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COMMUTING MULTIPLICATIVE GENERALIZED DERIVATIONS ON LIE IDEALS OF SEMIPRIME RINGS

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Abstract. Let R be a semiprime ring with characteristic different from two and L a noncentral square-closed Lie ideal of R. Suppose that R admits a multiplicative generalized derivation (F, d) satisfying $d(L) \subseteq L$. In the present paper, we shall prove that d is commuting on L if one of the following conditions holds: $(i) F([x, y]) = \pm[x, d(y)]$ $(ii) F(x \circ y) = \pm(x \circ d(y))$ $(iii) F([x, y]) = \pm(F(y)x)$ $(iv) F([x, y]) = \pm(F(x)y)$ $(v) F(x \circ y) = \pm(F(y)x)$ (vi) $F(x \circ y) = \pm(F(x)y)$ for all $x, y \in L$.

1. Introduction and preliminaries

In what follows, R will denote an associative ring with center $Z(R)$. For all $x, y \in R$, the symbol $[x, y]$ and $x \circ y$ will stand for Lie product $xy - yx$ and Jordan product $xy + yx$, respectively. For subsets A, B of R we let $[A, B]$ be the additive subgroup generated by all $[a, b]$ with $a \in A$ and $b \in B$. An additive subgroup L of R is said to be a Lie ideal of R if $[u, r] \in L$ for all $u \in L$ and $r \in R$ and it is called square-closed if $u^2 \in L$ for all $u \in L$. For a square-closed Lie ideal L of R, it is obvious that both $uv + vu = (u + v)^2 - u^2 - v^2 \in L$ and $uv - vu \in L$, so that $2uv \in L$ for all $u, v \in L$ and this relation will be used freely in this paper. Recall that a ring R is prime if for any $x, y \in R$, $xRy = 0$ implies $x = 0$ or $y = 0$ and is semiprime if for any $x \in R$, $xRx = 0$ implies $x = 0$. By a derivation on R we mean an additive mapping d on R such that $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping F of R is said to be a generalized derivation if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$, and d is called the associated derivation of F. In the particular case $d = 0$, that is, $F(xy) = F(x)y$ for all $x, y \in R$, such a generalized derivation F is called a left centralizer. Moreover, a mapping $f : R \longrightarrow R$ is called commuting on a subset S if $[f(x), x] = 0$ for all $x \in S$. The first important result on commuting mappings is Posner's theorem [17], which states that the existence of a nonzero commuting derivation d on a prime ring R forces that R is commutative. Thereafter, there has been considerable interest in commuting mappings and related mappings in prime and semiprime rings (see for instance [22]). For the development of the theory of

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commuting mappings and their applications on rings, we refer the reader to [9], where further references can be found.

Many papers in literature have studied the commutativity of prime and semiprime rings satisfying certain functional identities involving derivations or generalized derivations (see [20] for a partial bibliography). In [6], Daif and Bell proved a semiprime ring R must be commutative if it admits a derivation d such that $d([x, y]) = \pm[x, y]$ for all $x, y \in R$. Replacing Lie product by Jordan product, Ashraf and Rehman [1] proved a prime ring R is commutative if it admits a derivation d such that $d(x \circ y) = \pm(x \circ y)$ for all $x, y \in I$, a nonzero ideal of R. In view of above results, it is natural to ask what can we say about the commutativity of R if we replace the derivation d by a generalized derivation F. In this line of investigation, Quadri, Khan and Rehman [18] extended these results to generalized derivations. More, precisely, they proved a prime ring R is commutative if it admits a generalized derivation F associated with a nonzero derivation d such that any one of the following conditions holds: (i) $F([x, y]) = [x, y]$ (ii) $F([x, y]) + [x, y] = 0$ (iii) $F(x \circ y) = x \circ y$ (iv) $F(x \circ y) + x \circ y = 0$ for all x, y some nonzero ideal I of R. Further, Rehman [19] studied the above mentioned results of [1] and [6] in prime rings by replacing two-sided ideal with Lie ideals and derivations with generalized derivations. Very recently, several identities involving multiplicative generalized derivations on Lie ideals of semiprime rings are obtained in [15].

Now, recall the background of investigation about multiplicative (generalized)-derivations. Motivated by the work of Martindale [16], in the year 1991, Daif introduced the concept of multiplicative derivation. Following [4], a mapping $d : R \longrightarrow R$ is called a multiplicative derivation if it satisfies $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. Note that these mappings are not necessarily to be additive. Consider $R = C[0, 1]$, the ring of all continuous functions and define a mapping on R as follows: $d(f)(x) = f(x) \log |f(x)|$, if $f(x) \neq 0$; $d(f)(x) = 0$, if $f(x)$ is otherwise. Then it is obvious that d is a multiplicative derivation, but d is not additive. Later, these mapping were completely described in [12]. Inspired by the definition of multiplicative derivation, the concept of multiplicative derivations was extended to multiplicative generalized derivations by Daif and Tammam EI-Sayiad in [8] as follows: a mapping $F : R \longrightarrow R$ is called a multiplicative generalized derivation if there exists a derivation d such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. For completeness of notation, in the present paper, a multiplicative generalized derivation will be denoted by (F, d). Dhara and Ali [5] gave a slight generalization of this definition by taking d is any mapping (not necessarily additive or a derivation). Therefore, one may find that the concept of multiplicative generalized derivation covers the concept of multiplicative derivation as well as multiplicative left centralizers. The examples of multiplicative generalized derivation are multiplicative derivation and multiplicative centralizers can be found in [11]. It is clear that every generalized derivation is a multiplicative generalized derivation but the converse need not be true in general (see [5]). So, it should be interesting to extend some results concerning derivations and generalized derivations to multiplicative generalized derivations. However there are only a few papers about this subject (see [2, 3, 7, 13, 21] for a partial bibliography). The main objective of the present paper is to explore the cases when a multiplicative generalized derivation (F, d) satisfies the identities: $(i)F([x, y]) = \pm[x, d(y)]$ $(ii)F(x \circ y) = \pm(x \circ d(y))$ $(iii)F([x, y]) = \pm(F(y)x)$ $(iv)F([x, y]) = \pm(F(x)y)$ $(v)F(x \circ y) = \pm(F(y)x)$ (vi) $F(x \circ y) = \pm(F(x)y)$ for all x, y in some appropriate subsets of R.

In this paper, we shall do a great deal of calculation with Lie product and Jordan product and routinely use the following identities:

$$[xy, z] = x[y, z] + [x, z]y \text{ and } [x, yz] = y[x, z] + [x, y]z$$
$$xo(yz) = (xoy)z - y[x, z] = y(xoz) + [x, y]z$$
$$(xy)oz = x(yoz) - [x, z]y = (xoz)y + x[y, z].$$

The following lemmas are required for the proof of our main results.

Lemma 1.1. [15] *Let R be a semiprime ring with characteristic different from two, L a noncentral Lie ideal of R and $a \in L$. If $aLa = 0$, then $a = 0$.*

Lemma 1.2. [14] *Let R be a semiprime ring with characteristic different from two, L a Lie ideal of R such that $[L, L] \subseteq Z(R)$. Then $L \subseteq Z(R)$.*

Lemma 1.3. [10] *Let R be a prime ring with characteristic different from two and L a Lie ideal of R. If d is a nonzero derivation of R such that $d(L) = 0$, then $L \subseteq Z(R)$.*

Lemma 1.4. [10] *Let R be a prime ring with characteristic different from two and L a noncentral Lie ideal of R. If $a, b \in R$ and $aLb = 0$, then $a = 0$ or $b = 0$.*

2. Main results

Theorem 2.1. *Let R be a semiprime ring with characteristic different from two and L a noncentral square-closed Lie ideal of R. Suppose that R admits a multiplicative generalized derivation (F, d) satisfying $d(L) \subseteq L$ and $F([x, y]) = \pm[x, d(y)]$ for all $x, y \in L$, then d is commuting on U.*

Proof. By the assumption, we have

$$F([x, y]) = [x, d(y)] \text{ for all } x, y \in L. \tag{2.1}$$

Replacing x by $2xy$ in (2.1) and using the fact that $char(R) \neq 2$, we get

$$F([x, y])y + [x, y]d(y) = x[y, d(y)] + [x, d(y)]y \text{ for all } x, y \in L. \tag{2.2}$$

Combining (2.1) and (2.2), we arrive at

$$[x, y]d(y) = x[y, d(y)] \text{ for all } x, y \in L. \tag{2.3}$$

Using the fact $d(U) \subseteq U$, we replace x by $2d(y)x$ in (2.3) and use (2.3) to get

$$[d(y), y]xd(y) = 0 \text{ for all } x, y \in L. \tag{2.4}$$

Writing $2xy$ for x in (2.4) , we obtain that

$$[d(y), y]xyd(y) = 0 \text{ for all } x, y \in L. \tag{2.5}$$

Multiplying (2.4) by y on the right, we find that

$$[d(y), y]xd(y)y = 0 \text{ for all } x, y \in L. \tag{2.6}$$

Subtracting (2.5) from (2.6), we conclude that $[d(y), y]x[d(y), y] = 0$ for all $x, y \in L$. That is $[d(y, y]L[d(y), y] = 0$ for all $y \in L$. Application of Lemma 1.1 gives that $[d(y), y] = 0$ for all $y \in L$ and hence d is commuting on U. In a similar manner, we can prove that the same conclusion is true for $F([x, y]) + [x, d(y)] = 0$ for all $x, y \in L$. Therefore, the proof of the theorem is completed. □

Corollary 2.2. *Let R be a prime ring with characteristic different from two and L a square-closed Lie ideal of R. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero derivation d and $F([x, y]) = \pm[x, d(y)]$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Proof. Suppose on the contrary that $L \nsubseteq Z(R)$ and prove that a contradiction follows. As stated in the proof of Theorem 2.1, we have

$$[x, y]d(y) = x[y, d(y)] \text{ for all } x, y \in L. \tag{2.7}$$

Replacing x by $2xz$ in (2.7) and using (2.7) we have the following relation

$$[x, y]zd(y) = 0 \text{ for all } x, y, z \in L. \tag{2.8}$$

By Lemma Lemma 1.4 , for each $y \in L$ we have either $[x, y] = 0$ or $d(y) = 0$ for all $x, y \in L$. Let $L_1 = \{y \in L \mid [x, y] = 0\}$ and $L_2 = \{y \in L \mid d(y) = 0\}$. Then, L_1 and L_2 are both additive subgroups of L such that $L = L_1 \cup L_2$. By Brauer's trick, either $L_1 = L$ or $L_2 = L$. In the former case, $[L, L] = 0$. Then Lemma Lemma 1.2 yields that $L \subseteq Z(R)$, a contradiction. In the latter case, $d(U) = 0$. In light of Lemma Lemma 1.3 , $L \subseteq Z(R)$, again a contradiction. This completes the proof. □

If we replace Lie product by Jordan in Theorem 2.1, then we can prove the following:

Theorem 2.3. *Let R be a semiprime ring with characteristic different from two and L a noncentral square-closed Lie ideal of R. Suppose that R admits a multiplicative generalized derivation (F, d) satisfying $d(L) \subseteq L$ and $F(x \circ y) = \pm(x \circ d(y))$ for all $x, y \in L$, then d is commuting on U.*

Proof. First we consider the case

$$F(x \circ y) = x \circ d(y) \text{ for all } x, y \in L. \tag{2.9}$$

Writing $2xy$ for x in (2.9) and using (2.9) we have

$$(x \circ y)d(y) = x[y, d(y)] \text{ for all } x, y \in L. \tag{2.10}$$

Substituting $2d(y)x$ for x in (2.10) and using (2.10) we see that $[d(y), y]xd(y) = 0$ for all $x, y \in L$. This equation is same as (2.4) in the proof of Theorem 2.1 and we get required result. Using a similar approach one can prove the same conclusion is also true for the case $F(x \circ y) + x \circ d(y) = 0$ for all $x, y \in L$. Hence the theorem is now completed proved. □

Corollary 2.4. *Let R be a prime ring with characteristic different from two and L a square-closed Lie ideal of R. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero derivation d and $F(x \circ y) = \pm(x \circ d(y))$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Proof. By the same technique in the proof of Theorem 2.3 , we obtain the following

$$(x \circ y)d(y) = \pm x[y, d(y)] \text{ for all } x, y \in L. \tag{2.11}$$

Taking x by $2xz$ in (2.11) and using (2.11), we arrive at $[x, y]zd(y) = 0$ for all $x, y, z \in L$. Using the same arguments after equation (2.8), we get the required results. □

Theorem 2.5. *Let R be a semiprime ring with characteristic different from two and L a noncentral square-closed Lie ideal of R. Suppose that R admits a multiplicative generalized derivation (F, d) satisfying $d(L) \subseteq L$ and $F([x, y]) = \pm(F(y)x)$ for all $x, y \in L$, then d is commuting on U.*

Proof. By the given hypothesis, we have

$$F([x, y]) = F(y)x \text{ for all } x, y \in L. \tag{2.12}$$

Replacing y by $2yx$ in (2.12), we get

$$F([x, y])x + [x, y]d(x) = F(y)x^2 + yd(x)x \text{ for all } x, y \in L. \tag{2.13}$$

Right multiplying by x in (2.12) gives that

$$F([x, y])x = F(y)x^2 \text{ for all } x, y \in L. \tag{2.14}$$

Subtracting (2.14) from (2.13), we find that

$$[x, y]d(x) = yd(x)x \text{ for all } x, y \in L. \tag{2.15}$$

Taking y by $2d(x)y$ in (2.15) and using (2.15), we conclude that

$$[x, d(x)]yd(x) = 0 \text{ for all } x, y \in L. \tag{2.16}$$

This equation is similar as equation (2.4) and using the same methods we can get the required result. A similar conclusion holds for the case $F([x, y]) + F(y)x = 0$ for all $x, y \in L$. □

Corollary 2.6. *Let R be a prime ring with characteristic different from two and L a square-closed Lie ideal of R. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero derivation d and $F([x, y]) = \pm(F(y)x)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Proceeding on the same lines with necessary variations, we get the following:

Theorem 2.7. *Let R be a semiprime ring with characteristic different from two and L a noncentral square-closed Lie ideal of R. Suppose that R admits a multiplicative generalized derivation (F, d) satisfying $d(L) \subseteq L$ and $F([x, y]) = \pm(F(x)y)$ for all $x, y \in L$, then d is commuting on U.*

Corollary 2.8. *Let R be a prime ring with characteristic different from two and L a square-closed Lie ideal of R. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero derivation d and $F([x, y]) = \pm(F(x)y)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Similar arguments as above yield the following:

Theorem 2.9. *Let R be a semiprime ring with characteristic different from two and L a noncentral square-closed Lie ideal of R. Suppose that R admits a multiplicative generalized derivation (F, d) satisfying $d(L) \subseteq L$ and $F([x, y]) = \pm(F(x)y)$ for all $x, y \in L$, then d is commuting on U.*

Proof. We begin with the situation

$$F(x \circ y) = F(y)x \text{ for all } x, y \in L. \tag{2.17}$$

Putting $y = 2yx$ in (2.17)and using (2.17), we observe that

$$(x \circ y)d(x) = yd(x)x \text{ for all } x, y \in L. \tag{2.18}$$

Replace y by $2d(x)y$ in (2.18) to get

$$d(x)(x \circ y)d(x) + [x, d(x)]yd(x) = d(x)yd(x)x \text{ for all } x, y \in L. \tag{2.19}$$

Using (2.18), then equation (2.19) reduces to $[x, d(x)]yd(x) = 0$, which is same as (2.16). So we can get the required result. By the same arguments, the conclusion can be obtained when $F(x \circ y) + F(y)x = 0$ for all $x, y \in L$. □

Corollary 2.10. *Let R be a prime ring with characteristic different from two and L a square-closed Lie ideal of R. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero derivation d and $F(x \circ y) = \pm(F(y)x)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Similarly, in view of Theorem 2.9, we obtain the following result:

Theorem 2.11. *Let R be a semiprime ring with characteristic different from two and L a noncentral square-closed Lie ideal of R. Suppose that R admits a multiplicative generalized derivation (F, d) satisfying $d(L) \subseteq L$ and $F(x \circ y) = \pm(F(x)y)$ for all $x, y \in L$, then d is commuting on U.*

Corollary 2.12. *Let R be a prime ring with characteristic different from two and L a square-closed Lie ideal of R. Suppose that R admits a multiplicative generalized derivation F associated with a nonzero derivation d and $F(x \circ y) = \pm(F(x)y)$ for all $x, y \in L$, then $L \subseteq Z(R)$.*

Now we present an example which shows that the primeness of above corollaries is essential.

Example 2.13. Let $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} | a, b, c \in S \right\}$, $L = \left\{ \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} | b \in S \right\}$. Then L is a square-closed Lie ideal of R, where S is a set of integers. Define maps F and d of R as follows: $F \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a^2c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $d \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a^2 & bc \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Here, F is a multiplicative generalized derivation associated with a mapping d. The fact that $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = 0$ implies that R is not a prime ring. Here, we see that (F, d) satisfies the following conditions: $(i)F([X,Y]) = \pm[X, d(Y)]$ $(ii)F(X\circ Y) = \pm(X\circ d(Y))$ $(iii)F([X,Y]) = \pm(F(Y)X)(iv)$ $F([X,Y]) = \pm(F(X)Y)$ $(v)F(X\circ Y) = \pm(F(Y)X)$ $(vi)F(X\circ Y) = \pm(F(X)Y)$ for all $X, Y \in L$. However, L is not a central Lie ideal. Therefore, the hypothesis of primeness in the various corollaries is not superfluous.

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