfrak{R})$, then there are constants C and $\mu = \frac{d}{\varpi}(\frac{1}{\beta} - \frac{1}{r})$ such that

$$\|\ell\|_{0,r} \leq C \|\ell\|_{0,\beta}^{1-\mu} \|\ell\|_{\varpi,\beta}^{\mu}, \quad \text{holds for} \quad r \in \begin{cases} [\beta, \infty] & \text{if } \varpi - \frac{d}{\beta} > 0, \\ [\beta, \infty) & \text{if } \varpi - \frac{d}{\beta} = 0, \\ [\beta, -\frac{d}{\varpi - \frac{d}{\beta}}] & \text{if } \varpi - \frac{d}{\beta} < 0. \end{cases} \tag{2.13}$$

We also state the following result: Let $\vartheta_1, \vartheta_2, \vartheta_3 \in H^1(\mathfrak{R})$, $\kappa_1 = \vartheta_1 - \vartheta_2, \kappa_2 = \vartheta_1^m \vartheta_2^{n_1 - n_2}, n_1, n_2 = 0, 1, 2,$ and $n_1 - n_2 \geq 0$. We then have, for $d = 1, 2, 3$, that

$$\left| \int_{\mathfrak{R}} \kappa_1 \kappa_2 \vartheta_3 dx \right| \leq C \|\vartheta_1 - \vartheta_2\|_0 \|\vartheta_1\|_1^{n_2} \|\vartheta_2\|_1^{n_1 - n_2} \|\vartheta_3\|_1. \tag{2.14}$$

3. The regularization and equivalent weak formulations

We employ a regularization technique as utilized in [19], introducing a twice continuously differentiable function denoted as $\Theta_\epsilon \in C^2(\mathbb{R})$ for $\epsilon \in (0, 1)$. This function takes the form:

$$\Theta_\epsilon(\mathfrak{s}) = \begin{cases} \frac{\theta}{4\epsilon}\mathfrak{s}^2 + \frac{\theta}{2}\mathfrak{s} \ln \epsilon - \frac{\theta\epsilon}{4} & \text{if } \mathfrak{s} \leq \epsilon, \\ \Theta(\mathfrak{s}) \equiv \frac{\theta}{2}\mathfrak{s} \ln \mathfrak{s} & \text{if } \mathfrak{s} \geq \epsilon. \end{cases} \tag{3.1}$$

Next, we proceed to define the function $\psi_\epsilon \in C^2(\mathbb{R})$ in the following manner:

$$\psi_\epsilon(\mathfrak{s}) = \Theta_\epsilon(1 + \mathfrak{s}) + \Theta_\epsilon(1 - \mathfrak{s}) = \begin{cases} \Theta(1 + \mathfrak{s}) + \Theta_\epsilon(1 - \mathfrak{s}) & \text{if } \mathfrak{s} \geq 1 - \epsilon, \\ \psi(\mathfrak{s}) \equiv \Theta(1 + \mathfrak{s}) + \Theta(1 - \mathfrak{s}) & \text{if } |\mathfrak{s}| \leq 1 - \epsilon, \\ \Theta_\epsilon(1 + \mathfrak{s}) + \Theta(1 - \mathfrak{s}) & \text{if } \mathfrak{s} \leq -1 + \epsilon. \end{cases} \tag{3.2}$$

Therefore, for $i = 1, 2$, we apply regularization to the potential Ψ_i by introducing $\Psi_{\epsilon,i}$ belonging to $C^2(R)$.

$$\Psi_{\epsilon,i}(\mathfrak{s}) = \psi_\epsilon(\mathfrak{s}) + \frac{\theta_i}{2}(1 - \mathfrak{s}^2). \tag{3.3}$$

Moreover, we present the monotone odd function $\Xi_\epsilon : \mathbb{R} \to \mathbb{R}$ in the form

$$\Xi_\epsilon(\mathfrak{s}) = \psi'_\epsilon(\mathfrak{s}) = \begin{cases} \Theta'(1+\mathfrak{s})\Theta'_\epsilon(1-\mathfrak{s}) & \text{if } \mathfrak{s} \geq 1-\epsilon, \\ \psi'(\mathfrak{s}) \equiv \Xi(\mathfrak{s}) \equiv \Theta'(1+\mathfrak{s}) - \Theta'(1-\mathfrak{s}) & \text{if } |\mathfrak{s}| \leq 1-\epsilon, \\ \Theta'_\epsilon(1+\mathfrak{s}) - \Theta'(1-\mathfrak{s}) & \text{if } \mathfrak{s} \leq -1+\epsilon. \end{cases} \tag{3.4}$$

Here, we will outline certain properties of the aforementioned functions that will be referenced throughout the article. To begin with, for any ϵ within the interval (0, 1), it holds that

$$\Xi_\epsilon(\mathfrak{s}) \leq \Xi(\mathfrak{s}) \ \ \forall \ \mathfrak{s} \in [1-\epsilon, 1) \quad \text{and} \quad \Xi(\mathfrak{s}) \leq \Xi_\epsilon(\mathfrak{s}) \ \ \forall \ \mathfrak{s} \in (-1, -1+\epsilon]. \tag{3.5}$$

For $i = 1, 2$ and for $s_1, s_2 \in \mathbb{R}$, it follows that

$$\begin{aligned} \Psi_{\epsilon,i}(s_1)(s_2 - s_1) &= \psi'_\epsilon(s_1)(s_2 - s_1) - \theta_i s_1(s_2 - s_1) \leq \psi_\epsilon(s_2) - \psi_\epsilon(s_1) + \theta_i s_1(s_1 - s_2) \\ &= \Psi_{\epsilon,i}(s_2) - \Psi_{\epsilon,i}(s_1) + \frac{\theta_i}{2}(s_2 - s_1)^2, \end{aligned} \tag{3.6}$$

where we have noted the Taylor expansion, $\psi''_\epsilon \equiv \Xi'_\epsilon > 0$ and the identity

$$2s_1(s_1 - s_2) = s_1^2 - s_2^2 + (s_1 - s_2)^2. \tag{3.7}$$

For $\epsilon \leq \frac{1}{2}$ and for $s_1, s_2 \in \mathbb{R}$, we have that

$$\theta \leq \Xi'_\epsilon(s_1) \leq \frac{\theta}{\epsilon}, \tag{3.8}$$

$$\theta(s_1 - s_2)^2 \leq (\Xi_\epsilon(s_1) - \Xi_\epsilon(s_2))(s_1 - s_2), \tag{3.9}$$

$$(\Xi_\epsilon(s_1) - \Xi_\epsilon(s_2))^2 \leq \frac{\theta}{\epsilon}(\Xi_\epsilon(s_1) - \Xi_\epsilon(s_2))(s_1 - s_2). \tag{3.10}$$

From (3.10) and the monotonicity of Ξ_ϵ, it can be proved that

$$|\Xi_\epsilon(s_1) - \Xi_\epsilon(s_2)| \leq \frac{\theta}{\epsilon}|s_1 - s_2|. \tag{3.11}$$

and this confirms that the function Ξ_ϵ is a Lipschitz continuous with Lipschitz constant $\frac{\theta}{\epsilon}$. Moreover, if $s_1, s_2 > 1-\epsilon$ or $s_1, s_2 < -1+\epsilon$, then we obtain that

$$\frac{\theta}{2\epsilon}(s_1 - s_2)^2 \leq (\Xi_\epsilon(s_1) - \Xi_\epsilon(s_2))(s_1 - s_2). \tag{3.12}$$

It is followed, for any $\mathfrak{s} \in [s_1, s_2] \subset [-1+\epsilon, 1-\epsilon]$, that

$$\Xi'(\mathfrak{s}) = \Xi'_\epsilon(\mathfrak{s}) \leq \Xi'_\epsilon(\max\{|s_1|, |s_2|\}) = \Xi'(\max\{|s_1|, |s_2|\}). \tag{3.13}$$

For later use, we introduce some properties of the functions $\Xi_\epsilon^{-1} : \mathbb{R} \to \mathbb{R}$ and $\Xi^{-1} : \mathbb{R} \to (-1, 1)$. From (3.5) it can be shown that

$$\Xi_\epsilon^{-1}(\mathfrak{s}) \geq \Xi^{-1}(\mathfrak{s}) \quad \forall \mathfrak{s} \geq \Xi(1-\epsilon) = \Xi_\epsilon(1-\epsilon), \tag{3.14}$$

$$\Xi^{-1}(\mathfrak{s}) \geq \Xi_\epsilon^{-1}(\mathfrak{s}) \quad \forall \mathfrak{s} \leq \Xi(-1+\epsilon) = \Xi_\epsilon(-1+\epsilon). \tag{3.15}$$

From (3.9) we have, for $s_1, s_2 \in \mathbb{R}$ that

$$|\Xi_\epsilon^{-1}(s_1) - \Xi_\epsilon^{-1}(s_2)| \le \theta^{-1}|s_1 - s_2|. \tag{3.16}$$

The following lemma displays significant findings on $\Psi_{\epsilon,i}, \Xi$ and Ξ_ϵ.

Lemma 3.1.

$$\forall\ \epsilon \le \epsilon_0 = \min\{\frac{\theta}{4\theta_1}, \frac{\theta}{4\theta_2}\}, \quad \Psi_{\epsilon,i}(\mathfrak{s}) \ge -\frac{8\theta_i^2 + \theta^2}{16\theta_i} = -C_0 \quad i = 1,2 \ \ \textit{and} \ \ \mathfrak{s} \in \mathbb{R} \tag{3.17}$$

$$|\Xi_\epsilon^{-1}(\mathfrak{s}) - \Xi^{-1}(\mathfrak{s})| \le \frac{2\epsilon}{\theta}\Big([\mathfrak{s} - \Xi(1-\epsilon)]_+ + [-\mathfrak{s} - \Xi(1-\epsilon)]_+\Big) \ \mathfrak{s} \in \mathbb{R}, \tag{3.18}$$

where $[.]_+ = \max\{., 0\}$.

Now, we introduce a regularized version (Q_ϵ) of (Q):
(Q_ϵ) Find $\{\phi_{\epsilon,1}, \phi_{\epsilon,2}, w_{\epsilon,1}, w_{\epsilon,2}\} \in [H^1(\mathfrak{R})]^4$ such that for i=1, 2 $\phi_{\epsilon,i}(0) = \phi_i(0)$ and for a.e. $t \in (0,T)$ and all $\Pi \in H^1(\mathfrak{R})$

$$(\partial_t \phi_{\epsilon,i}, \Pi) = -(M(\phi_{\epsilon,i})\nabla w_{\epsilon,i}, \nabla \Pi), \tag{3.19}$$

$$(w_{\epsilon,i}, \Pi) = \gamma(\nabla \phi_{\epsilon,i}, \nabla \Pi) + (\Psi'_{\epsilon,i}(\phi_{\epsilon,i}), \Pi) + (f_D^{(i)}(\phi_{\epsilon,1}, \phi_{\epsilon,2}), \Pi). \tag{3.20}$$

4. Existence of Weak solutions

In this section, we establish the existence of a solution to the continuous problem (Q) based on the following assumptions concerning ϕ_1^0 and ϕ_2^0:
(A_1) Let $\{\phi_1^0, \phi_2^0\} \in H^1(\mathfrak{R}) \times H^1(\mathfrak{R})$ such that $\max\{|\phi_1^0|_{0,\infty}, |\phi_2^0|_{0,\infty}\} \le 1$ and for some given $\delta_0 \in (0,1), \max\{|m_1| = |⨍\, \phi_1^0|, |m_2| = |⨍\, \phi_2^0|\} \le 1 - \delta_0$ we will prove the existence relying one the classical Faed- Galerkin method.

Let $\{z_j\}_{j=1}^k$ be the orthonormal basis for $H^1(\mathfrak{R})$ and $L^2(\mathfrak{R})$ scaler product consisting of the eigenfunctions of the elliptic eigenvalue problem

$$-\Delta z_j + z_j = \mu_j z_j, \quad \text{in} \quad \mathfrak{R}, \frac{\partial z_j}{\partial \nu} = 0 \quad \text{in} \quad \partial\mathfrak{R}. \tag{4.1}$$

Upon noting the weak form (4.1) and the orthonormal basis property of $\{z_j\}_{j=1}^k$, i.e., $(zi, z_j) = \delta_{ij}$, we can deduce that $(\nabla z_i, \nabla z_j) = (\mu_i - 1)\delta_{ij}$. Let $V^k, k \ge 1$, be the finite-dimensional space generated by $\{z_j\}_{j=1}^k$, and let $P^k\Pi$ be the projection of $\Pi \in L^2(\mathfrak{R})$ onto V^k, satisfying:

$$P^k\Pi = \Sigma_{j=1}^k(\Pi, z_j)z_j. \tag{4.2}$$

For any $\Pi \in H^1(\mathfrak{R}) \subset L^2(\mathfrak{R})$, it is evident that this definition remains applicable. The subsequent features of the projection P^k from (4.2) can be readily ascertained:

$$(P^k\Pi, \Pi^k) = (\Pi, \Pi^k), \quad \forall \Pi^k \in V^k, \quad \Pi \in L^2(\mathfrak{R}), \tag{4.3}$$

$$(\nabla P^k\Pi, \nabla\Pi^k) = (\nabla\Pi, \nabla\Pi^k), \quad \forall \Pi^k \in V^k, \quad \Pi \in H^1(\mathfrak{R}), \tag{4.4}$$

It is clear, from (4.3) and (4.4), that

$$\|P^k\Pi\|_m \le \|\Pi\|_m, \tag{4.5}$$

$$\|P^k\Pi - \Pi\|_m \le \|\Pi^k - \Pi\|_m, \quad \forall \Pi^k \in V^k, \tag{4.6}$$

where $m = 0$ if $\Pi \in L^2(\mathfrak{R})$ and $m = 0, 1$ if $\Pi \in H^1(\mathfrak{R})$.
Using the result (4.6) and the fact that $\{V^k, k \geq 1\}$ is dense in $L^2(\mathfrak{R})$ and $H^1(\mathfrak{R})$, it follows that

$$P^k\Pi \to \Pi \quad \text{in} \quad L^2(\mathfrak{R}) \quad \text{and} \quad H^1(\mathfrak{R}), \tag{4.7}$$

where $\to$ represents the strong convergence.

Theorem 4.1. *Assuming the validity of the assumptions* (A_1), *it can be asserted that, for all* $\epsilon \leq \epsilon_0$, *the problem* $(Q\epsilon)$ *has a solution* $\phi_{\epsilon,1}, \phi_{\epsilon,2}, w_{\epsilon,1}, w_{\epsilon,2}$, *and for* $i = 1, 2$, *the following estimates hold independently of* ϵ*:*

$$\phi_{\epsilon,i}(x,t) \in L^\infty(0,T;(H^1(\mathfrak{R}))) \bigcap H^1(0,T;(H^1(\mathfrak{R}))') \bigcap L^2(0,T;H^1(\mathfrak{R}))$$

$$\bigcap C([0,T];L^2(\mathfrak{R})) \bigcap L^2(\mathfrak{R}_T), \tag{4.8}$$

$$w_{\epsilon,i} \in L^2(0,T;H^1(\mathfrak{R})), \tag{4.9}$$

$$\frac{\partial \phi_{\epsilon,i}}{\partial t} \in L^2(0,T;(H^1(\mathfrak{R}))'), \tag{4.10}$$

$$\Xi_\epsilon(\phi_{\epsilon,i}) \in L^2(\mathfrak{R}_T), \tag{4.11}$$

$$f_D^{(i)}(\phi_{\epsilon,1}, \phi_{\epsilon,2}) \in L^\infty(0,T;L^2(\mathfrak{R})). \tag{4.12}$$

Further, the solution satisfies, for $i = 1, 2$, *that*

$$\theta_\epsilon^{-1}\|\nabla\Xi_\epsilon(\phi_{\epsilon,i})\|^2_{L^2(\mathfrak{R}_T)} \leq \int_0^T (\nabla\phi_{\epsilon,i}, \nabla\Xi_\epsilon(\phi_{\epsilon,i}))dt \leq C. \tag{4.13}$$

Proof: For $k \geq 1$, we will employ the Faedo-Galerkin method [26]. This method ensures that, for $i = 1, 2$, $t \in [0,T]$, and for all $\Pi^k \in V^k$, the solution $\{\phi_{\epsilon,1}, \phi_{\epsilon,2}, w_{\epsilon,1}, w_{\epsilon,2}\} \in (V^k)^4$ solves the following weak form:

$$(\partial_t\phi^k_{\epsilon,i}, \Pi^k) = -(M(\phi^k_{\epsilon,i})\nabla w^k_{\epsilon,i}, \nabla\Pi^k), \tag{4.14}$$

$$(w^k_{\epsilon,i}, \Pi^k) = \gamma(\nabla\phi^k_{\epsilon,i}, \nabla\Pi^k) + ({\Psi'}_{\epsilon,i}(\phi^k_{\epsilon,i}), \Pi^k) + (f_D^{(i)}(\phi^k_{\epsilon,1}, \phi^k_{\epsilon,2}), \Pi^k), \tag{4.15}$$

$$\phi^k_{\epsilon,i}(.,0) = P^k\phi^0_i. \tag{4.16}$$

We will now establish the existence of a global solution, requiring a priori estimates for ϕ^k_i, w^k_i for $i = 1, 2$, which are independent of k. To begin, we consider the energy function as follows:

$$E(\phi^k_{\epsilon,1}, \phi^k_{\epsilon,2}) = \int_{\mathfrak{R}} \big(\frac{\gamma}{2}|\nabla\phi^k_{\epsilon,1}|^2 + \frac{\gamma}{2}|\nabla\phi^k_{\epsilon,2}|^2 + f_D(\phi^k_{\epsilon,1}, \phi^k_{\epsilon,2}) + \Psi_{\epsilon,1}(\phi^k_{\epsilon,1}) + \Psi_{\epsilon,2}(\phi^k_{\epsilon,2})\big)dx, \tag{4.17}$$

Given that $\frac{\partial \phi^k_{\epsilon,i}}{\partial t} \in V^k$ for $i = 1, 2$, differentiating $E(\phi^k_{\epsilon,1}, \phi^k_{\epsilon,2})$ with respect to t yields:

$$\begin{aligned}
&\frac{\partial}{\partial t} E(\phi^k_{\epsilon,1}, \phi^k_{\epsilon,2}) = \gamma(\nabla \phi^k_{\epsilon,1}(t), \nabla(\frac{\partial \phi^k_{\epsilon,1}}{\partial t})) + (\Psi'_{\epsilon,1}(\phi^k_{\epsilon,1}), \frac{\partial \phi^k_{\epsilon,1}}{\partial t}) \\
+ &\ 2D((\phi^k_{\epsilon,1} + \sigma_1)(\phi^k_{\epsilon,2} + \sigma_2)^2, \frac{\partial \phi^k_{\epsilon,1}}{\partial t}) + \gamma(\nabla \phi^k_{\epsilon,2}(t), \nabla(\frac{\partial \phi^k_{\epsilon,2}}{\partial t})) \\
+ &\ (\Psi'_{\epsilon,2}(\phi^k_{\epsilon,2}), \frac{\partial \phi^k_{\epsilon,2}}{\partial t}) + 2D((\phi^k_{\epsilon,1} + \sigma_1)^2(\phi^k_{\epsilon,2} + \sigma_2), \frac{\partial \phi^k_{\epsilon,2}}{\partial t}) \\
= &\ \gamma(\nabla \phi^k_{\epsilon,1}(t), \nabla(\frac{\partial \phi^k_{\epsilon,1}}{\partial t})) + (\Psi'_{\epsilon,1}(\phi^k_{\epsilon,1}), \frac{\partial \phi^k_{\epsilon,1}}{\partial t}) + (f^{(1)}_D(\phi^k_{\epsilon,1}, \phi^k_{\epsilon,2}), \frac{\partial \phi^k_{\epsilon,1}}{\partial t}) \\
+ &\ \gamma(\nabla \phi^k_{\epsilon,2}(t), \nabla(\frac{\partial \phi^k_{\epsilon,2}}{\partial t})) + (\Psi'_{\epsilon,2}(\phi^k_{\epsilon,2}), \frac{\partial \phi^k_{\epsilon,2}}{\partial t}) + (f^{(2)}_D(\phi^k_{\epsilon,1}, \phi^k_{\epsilon,2}), \frac{\partial \phi^k_{\epsilon,2}}{\partial t}).
\end{aligned} \tag{4.18}$$

By selecting $\Pi^k = w^k_{\epsilon,i}$ in (4.14) and $\Pi^k = \frac{\partial \phi^k_{\epsilon,i}}{\partial t}$ in (4.15) for $i = 1, 2$, we can represent (4.18) as

$$(w_{\epsilon,i}, \frac{\partial \phi_{\epsilon,i}}{\partial t}) = (\frac{\partial \phi_{\epsilon,i}}{\partial t}, w_{\epsilon,i}) = -(M(\phi_{\epsilon,i})\nabla w_{\epsilon,i}, \nabla w_{\epsilon,i}), \tag{4.19}$$

and

$$(w_{\epsilon,i}, \frac{\partial \phi_{\epsilon,i}}{\partial t}) = \gamma(\nabla \phi_{\epsilon,i}, \nabla \frac{\partial \phi_{\epsilon,i}}{\partial t}) + (\Psi'_{\epsilon,i}(\phi^k_{\epsilon,i}), \frac{\partial \phi_{\epsilon,i}}{\partial t}) + (f^{(i)}_D(\phi^k_{\epsilon,1}, \phi^k_{\epsilon,2}), \frac{\partial \phi_{\epsilon,i}}{\partial t}). \tag{4.20}$$

By substituting (4.19) into (4.20) and summing over $i = 1, 2$, we deduce that

$$\begin{aligned}
&\gamma(\nabla \phi_{\epsilon,1}, \nabla \frac{\partial \phi_{\epsilon,1}}{\partial t}) + (\Psi'_{\epsilon,1}(\phi^k_{\epsilon,1}), \frac{\partial \phi_{\epsilon,1}}{\partial t}) + (f^{(1)}_D(\phi^k_{\epsilon,1}, \phi^k_{\epsilon,2}), \frac{\partial \phi_{\epsilon,1}}{\partial t}) \\
+ &\ \gamma(\nabla \phi_{\epsilon,2}, \nabla \frac{\partial \phi_{\epsilon,2}}{\partial t}) + (\Psi'_{\epsilon,2}(\phi^k_{\epsilon,2}), \frac{\partial \phi_{\epsilon,2}}{\partial t}) + (f^{(2)}_D(\phi^k_{\epsilon,1}, \phi^k_{\epsilon,2}), \frac{\partial \phi_{\epsilon,2}}{\partial t}) \\
= &\ -(M(\phi_{\epsilon,1})\nabla w_{\epsilon,1}, \nabla w_{\epsilon,1}) - (M(\phi_{\epsilon,2})\nabla w_{\epsilon,2}, \nabla w_{\epsilon,2}).
\end{aligned} \tag{4.21}$$

From (4.21) and (4.18), it follows that

$$\frac{\partial}{\partial t} E(\phi^k_{\epsilon,1}, \phi^k_{\epsilon,2}) = -(M(\phi_{\epsilon,1})\nabla w_{\epsilon,1}, \nabla w_{\epsilon,1}) - (M(\phi_{\epsilon,2})\nabla w_{\epsilon,2}, \nabla w_{\epsilon,2}) \le 0. \tag{4.22}$$

Thus, E is a Lyapunov functional. Now, by integrating (4.22) over the interval $(0, t)$ and considering (4.16), we conclude that

$$E(\phi^k_{\epsilon,1}(t), \phi^k_{\epsilon,2}(t)) + \int_{\mathfrak{R}_T} [M(\phi^k_{\epsilon,1})|\nabla w^k_{\epsilon,1}|^2 + M(\phi^k_{\epsilon,2})|\nabla w^k_{\epsilon,2}|^2] dxdt \le E(P^k \phi^0_1, P^k \phi^0_2). \tag{4.23}$$

To bound $E(P^k\phi_1^0, P^k\phi_2^0)$, we utilize (4.17) and (4.7) in the following manner:

$$\begin{aligned} E(P^k\phi_1^0, P^k\phi_2^0) &= \int_{\mathfrak{R}} \big(\frac{\gamma}{2}|\nabla P^k\phi_1^0|^2 + \frac{\gamma}{2}|\nabla P^k\phi_2^0|^2 + f_D(P^k\phi_1^0, P^k\phi_2^0) \\ &+ \Psi_{\epsilon,1}(P^k\phi_1^0) + \Psi_{\epsilon,2}(P^k\phi_2^0)\big)dx \\ &= \int_{\mathfrak{R}} \frac{\gamma}{2}|\nabla P^k\phi_1^0|^2 dx + \int_{\mathfrak{R}} \frac{\gamma}{2}|\nabla P^k\phi_2^0|^2 dx + \int_{\mathfrak{R}} f_D(P^k\phi_1^0, P^k\phi_2^0)dx \\ &+ \int_{\mathfrak{R}} \Psi_{\epsilon,1}(P^k\phi_1^0)dx + \int_{\mathfrak{R}} \Psi_{\epsilon,2}(P^k\phi_2^0)dx. \end{aligned} \tag{4.24}$$

To bound the fourth and fifth terms of (4.24), we select $s = \phi_i^0$ and $r = P^k\phi_i^0$ for $i = 1, 2$ in (3). Noting that $\Psi'_{\epsilon,i}(r) = \Xi_\epsilon(r) - \theta_i r$, and leveraging the Lipschitz continuity of Ξ_ϵ (3.11) along with (4.5), we find that

$$\begin{aligned} &\int_{\mathfrak{R}} \Psi_{\epsilon,i}(P^k\phi_i^0)dx = (\Psi_{\epsilon,i}(P^k\phi_i^0), 1) = (\Psi_{\epsilon,i}(P^k\phi_i^0) - \Psi_{\epsilon,i}(\phi_i^0) + \Psi_{\epsilon,i}(\phi_i^0), 1) \\ \le\ & (\Psi_{\epsilon,i}(P^k\phi_i^0), P^k\phi_i^0 - \phi_i^0) + \frac{\theta_i}{2}((P^k\phi_i^0 - \phi_i^0)^2, 1) + (\Psi_{\epsilon,i}(\phi_i^0), 1) \\ =\ & (\Xi_\epsilon(P^k\phi_i^0) - \theta_i(P^k\phi_i^0), P^k\phi_i^0 - \phi_i^0) + \frac{\theta_i}{2}((P^k\phi_i^0 - \phi_i^0)^2, 1) + (\Psi_{\epsilon,i}(\phi_i^0), 1) \\ \le\ & \|\Xi_\epsilon(P^k\phi_i^0) - \theta_i(P^k\phi_i^0)\|_0\|P^k\phi_i^0 - \phi_i^0\|_0 + \frac{\theta_i}{2}\|P^k\phi_i^0 - \phi_i^0\|_0^2 + (\Psi_{\epsilon,i}(\phi_i^0), 1) \\ \le\ & (\|\Xi_\epsilon(P^k\phi_i^0)\|_0 + \theta_i\|P^k\phi_i^0\|_0)\|P^k\phi_i^0 - \phi_i^0\|_0 + \frac{\theta_i}{2}\|P^k\phi_i^0 - \phi_i^0\|_0^2 + (\Psi_{\epsilon,i}(\phi_i^0), 1) \\ \le\ & (\frac{\theta}{\epsilon} + \theta_i)\|\phi_i^0\|_0\|P^k\phi_i^0 - \phi_i^0\|_0 + \frac{\theta_i}{2}\|P^k\phi_i^0 - \phi_i^0\|_0^2 + (\Psi_{\epsilon,i}(\phi_i^0), 1). \end{aligned} \tag{4.25}$$

Therefore, employing (4.7), the assumptions of (A_1), (3.3), and the observation that $\psi_\epsilon(r) \le \psi_\epsilon(1),\ \forall r \in [-1, 1]$, we can deduce that

$$\limsup_{k\to\infty}(\Psi_{\epsilon,i}(P^k\phi_i^0), 1) \le (\Psi_{\epsilon,i}(\phi_i^0), 1) \le (\psi_\epsilon(1) + \frac{\theta_i}{2}, 1) \le (\theta\ln 2 + \frac{\theta_i}{2})|\mathfrak{R}|. \tag{4.26}$$

Furthermore, to bound the third term of (4.24), recalling (1.25), (4.5), a generalized Hölder's inequality, (2.13), the embedding $H^1(\mathfrak{R}) \hookrightarrow L^4(\mathfrak{R})$, and (4.7), we obtain that

$$\begin{aligned} \int_{\mathfrak{R}} f_D(P^k\phi_1^0, P^k\phi_2^0)dx &= (f_D(P^k\phi_1^0, P^k\phi_2^0, 1) \\ &= (D(P^k\phi_1^0 + \sigma_1)^2(P^k\phi_2^0 + \sigma_2)^2, 1) \le D|(P^k\phi_1^0 + \sigma_1)^2(P^k\phi_2^0 + \sigma_2)^2|_{0,1} \\ &\le\ \le D|P^k\phi_1^0 + \sigma_1|_{0,4}^2|P^k\phi_2^0 + \sigma_2|_{0,4}^2 \\ &\le C\|P^k\phi_1^0 + \sigma_1\|_1^2\|P^k\phi_2^0 + \sigma_2\|_1^2 \\ &\le\ \le C(\|\phi_1^0 + 1\|_1^2)(\|\phi_2^0 + 1\|_1^2). \end{aligned} \tag{4.27}$$

By combining (4.27), (4.26), (4.24), and taking into account (4.7) along with the assumption (A_1), we arrive at

$$\begin{aligned} E(P^k\phi_1^0, P^k\phi_2^0) &= \int_{\mathfrak{R}} \frac{\gamma}{2}|\nabla P^k\phi_1^0|^2 dx + \int_{\mathfrak{R}} \frac{\gamma}{2}|\nabla P^k\phi_2^0|^2 dx + \int_{\mathfrak{R}} f_D(P^k\phi_1^0, P^k\phi_2^0) dx \\ &+ \int_{\mathfrak{R}} \Psi_{\epsilon,1}(P^k\phi_1^0) dx + \int_{\mathfrak{R}} \Psi_{\epsilon,2}(P^k\phi_2^0) dx \\ &\leq \frac{\gamma}{2}\nabla\|\phi_1^0\|_1^2 + \frac{\gamma}{2}\nabla\|\phi_2^0\|_1^2 + \|\phi_1^0 + 1\|_1^2)(\|\phi_2^0 + 1\|_1^2) + 2(\theta\ln 2 + \frac{\theta_i}{2})|\mathfrak{R}| \\ &\leq C. \end{aligned} \tag{4.28}$$

Therefore, based on (4.28) and (4.23), it can be concluded that

$$E(\phi_{\epsilon,1}^k(t), \phi_{\epsilon,2}^k(t)) + \int_{\mathfrak{R}_T} [M(\phi_{\epsilon,1}^k)|\nabla w_{\epsilon,1}^k|^2 + M(\phi_{\epsilon,2}^k)|\nabla w_{\epsilon,2}^k|^2] dxdt \leq C. \tag{4.29}$$

Following (4.29) and (4.17), we find that

$$\begin{aligned} &\int_{\mathfrak{R}} \big(\frac{\gamma}{2}|\nabla\phi_{\epsilon,1}^k(t)|^2 + \frac{\gamma}{2}|\nabla\phi_{\epsilon,2}^k(t)|^2 + f_D(\phi_{\epsilon,1}^k(t), \phi_{\epsilon,2}^k(t)) + \Psi_{\epsilon,1}(\phi_{\epsilon,1}^k(t)) \\ &+ \Psi_{\epsilon,2}(\phi_{\epsilon,2}^k(t))\big) dx + \int_{\mathfrak{R}_T} [M(\phi_{\epsilon,1}^k)|\nabla w_{\epsilon,1}^k|^2 + M(\phi_{\epsilon,2}^k)|\nabla w_{\epsilon,2}^k|^2] dxdt \leq C. \end{aligned} \tag{4.30}$$

Given the positivity of the functions $\Psi_{\epsilon,i}(\phi_{\epsilon,i}^k(t))$ and $f_D(\phi_{\epsilon,1}^k(t), \phi_{\epsilon,2}^k(t))$, we consequently reach the conclusion that

$$\frac{\gamma}{2}\|\phi_{\epsilon,1}^k(t)\|_1^2 + \frac{\gamma}{2}\|\phi_{\epsilon,2}^k(t)\|_1^2 + \int_{\mathfrak{R}_T} [M(\phi_{\epsilon,1}^k)|\nabla w_{\epsilon,1}^k|^2 + M(\phi_{\epsilon,2}^k)|\nabla w_{\epsilon,2}^k|^2] dxdt \leq C, \tag{4.31}$$

where C is not dependent of T and k. Subsequently, by choosing $\Pi^k = 1$ in (4.14), we can ascertain, for $i = 1, 2$, the following:

$$(\frac{\partial\phi_{\epsilon,i}^k}{\partial t}, 1) = -(M(\phi_{\epsilon,i}^k)\nabla w_{\epsilon,i}^k, \nabla 1) = 0. \tag{4.32}$$

Then, by integrating both sides of (4.32) over the interval $(0, t)$, we obtain that

$$(\phi_{\epsilon,i}^k(t), 1) = (\phi_{\epsilon,i}^k(0), 1) = (P^k\phi_i^0, 1) = (\phi_i^0, 1) \leq C, \tag{4.33}$$

which implies to the following result:

$$|(\phi_{\epsilon,i}^k(t), 1)| \leq C. \tag{4.34}$$

By using the Poincaré inequality (2.5), (4.31) and (4.34), we obtain that

$$\|\phi_{\epsilon,i}^k(t)\|_0 \leq C_p(|\phi_{\epsilon,i}^k(t)|_1 + |(\phi_{\epsilon,i}^k(t), 1)|) \leq C, \tag{4.35}$$

Hence, we determine that

$$\sup_{t\in(0,T)} \|\phi_{\epsilon,i}^k(t)\|_1 \leq C,$$

i.e.

$$\phi_{\epsilon,i}^k(t) \in L^\infty(0, T; H^1(\mathfrak{R})). \tag{4.36}$$

By integrating (4.14) over the interval $(0,t)$ and employing Hölder's inequality (2.11) along with (4.31), we deduce, for any $\Pi^k \in L^2(0,T;H^1(\mathfrak{R}))$, that

$$\begin{aligned}
&\left|\int_0^T\int_{\mathfrak{R}} \partial_t\phi^k_{\epsilon,i}\Pi^k dxdt\right| = \left|\int_0^T\int_{\mathfrak{R}} M(\phi^k_{\epsilon,i})\nabla w^k_{\epsilon,i}\nabla\Pi^k dxdt\right| \\
\leq\ & \left(\int_0^T\int_{\mathfrak{R}} (M(\phi^k_{\epsilon,i})|\nabla w^k_{\epsilon,i}|)^2 dxdt\right)^{\frac{1}{2}}\left(\int_0^T\int_{\mathfrak{R}} |\nabla\Pi^k|^2 dxdt\right)^{\frac{1}{2}} \\
\leq\ & C\|\Pi^k\|_{L^2(0,T;H^1(\mathfrak{R}))}, \qquad (4.37)
\end{aligned}$$

then, we have that

$$\|\partial_t\phi^k_{\epsilon,i}\|_{L^2(0,T;(H^1(\mathfrak{R}))')} = \sup_{\Pi^k\neq 0}\frac{|\int_0^T\int_{\mathfrak{R}}\partial_t\phi^k_{\epsilon,i}\Pi^k dxdt|}{\|\Pi^k\|_{L^2(0,T;H^1(\mathfrak{R}))}} \leq C,$$

i.e.

$$\partial_t\phi^k_{\epsilon,i} \in L^2(0,T;(H^1(\mathfrak{R}))'), \qquad \forall i = 1,2, \qquad (4.38)$$

and this proves (4.10). To demonstrate the boundedness of $\phi^k_{\epsilon,i}(t)$ in $L^2(0,T;(H^1(\mathfrak{R}))')$, it is necessary to establish the boundedness of $\phi^k_{\epsilon,i}(t) - \fint \phi^k_{\epsilon,i}(t)$ in the same space. This hinges on the observations provided in (2.3) and (4.33), yielding that

$$\begin{aligned}
&\phi^k_{\epsilon,i}(t) - \fint \phi^k_{\epsilon,i}(t) = \phi^k_{\epsilon,i}(t) - \frac{1}{|\mathfrak{R}|}(\phi^k_{\epsilon,i}(t),1) \\
=\ & \phi^k_{\epsilon,i}(t) - \frac{1}{|\mathfrak{R}|}(\phi^k_{\epsilon,i}(0),1) \\
=\ & \int_0^t \frac{\partial}{\partial s}\phi^k_{\epsilon,i}(s)ds + \phi^k_{\epsilon,i}(0) - \frac{1}{|\mathfrak{R}|}(\phi^k_{\epsilon,i}(0),1). \qquad (4.39)
\end{aligned}$$

Hence, on utilising Young's inequality and setting $t=T$ in the integration on the right hand side and noting (4.38) and (2.8), we obtain that

$$\begin{aligned}
\left\|\phi^k_{\epsilon,i}(t) - \fint \phi^k_{\epsilon,i}(t)\right\|_{-1} &= \left(\left\|\int_0^T \frac{\partial\phi^k_{\epsilon,i}}{\partial t}dt + \phi^k_{\epsilon,i}(0) - \frac{1}{|\mathfrak{R}|}(\phi^k_{\epsilon,i}(0),1)\right\|_{-1}\right)^2 \\
&\leq \left(\left\|\int_0^T \frac{\partial\phi^k_{\epsilon,i}}{\partial t}dt\right\|_{-1} + C_p\left\|\phi^k_{\epsilon,i}(0) - \frac{1}{|\mathfrak{R}|}(\phi^k_{\epsilon,i}(0),1)\right\|_{-1}\right)^2 \\
&\leq \left\|\int_0^T \frac{\partial\phi^k_{\epsilon,i}}{\partial t}dt\right\|^2_{-1} + C\left\|\phi^k_{\epsilon,i}(0)\right\|^2_0 + C\left\|(\phi^k_{\epsilon,i}(0),1)\right\|^2_0 \\
&\leq \left\|\int_0^T \frac{\partial\phi^k_{\epsilon,i}}{\partial t}dt\right\|^2_{-1} + C \quad \leq C. \qquad (4.40)
\end{aligned}$$

By squaring the the above result, integrating over $(0,T)$ and using (2.9), we obtain that

$$\left\|\phi^k_{\epsilon,i}(t) - \fint \phi^k_{\epsilon,i}(t)\right\|^2_{L^2(0,T;(H^1(\mathfrak{R}))')} \leq \int_0^T \left\|\phi^k_{\epsilon,i}(t) - \fint \phi^k_{\epsilon,i}(t)\right\|^2_{-1} dt \leq C\int_0^T dt \leq C(T). \qquad (4.41)$$

Hence, (4.38) and (4.41) imply that

$$\|\phi^k_{\epsilon,i}\|^2_{H^1(0,T;(H^1(\mathfrak{R}))')} \leq C. \tag{4.42}$$

We have to demonstrate that $\|w^k_{\epsilon,i}\|_1$ is bounded. By using the Poincaré inequality (2.5), we have that

$$\|w^k_{\epsilon,i}\|_1^2 = \|w^k_{\epsilon,i}\|_0^2+|w^k_{\epsilon,i}|_1^2 \leq 2C_p(|w^k_{\epsilon,i}|_1^2+|(w^k_{\epsilon,i},1)|^2)+|w^k_{\epsilon,i}|_1^2 \leq C(|w^k_{\epsilon,i}|_1^2+|(w^k_{\epsilon,i},1)|^2). \tag{4.43}$$

Thus, on noting (4.31), it is enough to bound $|(w^k_{\epsilon,i},1)|$ to conclude $\|w^k_{\epsilon,i}\|_1$ is bound. Taking $\Pi^k = 1$ in (4.15) we have, for $i = 1, 2$, that

$$(w^k_{\epsilon,,i},1) = (\Psi'_{\epsilon,i}(\phi^k_{\epsilon,i}),1) + (f^{(i)}_D(\phi^k_{\epsilon,1},\phi^k_{\epsilon,2}),1),$$

which implies that

$$|(w^k_{\epsilon,i},1)| \leq |(\Psi'_{\epsilon,i}(\phi^k_{\epsilon,i}),1)| + |(f^{(i)}_D(\phi^k_{\epsilon,1},\phi^k_{\epsilon,2},1)|. \tag{4.44}$$

To bound (4.44), let us first bound the first term of (4.44) by noting $\Psi'_{\epsilon,i}(r) = \Xi_\epsilon(r)-\theta_i r$, along with the Lipschitz continuity of Ξ_ϵ (3.11) and (4.35) for $i = 1, 2$. This allows us to find that

$$|(\Psi'_{\epsilon,i}(\phi^k_{\epsilon,i}),1)| = \|\Xi_\epsilon(\phi^k_{\epsilon,i}) - \theta_i(\phi^k_{\epsilon,i})\|_0 \leq \frac{\theta}{\epsilon}\|\phi^k_{\epsilon,i}\|_0 + \theta_i\|\phi^k_{\epsilon,i}\|_0 \leq C. \tag{4.45}$$

Thus, by using (1.26), a generalized Hölder's inequality, (2.13), $H^1(\mathfrak{R}) \hookrightarrow L^4(\mathfrak{R})$ and (4.36) for $i = 1, 2, i \neq j$ yields that

$$\begin{aligned} |(f^{(i)}_D(\phi^k_{\epsilon,1},\phi^k_{\epsilon,2}),1)| &= D|(\frac{\partial f_D(\phi^k_{\epsilon,1},\phi^k_{\epsilon,2})}{\partial \phi_{\epsilon,i^k},1})| \\ &= 2D|((\phi^k_{\epsilon,i}+\sigma_i)(\phi^k_{\epsilon,j}+\sigma_j)^2,1)| \\ &\leq 2D|\phi^k_{\epsilon,i}+\sigma_i|_{0,1}|\phi^k_{\epsilon,j}+\sigma_j|^2_{0,1} \\ &\leq 2D\|\phi^k_{\epsilon,i}+\sigma_i\|_{0,4}\|\phi^k_{\epsilon,j}+\sigma_j\|^2_{0,4} \\ &\leq 2D\|\phi^k_{\epsilon,i}+\sigma_i\|_1\|\phi^k_{\epsilon,j}+\sigma_j\|^2_1 \\ &\leq \leq C. \end{aligned} \tag{4.46}$$

Thus, from (4.44), (4.45) and (4.46), we conclude that

$$|(w^k_{\epsilon,i},1)| \leq C. \tag{4.47}$$

By inserting (4.47) into (4.43) and integrating the result over $(0,T)$, we obtain

$$\int_0^T \|w^k_{\epsilon,i}\|_1^2 ds \leq C\int_0^T |w^k_{\epsilon,i}|_1^2 ds + CT. \tag{4.48}$$

Now, we have to prove that $\int_0^t |w^k_{\epsilon,i}|_1^2 dt \leq C$. From (4.31), it follows, for $i = 1, 2$, that

$$\int_{\mathfrak{R}_T} M(\phi^k_{\epsilon,i})|\nabla w^k_{\epsilon,i}|^2 dxdt \leq C.$$

By using (1.3), then we find that

$$\int_0^t |w^k_{\epsilon,i}|_1^2 dt \leq \frac{C}{m_0} = C. \tag{4.49}$$

By substituting (4.49) in to (4.48), we have that

$$\|w_{\epsilon,i}^k\|^2_{L^2(0,T;H^1(\mathfrak{R}))} = \int_0^t \|w_{\epsilon,i}^k\|_1^2 dt \leq C\int_0^t |w_{\epsilon,i}^k|_1^2 dt + C\int_0^t dt \leq C + CT,$$

so, it follows finally that

$$\|w_{\epsilon,i}^k\|_{L^2(0,T;H^1(\mathfrak{R}))} \leq C(1+T^{\frac{1}{2}}). \tag{4.50}$$

Now, since $L^\infty(0,T;H^1(\mathfrak{R})) \subset L^2(0,T;H^1(\mathfrak{R}))$, then by using (4.36), we arrive at

$$\phi_{\epsilon,i}^k \in L^2(0,T;H^1(\mathfrak{R})). \tag{4.51}$$

Now since $H^1(0,T;(H^1(\mathfrak{R}))')$ and $L^2(0,T;H^1(\mathfrak{R}))$ are reflexive then by compactness arguments we deduce existence of subsequence such that

$$\phi_{\epsilon,i}^k \rightharpoonup \phi_{\epsilon,i}, \quad \text{in} \quad H^1(0,T;(H^1(\mathfrak{R}))')\bigcap L^2(0,T;(H^1(\mathfrak{R}))'), \tag{4.52}$$

$$w_{\epsilon,i}^k \rightharpoonup w_{\epsilon,i} \quad \text{in} \quad L^2(0,T;H^1(\mathfrak{R})), \tag{4.53}$$

$$\partial_t\phi_{\epsilon,i}^k \rightharpoonup \partial_t\phi_{\epsilon,i}, \quad \text{in} \quad L^2(0,T;(H^1(\mathfrak{R}))'). \tag{4.54}$$

Since $L^\infty(0,T;H^1(\mathfrak{R}))$ is dual of $L^1(0,T;(H^1(\mathfrak{R}))')$ which is separable we can extract a subsequence in $L^\infty(0,T;H^1(\mathfrak{R}))$ such that

$$\phi_{\epsilon,i}^k \rightharpoonup^* \phi_{\epsilon,i} \quad \text{in} \quad L^\infty(0,T;H^1(\mathfrak{R})). \tag{4.55}$$

Note that $H^1(\mathfrak{R})$ and $(H^1(\mathfrak{R}))'$ are reflexive and the injection of $H^1(\mathfrak{R})$ in to $L^2(\mathfrak{R})$ is compact. As a result of Lions' compactness theory, see Theorem 5.1 in [26] page 56, we may extract a subsequence in $L^2(0,T;L^2(\mathfrak{R}))$ such that

$$\phi_{\epsilon,i}^k \rightharpoonup \phi_{\epsilon,i} \quad \text{in} \quad L^2(\mathfrak{R}_T). \tag{4.56}$$

Moreover, since $\phi_{\epsilon,i}^k \in L^2(0,T;(H^1(\mathfrak{R}))$ and $\frac{\partial\phi_{\epsilon,i}^k}{\partial t} \in L^2(0,T;(H^1(\mathfrak{R}))')$ then $\phi_{\epsilon,i}^k \in C([0,T];L^2(\mathfrak{R}))$. This result together with (4.52) and the strong convergence of $P^k(\phi_i^0)$ to ϕ_i^0 in $L^2(\mathfrak{R})$ implies that $\phi_{\epsilon,i}(0) = \phi_i^0$. Thus, we have prove the following results:

$$\phi_{\epsilon,i}^k \in L^\infty(0,T;(H^1(\mathfrak{R})))\bigcap H^1(0,T;(H^1(\mathfrak{R}))')\bigcap L^2(0,T;H^1(\mathfrak{R}))$$

$$\bigcap C([0,T];L^2(\mathfrak{R}))\bigcap L^2(\mathfrak{R}_T),$$

$$w_{\epsilon,i}^k \in L^2(0,T;H^1(\mathfrak{R})),$$

$$\frac{\partial\phi_{\epsilon,i}^k}{\partial t} \in L^2(0,T;(H^1(\mathfrak{R}))').$$

Since $\phi_{\epsilon,i} \in L^2(0,T;H^1(\mathfrak{R}))$ and Ξ_ϵ is Lipschitz continuous and its first derivative is bounded by (3.8), we can test (3.20) by $\Pi = \Xi_\epsilon(\phi_{\epsilon,i}) \in H^1(\mathfrak{R})$ and using Hölder and Young inequalities, to yield, for $i = 1,2$ and a.e. $t \in (0,T)$, that

$$\begin{aligned}
& \gamma(\nabla\phi_{\epsilon,i}, \nabla\Xi_\epsilon(\phi_{\epsilon,i})) + \|\Xi_\epsilon(\phi_{\epsilon,i})\|_0^2 \\
= & (w_{\epsilon,i}, \Xi_\epsilon(\phi_{\epsilon,i})) + \theta_i((\phi_{\epsilon,i}), \Xi_\epsilon(\phi_{\epsilon,i})) - (f_D^{(i)}(\phi_{\epsilon,1},\phi_{\epsilon,2}), \Xi_\epsilon(\phi_{\epsilon,i})) \\
\leq & \|w_{\epsilon,i}\|_0\|\Xi_\epsilon(\phi_{\epsilon,i})\|_0 + \theta_i\|\phi_{\epsilon,i}\|_0\|\Xi_\epsilon(\phi_{\epsilon,i})\|_0 + \|f_D^{(i)}(\phi_{\epsilon,1},\phi_{\epsilon,2})\|_0\|\Xi_\epsilon(\phi_{\epsilon,i})\|_0 \\
\leq & C\|w_{\epsilon,i}\|_0^2 + \frac{1}{8}\|\Xi_\epsilon(\phi_{\epsilon,i})\|_0^2 + \frac{\theta_i}{2}\|\phi_{\epsilon,i}\|_0^2 + \frac{1}{8}\|\Xi_\epsilon(\phi_{\epsilon,i})\|_0^2 + C\|f_D^{(i)}(\phi_{\epsilon,1},\phi_{\epsilon,2})\|_0^2 \\
+ & \frac{1}{4}\|\Xi_\epsilon(\phi_{\epsilon,i})\|_0^2 \\
\leq & \frac{1}{2}\|\Xi_\epsilon(\phi_{\epsilon,i})\|_0^2 + C\left[\|w_{\epsilon,i}\|_0^2 + \|\phi_{\epsilon,i}\|_0^2 + \|f_D^{(i)}(\phi_{\epsilon,1},\phi_{\epsilon,2})\|_0^2\right] \qquad (4.57)
\end{aligned}$$

As $\Xi'_\epsilon > 0$, then the first term of (4.57) is positive. Next, by utilising a generalized Hölder's inequality, (2.13), $H^1(\mathfrak{R}) \hookrightarrow L^6(\mathfrak{R})$ and (4.1), we have, $i = 1,2, i \neq j$, that

$$\begin{aligned}
\|f_D^{(i)}(\phi_{\epsilon,i},\phi_{\epsilon,j})\|_0^2 &= \|2D(\phi_{\epsilon,i}+\sigma_i)(\phi_{\epsilon,j}+\sigma_j)^2\|_0^2 \\
&= 4D^2|(\phi_{\epsilon,i}+\sigma_i)^2(\phi_{\epsilon,j}+\sigma_j)^4|_{0,1} \\
&\leq 4D^2\|\phi_{\epsilon,i}+\sigma_i\|_{0,6}^2\|\phi_{\epsilon,j}+\sigma_j\|_{0,6}^4 \\
&\leq C\|\phi_{\epsilon,i}+\sigma_i\|_1^2\|\phi_{\epsilon,j}+\sigma_j\|_1^4. \qquad (4.58)
\end{aligned}$$

Now, from (4.58) and the bound (4.1) we can find for $i = 1,2$ that

$$\|f_D^{(i)}(\phi_{\epsilon,1},\phi_{\epsilon,2})\|_{L^\infty(0,T;L^2(\mathfrak{R}))} \leq C. \qquad (4.59)$$

Integrating (4.57) over the interval $(0,T)$ and employing the estimates (4.1), (4.9), and (4.59), results in the estimate (4.11). The second inequality in (4.13) is also derived from this integration, while the first one is a consequence of the property (3.8) of Ξ_ϵ.

From an application of the Lions-Aubin theorem with $X_0 = H^1(\mathfrak{R}), X = L^2(\mathfrak{R}), X_1 = (H^1(\mathfrak{R}))'$ and $p_0 = p_1 = 2$ we can extract subsequences, still denoted $\{\phi_{\epsilon,i}^k\}$, such that for $i = 1,2$

$$\phi_{\epsilon,i}^k \to \phi_{\epsilon,i} \quad \text{in} \quad L^2(\mathfrak{R}_T). \qquad (4.60)$$

□

4.1. **Passage to the limit.** In the finite weak forms (4.14) and (4.15), we proceed to the limit. To achieve this, we consider any function $\Pi \in L^2(0,T;H^1(\mathfrak{R}))$ and substitute $\Pi^k = P^k\Pi$ into (4.14) and (4.15). Upon integration over $(0,T)$, we obtain that

$$\int_0^T (\partial_t\phi_{\epsilon,i}^k, P^k\Pi)dt = -\int_0^T (M(\phi_{\epsilon,i}^k)\nabla w_{\epsilon,i}^k, \nabla P^k\Pi)dt, \qquad (4.61)$$

$$\int_0^T (w_{\epsilon,i}^k, P^k\Pi)dt = \int_0^T \left[\gamma(\nabla\phi_{\epsilon,i}^k, \nabla P^k\Pi) + (\Psi'_{\epsilon,i}(\phi_{\epsilon,i}^k), P^k\Pi) + (f_D^{(i)}(\phi_{\epsilon,1}^k,\phi_{\epsilon,2}^k), P^k\Pi)\right]dt. \qquad (4.62)$$

We solely illustrate the convergence of the nonlinear terms, as the transition to the limit for linear terms is straightforward through the convergence properties of $\phi^k_{\epsilon,i}$, $w^k_{\epsilon,i}$ for $i = 1, 2$, and P^k.

By adding and subtracting the term $(\Xi_\epsilon(\phi^k_{\epsilon,i}), \Pi)$ and using (3.11) and the strong convergence (4.7) and (4.56) yields, for $i = 1, 2$, that

$$\begin{aligned}
&\Big|\int_0^T [(\Xi_\epsilon(\phi^k_{\epsilon,i}), P^k\Pi) - (\Xi_\epsilon(\phi_{\epsilon,i}), \Pi)]dt\Big| \\
= &\Big|\int_0^T [(\Xi_\epsilon(\phi^k_{\epsilon,i}), P^k\Pi) - (\Xi_\epsilon(\phi^k_{\epsilon,i}), \Pi) + (\Xi_\epsilon(\phi^k_{\epsilon,i}), \Pi) - (\Xi_\epsilon(\phi_{\epsilon,i}), \Pi)]dt\Big| \\
= &\Big|\int_0^T [(\Xi_\epsilon(\phi^k_{\epsilon,i}), P^k\Pi - \Pi) + (\Xi_\epsilon(\phi^k_{\epsilon,i}) - \Xi_\epsilon(\phi_{\epsilon,i}), \Pi)]dt\Big| \\
\le &\|\Xi_\epsilon(\phi^k_{\epsilon,i})\|_{L^2(\mathfrak{R}_T)}\|P^k\Pi - \Pi\|_{L^2(\mathfrak{R}_T)} + \|\Xi_\epsilon(\phi^k_{\epsilon,i}) - \Xi_\epsilon(\phi_{\epsilon,i})\|_{L^2(\mathfrak{R}_T)}\|\Pi\|_{L^2(\mathfrak{R}_T)} \\
\le &\frac{\theta}{\epsilon}\|\phi^k_{\epsilon,i}\|_{L^2(\mathfrak{R}_T)}\|P^k\Pi - \Pi\|_{L^2(\mathfrak{R}_T)} + \frac{\theta}{\epsilon}\|\phi^k_{\epsilon,i} - \phi_{\epsilon,i}\|_{L^2(\mathfrak{R}_T)}\|\Pi\|_{L^2(\mathfrak{R}_T)} \\
\to &\ 0, \quad \text{as} \quad k \to \infty,
\end{aligned} \tag{4.63}$$

Noting the above result, we can infer, based on $\Psi'_{\epsilon,i}(r) = \Xi_\epsilon(r) - \theta_i r$ for $i = 1, 2$, that

$$\int_0^T (\Psi'_{\epsilon,i}(\phi^k_{\epsilon,i}), P^k\Pi)dt \to \int_0^T (\Psi'_{\epsilon,i}(\phi_{\epsilon,i}), \Pi)dt \ \text{ as } \ k \to \infty. \tag{4.64}$$

The D-coupling term is treated for $i = 1, 2$ in the following manner:

$$\begin{aligned}
&\Big|\int_0^T [(f^{(i)}_D(\phi^k_{\epsilon,1}, \phi^k_{\epsilon,2}), P^k\Pi) - (f^{(i)}_D(\phi_{\epsilon,1}, \phi_{\epsilon,2}), \Pi)]dt\Big| \\
= &\Big|\int_0^T [(f^{(i)}_D(\phi^k_{\epsilon,1}, \phi^k_{\epsilon,2}), P^k\Pi) - (f^{(i)}_D(\phi^k_{\epsilon,1}, \phi^k_{\epsilon,2}), \Pi) + (f^{(i)}_D(\phi^k_{\epsilon,1}, \phi^k_{\epsilon,2}), \Pi) \\
- &(f^{(i)}_D(\phi_{\epsilon,1}, \phi_{\epsilon,2}), \Pi)]dt\Big| \\
\le &\int_0^T \|(f^{(i)}_D(\phi^k_{\epsilon,1}, \phi^k_{\epsilon,2}), P^k\Pi - \Pi)\|_0 dt + \int_0^T \|(f^{(i)}_D(\phi^k_{\epsilon,1}, \phi^k_{\epsilon,2}) - f^{(i)}_D(\phi_{\epsilon,1}, \phi_{\epsilon,2}), \Pi)\|_0 dt. \\
= &\ I^k_1 + I^k_2.
\end{aligned} \tag{4.65}$$

From the bound (4.46) and (4.7), it can be inferred that

$$\begin{aligned}
I^k_1 &= \int_0^T \|(f^{(i)}_D(\phi^k_{\epsilon,1}, \phi^k_{\epsilon,2}), P^k\Pi - \Pi)\|_0 dt \\
&\le \int_0^T \|f^{(i)}_D(\phi^k_{\epsilon,1}, \phi^k_{\epsilon,2})\|_0\|P^k\Pi - \Pi\|_0 dt \to 0 \ \text{ as } \ k \to \infty.
\end{aligned} \tag{4.66}$$

We note, for any $\iota_1, \iota_2, \varsigma_1, \varsigma_2 \in R$ and for $i, j = 1, 2$ with $i \neq j$, that

$$\begin{aligned}
f_D^{(i)}(\iota_1,\iota_2) - f_D^{(i)}(\varsigma_1,\varsigma_2) &= 2D[(\iota_i+\sigma_i)(\iota_j+\sigma_j)^2 - (\varsigma_i+\sigma_i)(\varsigma_j+\sigma_j)^2] \\
&= 2D[(\iota_i+\sigma_i)(\iota_j+\sigma_j)^2 - (\iota_j+\sigma_j)^2(\varsigma_i+\sigma_i) \\
&+ (\iota_j+\sigma_j)^2(\varsigma_i+\sigma_i) - (\varsigma_i+\sigma_i)(\varsigma_j+\sigma_j)^2] \\
&= 2D[(\iota_j+\sigma_j)^2(\iota_i-\varsigma_i) + (\varsigma_i+\sigma_i)((\iota_j+\sigma_j)^2 - (\varsigma_j+\sigma_j)^2)] \\
&= 2D(\iota_j+\sigma_j)^2(\iota_i-\varsigma_i) \\
&+ 2D(\varsigma_i+\sigma_i)(\iota_j+\varsigma_j+2\sigma_j)(\iota_j-\varsigma_j)\ , \qquad (4.67)
\end{aligned}$$

Now, using the above result with $\iota_i = \phi_{\epsilon,i}^k$ and $\varsigma_i = \phi_{\epsilon,i}, i = 1, 2$, the generalized Hölder's inequality the continuous embedding $H^1(\mathfrak{R}) \hookrightarrow L^6(\mathfrak{R})$, the bounds (4.35), (4.1) and (4.56) we obtain for $i = 1, 2$ with $i \neq j$ that

$$\begin{aligned}
I_2^k &= \int_0^T |(f_D^{(i)}(\phi_{\epsilon,1}^k,\phi_{\epsilon,2}^k) - f_D^{(i)}(\phi_{\epsilon,1},\phi_{\epsilon,2}),\Pi)|dt \\
&\leq 2D\int_0^T \|\phi_{\epsilon,j}^k+\sigma_j\|_{0,6}^2\|\Pi\|_{0,6}\|\phi_{\epsilon,i}^k-\phi_{\epsilon,i}\|_0 dt \\
&+ 2D\int_0^T \|\phi_{\epsilon,i}+\sigma_i\|_{0,6}\|\phi_{\epsilon,j}^k+\phi_{\epsilon,j}+2\sigma_j\|_{0,6}\|\Pi\|_{0,6}\|\phi_{\epsilon,j}^k-\phi_{\epsilon,j}\|_0 dt \\
&+ C\|\Pi\|_{L^2(0,T;H^1(\mathfrak{R}))}\|\phi_{\epsilon,i}^k-\phi_{\epsilon,i}\|_{L^2(\mathfrak{R}_T)} + C\|\Pi\|_{L^2(0,T;H^1(\mathfrak{R}))}\|\phi_{\epsilon,j}^k-\phi_{\epsilon,j}\|_{L^2(\mathfrak{R}_T)} \\
&\to 0 \text{ as } k\to\infty. \qquad (4.68)
\end{aligned}$$

Thus, from (4.65), (4.66) and (4.68) it follows, as $k \to 0$, that

$$\int_0^T (f_D^{(i)}(\phi_{\epsilon,1}^k,\phi_{\epsilon,2}^k), P^k\Pi)dt \to \int_0^T (f_D^{(i)}(\phi_{\epsilon,1},\phi_{\epsilon,2}),\Pi)dt. \qquad (4.69)$$

Theorem 4.2. *Assuming the validity of the conditions (A_1), then there exists a solution ϕ_1, ϕ_2, w_1, w_2 to the system (Q), such that*

$$\phi_1,\phi_2 \in L^\infty(0,T;H^1(\mathfrak{R})) \cap H^1(0,T;(H^1(\mathfrak{R}))'), \qquad (4.70)$$

$$w_1,w_2 \in L^2(0,T;H^1(\mathfrak{R})), \qquad (4.71)$$

$$\frac{\partial\phi_1}{\partial t},\frac{\partial\phi_2}{\partial t} \in L^2(0,T;(H^1(\mathfrak{R}))'), \qquad (4.72)$$

$$\Xi(\phi_1),\Xi(\phi_2) \in L^2(\mathfrak{R}_T), \qquad (4.73)$$

$$f_D^{(1)}(\phi_1,\phi_2), f_D^{(2)}(\phi_1,\phi_2) \in L^\infty(0,T;L^2(\mathfrak{R})), \qquad (4.74)$$

$$\max\{|\phi_1|,|\phi_2|\} < 1 \quad a.e. \ in \quad \mathfrak{R}_T. \qquad (4.75)$$

Proof: The bounds (4.1) - (4.11) are not dependent on the epsilon parameter. Consequently, through compactness arguments, we can extract subsequences represented by $\{\phi_{\epsilon,i}\}$ and $\{w_{\epsilon,i}\}$ for $i = 1, 2$, such that

$$\phi_{\epsilon,i} \rightharpoonup \phi_i \ in \quad L^2(0,T;H^1(\mathfrak{R})) \cap H^1(0,T;(H^1(\mathfrak{R}))'), \qquad (4.76)$$

$$\phi_{\epsilon,i} \rightharpoonup^* \phi_i \ in \quad L^\infty(0,T;H^1(\mathfrak{R})), \qquad (4.77)$$

$$w_{\epsilon,i} \rightharpoonup w_i \ in \ Ł^2(0,T;H^1(\mathfrak{R})), \qquad (4.78)$$

$$\frac{\partial \phi_{\epsilon,i}}{\partial t} \rightharpoonup \frac{\partial \phi_i}{\partial t} \;\; in \quad L^2(0,T;(H^1(\mathfrak{R}))'), \tag{4.79}$$

$$\Xi_\epsilon(\phi_{\epsilon,i}) \rightharpoonup \dot{\Pi}_i \;\; in \quad L^2(\mathfrak{R}_T). \tag{4.80}$$

Furthermore, employing a reasoning analogous to that presented in Theorem (4.1), we can show that

$$\phi_{\epsilon,i} \to \phi_i \;\; in \quad L^2(\mathfrak{R}_T). \tag{4.81}$$

Now, our aim is to establish that $\dot{\Pi}_i = \Xi(\phi_i)$ for $i = 1, 2$. It's important to note that demonstrating $\phi_i = \Xi^{-1}(\dot{\Pi}_i)$ almost everywhere in $\mathfrak{R}_T$ would immediately satisfy our objective and ensure $|\phi i| < 1$ almost everywhere in $\mathfrak{R}_T$, as $\Xi^{-1}(r) \in (-1, 1)$ for all $r \in \mathbb{R}$. To accomplish this, we first prove that

$$I_i(\varrho) = \int_0^T (\phi_i - \Xi^{-1}(\varrho), \dot{\Pi}_i - \varrho)dt \geq 0 \quad \forall \varrho \in L^2(\mathfrak{R}_T). \tag{4.82}$$

For $i = 1, 2$ and a.e. $t \in (0, T)$, selecting $s = \phi_{\epsilon,i}$ and $r = \Xi_\epsilon^{-1}(\varrho)$ in (3.9), results that

$$\theta(\phi_{\epsilon,i} - \Xi_\epsilon^{-1}(\varrho))^2 \leq (\Xi_\epsilon(\phi_{\epsilon,i}) - \Xi_\epsilon(\Xi_\epsilon^{-1}(\varrho)))(\phi_{\epsilon,i} - \Xi_\epsilon^{-1}(\varrho))$$

$$\to (\phi_{\epsilon,i} - \Xi_\epsilon^{-1}(\varrho), \Xi_\epsilon(\phi_{\epsilon,i}) - \varrho) \geq \theta\|\phi_{\epsilon,i} - \Xi_\epsilon^{-1}(\varrho)\|_0^2 \geq 0, \tag{4.83}$$

therefore, it immediately follows that

$$I_{\epsilon,i}(\varrho) = \int_0^T (\phi_{\epsilon,i} - \Xi_\epsilon^{-1}(\varrho), \Xi_\epsilon(\phi_{\epsilon,i}) - \varrho)dt \geq 0 \quad \forall \varrho \in L^2(\mathfrak{R}_T). \tag{4.84}$$

To ensure the well-defined of this integral, we utilize (3.16) with $s = \Xi_\epsilon(\phi_{\epsilon,i})$ and $r = \varrho$, incorporating the bound (4.11). This allows us to establish, for $i = 1, 2$, the following:

$$\begin{aligned} I_{\epsilon,i}(\varrho) \;&\leq\; \int_0^T \|(\phi_{\epsilon,i} - \Xi_\epsilon^{-1}(\varrho)\|_0 \|\Xi_\epsilon(\phi_{\epsilon,i}) - \varrho\|_0 dt \\ &\leq\; \theta^{-1} \int_0^T \|\Xi_\epsilon(\phi_{\epsilon,i}) - \varrho\|_0^2 dt = \theta^{-1}\|\Xi_\epsilon(\phi_{\epsilon,i}) - \varrho\|_{L^2(\mathfrak{R}_T)} < \infty. \end{aligned} \tag{4.85}$$

Now, it is imperative to establish that $I_{\epsilon,i} \to I_i$ as $\epsilon \to 0$ to attain (4.82). Utilizing the strong convergence (4.81), the constraint (4.11) on $\Xi_\epsilon(\phi_{\epsilon,i})$, and the weak convergence (4.80), we deduce, for all $\varrho \in L^2(\mathfrak{R}_T)$ and $i = 1, 2$, the following

$$
\begin{aligned}
& |I_{\epsilon,i}(\varrho) - I_i(\varrho)| \\
= \;& \Big| \int_0^T \Big[(\phi_{\epsilon,i} - \Xi_\epsilon^{-1}(\varrho), \Xi_\epsilon(\phi_{\epsilon,i}) - \varrho) - (\phi_i - \Xi^{-1}(\varrho), \dot{\Pi}_i - \varrho) \Big] dt \Big| \\
= \;& \Big| \int_0^T \Big[(\phi_{\epsilon,i} - \Xi_\epsilon^{-1}(\varrho), \Xi_\epsilon(\phi_{\epsilon,i}) - \varrho) - (\phi_i - \Xi^{-1}(\varrho), \Xi_\epsilon(\phi_{\epsilon,i}) - \varrho) \\
+ \;& (\phi_i - \Xi^{-1}(\varrho), \Xi_\epsilon(\phi_{\epsilon,i}) - \varrho) - (\phi_i - \Xi^{-1}(\varrho), \dot{\Pi}_i - \varrho) \Big] dt \Big| \\
= \;& \Big| \int_0^T \Big[(\phi_{\epsilon,i} - \Xi_\epsilon^{-1}(\varrho) - \phi_i + \Xi^{-1}(\varrho), \Xi_\epsilon(\phi_{\epsilon,i}) - \varrho) + (\phi_i - \Xi^{-1}(\varrho), \Xi_\epsilon(\phi_{\epsilon,i}) - \dot{\Pi}_i) \Big] dt \Big| \\
= \;& \Big| \int_0^T \Big[(\phi_{\epsilon,i} - \phi_i, \Xi_\epsilon(\phi_{\epsilon,i}) - \varrho) + (\Xi^{-1}(\varrho) - \Xi_\epsilon^{-1}(\varrho), \Xi_\epsilon(\phi_{\epsilon,i}) - \varrho) \\
+ \;& (\phi_i - \Xi^{-1}(\varrho), \Xi_\epsilon(\phi_{\epsilon,i}) - \dot{\Pi}_i) \Big] dt \Big| \\
\leq \;& \Big| \int_0^T (\phi_{\epsilon,i} - \phi_i, \Xi_\epsilon(\phi_{\epsilon,i}) - \varrho) dt \Big| + \Big| \int_0^T (\Xi^{-1}(\varrho) - \Xi_\epsilon^{-1}(\varrho), \Xi_\epsilon(\phi_{\epsilon,i}) - \varrho) dt \Big| \\
+ \;& \Big| \int_0^T (\phi_i - \Xi^{-1}(\varrho), \Xi_\epsilon(\phi_{\epsilon,i}) - \dot{\Pi}_i) dt \Big| \\
\leq \;& \Big| \int_0^T \Big[\|\phi_{\epsilon,i} - \phi_i\|_0 \|\Xi_\epsilon(\phi_{\epsilon,i}) - \varrho\|_0 \\
+ \;& \|\Xi^{-1}(\varrho) - \Xi_\epsilon^{-1}(\varrho)\|_0 \|\Xi_\epsilon(\phi_{\epsilon,i}) - \varrho\|_0 \Big] dt + \Big| \int_0^T (\phi_i - \Xi^{-1}(\varrho), \Xi_\epsilon(\phi_{\epsilon,i}) - \dot{\Pi}_i) dt \Big| \\
= \;& \|\phi_{\epsilon,i} - \phi_i\|_{L^2(\mathfrak{R}_T)} \|\Xi_\epsilon(\phi_{\epsilon,i}) - \varrho\|_{L^2(\mathfrak{R}_T)} + \|\Xi^{-1}(\varrho) - \Xi_\epsilon^{-1}(\varrho)\|_{L^2(\mathfrak{R}_T)} \|\Xi_\epsilon(\phi_{\epsilon,i}) - \varrho\|_{L^2(\mathfrak{R}_T)} \\
+ \;& \Big| \int_0^T (\phi_i - \Xi^{-1}(\varrho), \Xi_\epsilon(\phi_{\epsilon,i}) - \dot{\Pi}_i) dt \Big| \to 0 \quad as \quad \epsilon \to 0. \qquad (4.86)
\end{aligned}
$$

Therefore, it follows that

$$I_i(\varrho) = \lim_{\epsilon \to 0} I_{\epsilon,i}(\varrho) \geq 0 \quad \forall \varrho \in L^2(\mathfrak{R}_T).$$

By substituting $\dot{\Pi}_i \pm \beta\varrho \in L^2(\mathfrak{R}T)$ into I_i and utilizing (4.82), it follows, for all $\beta \in \mathbb{R}^{>0}$ and all $\varrho \in L^2(\mathfrak{R}_T)$, that

$$\int_0^T (\phi_i - \Xi^{-1}(\dot{\Pi}_i + \beta\varrho), -\beta\varrho) dt \geq 0, \quad \text{and} \quad \int_0^T (\phi_i - \Xi^{-1}(\dot{\Pi}_i - \beta\varrho), \beta\varrho) dt \geq 0.$$

Dividing the first inequality by $-\beta$ and the second by β, it follows that:

$$\int_0^T (\phi_i - \Xi^{-1}(\dot{\Pi}_i + \beta\varrho), \varrho) dt \leq 0, \quad \text{and} \quad \int_0^T (\phi_i - \Xi^{-1}(\dot{\Pi}_i - \beta\varrho), \varrho) dt \geq 0.$$

Upon observing the continuity of Ξ^{-1} and considering the limit as $\beta \to 0$, we obtain, and subsequently, taking the limit as $\beta \to 0$ and acknowledging the continuity of Ξ^{-1}, we have that:

$$\int_0^T (\phi_i - \Xi^{-1}(\dot{\Pi}_i), \varrho)dt \leq 0, \quad \text{and} \quad \int_0^T (\phi_i - \Xi^{-1}(\dot{\Pi}_i), \varrho)dt \geq 0.$$

which leads, for $i = 1, 2$, to

$$\int_0^T (\phi_i - \Xi^{-1}(\dot{\Pi}_i), \varrho)dt = 0 \quad \forall \varrho \in L^2(\mathfrak{R}_T). \tag{4.87}$$

We choose $\varrho = \phi_i - \Xi^{-1}(\dot{\Pi}_i) \in L^2(\mathfrak{R}_T)$ in (4.87), yielding, for $i = 1, 2$, the following result:

$$\|\phi_i - \Xi^{-1}(\dot{\Pi}_i)\|^2_{L^2(\mathfrak{R}_T)} = \int_0^T \|\phi_i - \Xi^{-1}(\dot{\Pi}_i)\|_0^2 dt = \int_0^T (\phi_i - \Xi^{-1}(\dot{\Pi}_i), \phi_i - \Xi^{-1}(\dot{\Pi}_i))dt = 0$$

which shows that $\phi_i = \Xi^{-1}(\dot{\Pi}_i)$ a.e. in $\mathfrak{R}_T$, thus, for $i = 1, 2$, it follows that $|\phi_i| < 1$ a.e. in $\mathfrak{R}_T$ and $\dot{\Pi}_i = \Xi(\phi_i)$.

Similarly to Theorem (4.1), one can take the limit as $\epsilon \to 0$ in (Q_ϵ) to determine that ϕ_1, ϕ_2, w_1, w_2 solves (Q) by applying the results (4.76)-(4.78). □

5. Regularity

Under the subsequent, more stringent assumptions on ϕ_1^0, ϕ_2^0, we will delve into studying the problem (Q):

(A_2) Let $\{\phi_1^0, \phi_2^0\} \in H^2(\mathfrak{R}) \times H^2(\mathfrak{R})$, $|\Delta\phi_1^0| + |\Delta\phi_2^0| \leq C$, $\frac{\partial \phi_1^0}{\partial \nu} = \frac{\partial \phi_2^0}{\partial \nu} = 0$ on $\partial\mathfrak{R}$ and $\max\{|\phi_1^0|_{0,\infty}, |\phi_2^0|_{0,\infty}\} \leq 1 - \delta_0$ for some given $\delta_0 \in (0, 1)$. We recall that if $u \in H^1(\mathfrak{R})$ is a solution of the variational equation.

$$(\nabla u, \nabla \Pi) + (u, \Pi) = (f, \Pi), \forall \Pi \in H^1(\mathfrak{R}),$$

where $f \in L^2(\mathfrak{R})$ and if u is a convex polygonal $\partial u \in C^2$, then from the standard regularity theory of elliptic problem $u \in H^2(\mathfrak{R})$ and

$$\|u\|_2 \leq C\|f\|_0.$$

Hence, by the weak form of (4.1) we have $z_j \in H^2(\mathfrak{R}), 1 \leq j \leq k$ (k, fixed and finite) and thus $V^k \subset H^2(\mathfrak{R})$.

Lemma 5.1. *If $\Pi \in H^2(\mathfrak{R})$, then there are constants $\alpha = d(\frac{1}{2} - \frac{1}{\varepsilon})$ and C such that*

$$\|\nabla \Pi\|_{0,\varepsilon} \leq C|\Pi|_1^{1-\alpha}\|\Pi\|_2^{\alpha} \leq C\|\Pi\|_2 \qquad \text{holds for} \qquad \varepsilon \in \begin{cases} [2, \infty] & \text{if } d = 1, \\ [2, \infty) & \text{if } d = 2, \\ [2, 6] & \text{if } d = 3. \end{cases} \tag{5.1}$$

Proof: Applying the Sobolev interpolation result (2.13) results in:

$$\|\nabla \Pi\|_{0,\varepsilon}^{\varepsilon} = \int_{\mathfrak{R}} \Big(\sum_{i=1}^d \Big| \frac{\partial \Pi}{\partial x_i} \Big|^2 \Big)^{\frac{\varepsilon}{2}} dx \leq C \int_{\mathfrak{R}} \Big(\sum_{i=1}^d \Big| \frac{\partial \Pi}{\partial x_i} \Big|^{\varepsilon} \Big) dx = C \sum_{i=1}^d \Big\| \frac{\partial \Pi}{\partial x_i} \Big\|_{0,\varepsilon}^{\varepsilon}$$

$$\leq C\sum_{i=1}^{d}\Big\|\frac{\partial \Pi}{\partial x_i}\Big\|_0^{\varepsilon(1-\alpha)}\Big\|\frac{\partial \Pi}{\partial x_i}\Big\|_1^{\varepsilon\alpha} \leq C\Big(\sum_{i=1}^{d}\Big\|\frac{\partial \Pi}{\partial x_i}\Big\|_0^2\Big)^{\frac{\varepsilon}{2}(1-\alpha)}\|\Pi\|_2^{\varepsilon\alpha} = C|\Pi|_1^{\varepsilon(1-\alpha)}X\|\Pi\|_2^{\varepsilon\alpha},$$

and the second inequality follows directly from the embedding $H^2(\mathfrak{R}) \hookrightarrow H^1(\mathfrak{R})$. □

Theorem 5.2. *Assuming that* $(A2)$ *holds, and considering* $\mathfrak{R}$ *as a convex polygonal domain or* $\partial\mathfrak{R} \in C^2$*, the solution of* (Q) *exhibits the following additional regularity results:*

$$\phi_1, \phi_2 \in L^2(0,T;H^2(\mathfrak{R})), \tag{5.2}$$

$$f_D^{(i)}(\phi_1,\phi_2) \in L^2(0,T;H^1(\mathfrak{R})). \tag{5.3}$$

Proof: From Theorem 4.1, we have for $i = 1, 2$ and a.e.$t \in (0,T)$ that $\phi_i \in H^1(\mathfrak{R})$ is a solution of elliptic variational equation

$$\gamma(\nabla\phi_i, \nabla\Pi) + (\Xi(\phi_i) - \theta_i\phi_i + f_D^{(i)}(\phi_1, u-2) - w_i, \Pi) = 0 \quad \forall \Pi \in H^1(\mathfrak{R}).$$

Hence, employing the estimates provided by Theorem 4.1 and applying the standard regularity theory for elliptic problems, it follows, for almost every $t \in (0,T)$, $\phi_i \in H^2(\mathfrak{R})$, that

$$\|\phi_i\|_2 \leq C\|w_i - \Xi(\phi_i) + \theta_i\phi_i - f_D^{(i)}(\phi_1,\phi_2) + \phi_i\|_0,$$

Therefore, by squaring this inequality and integrating over the interval $(0,T)$, we obtain that

$$\|\phi_i\|^2_{L^2(0,T;H^2(\mathfrak{R}))} \leq C\|w_i - \Xi(\phi_i) + \theta_i\phi_i - f_D^{(i)}(\phi_1,\phi_2) + \phi_i\|^2_{L^2(\mathfrak{R}_T)} \leq C. \tag{5.4}$$

The result (5.3) can be demonstrated by utilizing the generalized Hölder's inequality, $H^1(\mathfrak{R}) \hookrightarrow L^6(\mathfrak{R})$ and (5.1). This leads to the following outcome for $\phi_1, \phi_2 \in H^2(\mathfrak{R})$, where $i = 1, 2$, and $i \neq j$:

$$\begin{aligned}
|f_D^{(i)}(\phi_1,\phi_2)|_1^2 &= \|\nabla f_D^{(i)}(\phi_1,\phi_2)\|_0^2 \\
&= \|\nabla 2D(\phi_i+\sigma_i)(\phi_j+\sigma_j)^2\|_0^2 \\
&= 4D^2\|\nabla(\phi_i+\sigma_i)(\phi_j+\sigma_j)^2\|_0^2 \\
&= 4D^2\|(\phi_j+\sigma_j)^2\nabla\phi_i + 2(\phi_i+\sigma_i)(\phi_j+\sigma_j)\nabla\phi_j\|_0^2 \\
&\leq 8D^2\|(\phi_j+\sigma_j)^4(\nabla\phi_i)^2\|_{0,1} + 32D^2\|(\phi_i+\sigma_i)^2(\phi_j+\sigma_j)^2(\nabla\phi_j)^2\|_{0,1} \\
&\leq 8D^2\|\phi_j+\sigma_j\|_{0,6}^4\|\nabla\phi_i\|_{0,6}^2 + 32D^2\|\phi_i+\sigma_i\|_{0,6}^2\|\phi_j+\sigma_j\|_{0,6}^2\|\nabla\phi_j\|_{0,6}^2 \\
&\leq C\|\phi_j+\sigma_j\|_1^4\|\phi_i\|_2^2 + C\|\phi_i+\sigma_i\|_1^2\|\phi_j+\sigma_j\|_1^2\|\phi_j\|_2^2. \qquad (5.5)
\end{aligned}$$

Consequently, through integration over the interval $(0,T)$ and considering (4.1) and (5.4), it follows that for $i, j = 1, 2$ and $i \neq j$:

$$\begin{aligned}
\int_0^T |f_D^{(i)}(\phi_1,\phi_2)|_1^2 dt &\leq C\int_0^T \|\phi_j+\sigma_j\|_1^4\|\phi_i\|_2^2 dt + C\int_0^T \|\phi_i+\sigma_i\|_1^2\|\phi_j+\sigma_j\|_1^2\|\phi_j\|_2^2 dt \\
&\leq C\int_0^T \|\phi_i\|_2^2 dt + C\int_0^T \|\phi_j\|_2^2 dt \leq C. \qquad (5.6)
\end{aligned}$$

Therefore, we can derive the desired result (5.3) by utilizing the aforementioned estimate and (5.4), considering that $f_D^{(i)}(\phi_1,\phi_2) \in L^2(0,T;H^1(\mathfrak{R}))$. □

6. Conclusions

We investigated a system of two interconnected Cahn-Hilliard equations in a spatial dimension of $d \leq 3$. The equations featured a logarithmic potential, a diffusional mobility dependent on concentration, and zero Neumann boundary conditions. Under specific assumptions (A_1) on the initial conditions, we established the existence of a weak solution and provided stability estimates.

Our approach involved introducing a regularized problem $((Q_\epsilon)$ by smoothly replacing the logarithmic potential to address the challenges posed by the original continuous problem (Q). Utilizing the Faedo-Galerkin technique and compactness arguments, we demonstrated the existence of a solution to (Q_ϵ). By subsequently taking the limit as ϵ approached zero, we inferred the existence of a solution to the original problem (Q).

Furthermore, we enhanced the regularity of the weak solutions to (Q) and (Q_ϵ). Employing the standard regularity theory for elliptic problems, we showed that the weak solutions belong to higher-order Sobolev spaces. This was achieved by imposing additional requirements on the domain's boundary and the initial conditions.

There are still mathematical work to be done in the future. In this article, we were not able to conduct a study of the uniqueness of the solutions due to severe mathematical difficulties. Studying the uniqueness of the solutions requires proving that solutions belong to higher spaces, and this we could not derive in this article. Also, the mobility function that was adopted in this article is nondegenerate, so the current study can be generalized and developed by adopting a degenerate mobility function.

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