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APPROXIMATE CONTROLLABILITY OF IMPULSIVE EVOLUTION STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS DRIVEN BY A FRACTIONAL BROWNIAN MOTION

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ABSTRACT. In this paper, we study the approximate controllability of certain class of impulsive evolution stochastic functional differential equations, with variable delays, driven by a fractional Brownian motion in a separable real Hilbert space. We derive a new set of sufficient conditions for approximate controllability using a stochastic analysis of fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ and a Schaefer's fixed point theorem. An example is considered at the end of the paper to illustrate the obtained abstract results.

1. INTRODUCTION

Controllability refers to the ability to guide a dynamic control system from an initial state to an arbitrary end state using a set of admissible controls. Approximate controllability, on the other hand, allows the system to be directed towards a small neighborhood of the final state. With the growing prevalence of approximate controllable systems, it is crucial to explore their applicability in various situations.

Studies on approximate controllability are typically conducted in two stages. The first involves demonstrating the existence of a sequence of mild solutions for the equation under study, and the second involves showing that the sequence converges to the final state. In previous research [15, 7, 11, 8], researchers employed the contraction mapping principle to establish the existence of the sequence. However, the existence of terms in the sequence x_λ was conditioned on a fraction being less than 1, which was only verified for the initial terms due to the lambda value tending towards 0^+ and being in the denominator. In this work, we present a method for demonstrating the existence of the sequence without using the contraction approach. Instead, we utilize Schaefer's fixed-point theorem, with conditions established using semigroup theory, under the assumption that the linear part of the associated nonlinear system is approximately controllable.

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In modeling systems, perturbations are typically introduced via Brownian motion, as it is Gauss-Markov and has independent increments. However, empirical evidence from various physical phenomena suggests that Brownian motion may not always be an effective modeling choice. A family of processes that has shown broad applicability in physics is fractional Brownian motion, introduced by Kolmogorov [9] and further studied by Mandelbrot and VanNess [13], who obtained a stochastic integral representation in terms of a standard Brownian motion. As fBm is not a semimartingale when $H \neq \frac{1}{2}$ (see Biagini et al. [2]), classical Itô theory cannot be used to construct a stochastic calculation with respect to fBm. Currently, a general theory for stochastic differential equations driven by fractional Brownian motion is not established, and only a few results have been proven. Recently, stochastic functional differential equations driven by fractional Brownian motion have garnered the interest of many researchers [3, 5, 10, 6] and the literature concerning the existence and qualitative properties of solutions of time-dependent functional stochastic differential equations is limited to a few articles.

Motivated by the above works, this paper is concerned with the approximate controllability results for a class of evolution stochastic functional differential equations driven by a fractional Brownian motion described in the form:

$$\begin{cases} dx(t) = [A(t)x(t) + f(t, x(t-\rho(t))) + Lu(t)]dt + \sigma(t)dB^H(t), \; t \in [0,T], \\ \Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), k = 1, ..., m, m \in \mathbb{N} \\ x(.) = \varphi(.) \in \mathcal{C}([-\tau, 0], \mathbb{L}^2(\Omega, X)), \quad \tau > 0, \end{cases} \tag{1.1}$$

in a real Hilbert space X with inner product $< .,. >$ and norm $\|.\|$, where $\{\ A(t),\ t \in [0,T]\}$ is a family of linear closed operators from a space X into X that generates an evolution system of operators $\{U(t,s),\ 0 \leq s \leq t \leq T\}$. B^H is a fractional Brownian motion on a real and separable Hilbert space Y, $r,\ \rho\ : [0,+\infty) \to [0,\tau]$ $(\tau > 0)$ are continuous and $f : [0,+\infty) \times X \to X,\ \ \sigma : [0,+\infty) \to \mathcal{L}_2^0(Y,X)$ are appropriate functions. Here $\mathcal{L}_2^0(Y,X)$ denotes the space of all Q-Hilbert-Schmidt operators from Y into X (see section 2 below). The control function $u(.)$ taking values in $\mathbb{L}^2([0,T],U)$ of admissible control functions for a separable Hilbert space U, L is a bounded linear operator from U into X. Here, $I_k \in C(X,X)(k = 1,2,...,m)$ are bounded functions and the fixed times t_k satisfies $0 < t_0 < t_1 < t_2 < ... < t_m < T$; $x(t_k^+)$ and $x(t_k^-)$ denote the right and left limits of $x(t)$ at time t_k. And $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ represents the jump in the state x at time t_k, where I_k determines the size of the jump.

The remainder of this paper is structured as follows. Section 2 recaps some notations, basic concepts, and basic results about fractional Brownian motion, Wiener integral over Hilbert spaces, and preliminary results about evolution operator. Section 3 provides enough conditions to demonstrate the approximate controllability result for the problem (1.1). Section 4 provides an example to demonstrate the effectiveness of the obtained result.

2. Preliminaries

2.1. **Evolution families.** In this subsection we introduce the notion of evolution family.

Definition 2.1. A set $\{U(t,s) : 0 \le s \le t \le T\}$ of bounded linear operators on a Hilbert space X is called an *evolution family* if

(a) $(t,s) \to U(t,s)x$ is strongly continuous for $t > s$,
(b) $U(t,s)U(s,r) = U(t,r)$, $U(s,s) = I$ if $r \le s \le t$.

Let $\{A(t),\, t \in [0,T]\}$ be a family of closed densely defined linear unbounded operators on the Hilbert space X and with domain $D(A(t))$ independent of t, satisfying the following conditions introduced by [1].

There exist constants $\theta \in (\frac{\pi}{2}, \pi)$, $\lambda_0 \ge 0$, K, $l \ge 0$, and ν, $\mu \in (0,1]$ with $\nu + \mu > 1$ such that

$$\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0), \quad \|R(\lambda, A(t) - \lambda_0)\| \le \frac{K}{|\lambda| + 1} \tag{2.1}$$

and

$$\|(A(t) - \lambda_0)R(\lambda, A(t) - \lambda_0)\big[R(\lambda_0, A(t)) - R(\lambda_0, A(s))\big]\| \le l|t-s|^{\mu}|\lambda|^{-\nu}, \tag{2.2}$$

for t, $s \in \mathbb{R}$, $\lambda \in \Sigma_\theta$ where $\Sigma_\theta := \big\{\lambda \in \mathbb{C} - \{0\} : |\arg \lambda| \le \theta\big\}$.

It is well known, that this assumption implies that there exists a unique evolution family $\{U(t,s) : 0 \le s \le t \le T\}$ on X such that $(t,s) \to U(t,s) \in \mathcal{L}(X)$ is continuous for $t > s$, $U(\cdot, s) \in \mathcal{C}^1((s,\infty), \mathcal{L}(X))$, $\partial_t U(t,s) = A(t)U(t,s)$, and

$$\|A(t)^k U(t,s)\| \le C(t-s)^{-k} \tag{2.3}$$

for $0 < t - s \le 1$, $k = 0, 1$, $0 \le \alpha < \mu$, $x \in D((\lambda_0 - A(s))^\alpha)$, and a constant C depending only on the constants in (2.1)-(2.2). Moreover, $\partial_s^+ U(t,s)x = -U(t,s)A(s)x$ for $t > s$ and $x \in D(A(s))$ with $A(s)x \in \overline{D(A(s))}$. We say that $A(\cdot)$ generates $\{U(t,s) : 0 \le s \le t \le T\}$. Note that $U(t,s)$ is exponentially bounded by (2.3) with $k = 0$. For additional details on evolution system and their properties, we refer the reader to [14].

2.2. **Fractional Brownian Motion.** We present various notions, concepts, and lemmas regarding the Wiener integral with respect to an infinite-dimensional fractional Brownian motion. Additionally, we recall some basic results that will be used throughout our analysis.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a complete probability space. A fractional Brownian motion (fBm) $\beta^H(t), t \in \mathbb{R}^+$ is defined with a Hurst parameter $H \in (0,1)$. It is a zero-mean Gaussian process characterized by continuous sample paths, satisfying the following properties:

$$R_H(t,s) = \mathbb{E}[\beta^H(t)\beta^H(s)] = \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big), \qquad s, t \in \mathbb{R}^+. \tag{2.4}$$

Consider two real, separable Hilbert spaces X and Y, and let $\mathcal{L}(Y,X)$ denote the space of bounded linear operators from Y to X. To simplify notation, we will use the same symbols to represent the norms in X, Y, and $\mathcal{L}(Y,X)$.

Let $Q \in \mathcal{L}(Y,Y)$ be an operator defined as $Qe_n = \lambda_n e_n$, where $\lambda_n \ge 0$ for $n = 1, 2, \ldots$, e_n forms a complete orthonormal basis in Y, and the operator Q has a finite trace denoted as $trQ = \sum_{n=1}^{\infty} \lambda_n < \infty$.

We define the infinite dimensional fBm on Y with covariance Q as

$$B^H(t) = B^H_Q(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta^H_n(t),$$

where β^H_n are real, independent one dimensional fBm's. This process is Gaussian, it starts from 0, has zero mean and covariance:

$$E\langle B^H(t), x\rangle \langle B^H(s), y\rangle = R(s,t)\langle Q(x), y\rangle \text{ for all } x, y \in Y \text{ and } t, s \in [0,T].$$

To define Wiener integrals with respect to the Q-fractional Brownian motion (Q-fBm), we introduce the space $\mathcal{L}^0_2 := \mathcal{L}^0_2(Y,X)$ of all Q-Hilbert-Schmidt operators $\psi : Y \to X$. Here, $\psi \in \mathcal{L}(Y,X)$ is referred to as a Q-Hilbert-Schmidt operator if it satisfies the condition: $\|\psi\|^2_{\mathcal{L}^0_2} := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\psi e_n\|^2 < \infty$, where λ_n are non-negative real numbers and e_n forms a complete orthonormal basis in Y. The space $\mathcal{L}^0_2$ is equipped with the inner product $\langle \varphi, \psi\rangle_{\mathcal{L}^0_2} = \sum_{n=1}^{\infty}\langle \varphi e_n, \psi e_n\rangle$, making it a separable Hilbert space. Consider a function $\phi(s)$ with values in $\mathcal{L}^0_2(Y,X)$, defined for $s \in [0,T]$, such that $\sum_{n=1}^{\infty} \|K^*\phi Q^{\frac{1}{2}} e_n\|^2_{\mathcal{L}^0_2} < \infty$. The Wiener integral of ϕ with respect to B^H is then defined by:

$$\int_0^t \phi(s) dB^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}\phi(s) e_n d\beta^H_n(s). \tag{2.5}$$

We conclude this subsection with the following result which is fundamental to proving our result.

Lemma 2.2. [4] *If $\psi : [0,T] \to \mathcal{L}^0_2(Y,X)$ satisfies $\int_0^T \|\psi(s)\|^2_{\mathcal{L}^0_2} ds < \infty$, then the above sum in (2.5) is well-defined as an X-valued random variable and we have*

$$\mathbb{E}\|\int_0^t \psi(s) dB^H(s)\|^2 \leq 2Ht^{2H-1}\int_0^t \|\psi(s)\|^2_{\mathcal{L}^0_2} ds.$$

2.3. **Mild solution.** Now we introduce the concepts of a mild solution of the problem (1.1) and the meaning of approximate controllability of a stochastic functional differential equation.
Henceforth we will assume that the family $\{A(t),\, t \in [0,T]\}$ of linear operators generates an evolution system of operators $\{U(t,s),\, 0 \leq s \leq t \leq T\}$.

In order to define the concept of mild solution of the problem (1.1), we consider the following space: $PC([-\tau,T], \mathbb{L}^2(\Omega, X)) := \{x : [-\tau, T] \to \mathbb{L}^2(\Omega, X)$ such that $x(.)$ is continuous except for a finite number of points t_k at which the left and right limits exist and $x(t_k^+) = x(t_k)\}$. It is easy to verify that PC is a Banach space (see Lemma 2.6 in [17]) with the supremum norm

$$\|x\|_{PC} = \sup_{-\tau \leq t \leq T} (\mathbb{E}\|x(t)\|^2)^{\frac{1}{2}}.$$

Definition 2.3. An X-valued process $\{x(t),\ t \in [-\tau, T]\}$, is called a mild solution of equation (1.1) if

i) $x(.) \in PC([-\tau, T], \mathbb{L}^2(\Omega, X))$,
ii) $x(t) = \varphi(t),\ -\tau \leq t \leq 0$,

iii) For arbitrary $t \in [0,T]$, we have

$$\begin{aligned} x(t) &= U(t,0)\varphi(0) + \int_0^t U(t,s)f(s,x(s-\rho(s)))ds \\ &+ \int_0^t U(t,s)(Lu)(s)ds + \int_0^t U(t,s)\sigma(s)dB^H(s) \\ &+ \sum_{0<t_k<t} U(t,t_k)I_k(x(t_k^-)), \quad \mathbb{P}-a.s. \end{aligned} \tag{2.6}$$

Definition 2.4. The system (1.1) is said to be controllable on the interval $[-\tau,T]$, if for every initial stochastic process $\varphi \in \mathcal{C}([-\tau,0],\mathbb{L}^2(\Omega,X))$, there exists a stochastic control $u \in L^2([0,T],U)$ such that the mild solution $x(.)$ of (1.1) satisfies $x(T) = x_1$, where $x_1 \in \mathbb{L}^2(\Omega,X)$ and T are the preassigned terminal state and time, respectively.

In order to study the approximate controllability for the system (1.1), we introduce the following linear differential system:

$$\begin{cases} dx(t) = A(t)x(t) + Lu(t), & t \in [0,T], \\ x(0) = x_0. \end{cases} \tag{2.7}$$

It is convenient at this point to introduce the resolvent operators associated with (2.7), namely

$$\Gamma_0^T = \int_0^T U(T,s)LL^*U^*(T,s)ds,$$

and

$$R(\lambda,\Gamma_0^T) = (\lambda I + \Gamma_0^T)^{-1},$$

where L^* and U^* denote the adjoint of L and U, respectively.
Let $x(T;\varphi,u)$ be the state value of (1.1) at terminal state T, corresponding to the control u and the initial value φ. Denote by $R(T,\varphi) = \{x(T;\varphi,u) : u \in L^2([0,T],U)\}$ the reachable set of system (1.1) at terminal time T, its closure in X is denoted by $\overline{R(T,\varphi)}$.

Definition 2.5. The system (1.1) is said to be approximately controllable on the interval $[-\tau,T]$ if $\overline{R(T,\varphi)} = L^2(\Omega,X)$.

Lemma 2.6. [16] *The linear control system (2.7) is approximately controllable on* $[0,T]$ *if and only if* $\lambda(\lambda I + \Gamma_0^T)^{-1} \to 0$ *strongly as* $\lambda \to 0^+$.

Lemma 2.7. [16] *For any* $\bar{x}_T \in L^2(\Omega,X)$ *there exists* $\bar{\varphi} \in L^2(\Omega; L^2([0,T]; L_0^2))$ *such that* $\bar{x}_T = E\bar{x}_T + \int_0^T \bar{\varphi}(s)dB^H(s)$.

3. Approximate Controllability Result

We will work under the following assumptions to prove the existence of solutions for a stochastic control system (1.1):

($\mathcal{H}$.1) The evolution family $U(t,s)$ is a compact operator for $t-s>0$; and is exponentially stable, that is, there exist two constants $\beta > 0$ and $M \geq 1$ such that

$$\|U(t,s)\| \leq Me^{-\beta(t-s)}, \qquad \textit{for all} \quad t \geq s,$$

($\mathcal{H}$.2) The maps $f : [0,T] \times X \to X$ is a continuous function and there exist two positive constants C_1 and C_2, such that for all $t \in [0,T]$ and $x, y \in X$:
i) $\|f(t,x) - f(t,y)\| \leq C_1\|x - y\|$,
ii) $\|f(t,x)\|^2 \leq C_2(1 + \|x\|^2)$.

($\mathcal{H}$.3) *i)* The map $\sigma : [0,T] \longrightarrow \mathcal{L}_2^0(Y,X)$ is bounded, that is: there exists a positive constant C_3 such that $\|\sigma(t)\|_{\mathcal{L}_2^0(Y,X)} \leq C_3$ uniformly in $t \in [0,T]$.

ii) Moreover, we assume that the initial data $\varphi = \{\varphi(t) : -\tau \leq t \leq 0\}$ satisfies $\varphi \in \mathcal{C}([-\tau, 0], \mathbb{L}^2(\Omega, X))$.

($\mathcal{H}$.4) The functions $I_k : X \longrightarrow X$, $k = 1, 2, ..., m$ are continuous, and there exist constants $M_k > 0$, $\widetilde{M_k} \geq 0$ such that $\mathbb{E}\|I_k(x)\|^2 \leq \widetilde{M_k}$ and

$$\mathbb{E}\|I_k(x) - I_k(y)\|^2 \leq M_k\|x - y\|^2, \ \ \forall x, y \in X.$$

For any $\lambda > 0$ and $\bar{x}_T \in L^2(\Omega, X)$, we define the control function by

$$\begin{array}{rcl} u^\lambda(t,x) & = & L^*U^*(T,t)(\lambda I + \Gamma_0^T)^{-1}\{E\bar{x}_T - U(T,0)\varphi(0)\} \\ & + & L^*U^*(T,t)\int_0^T(\lambda I + \Gamma_0^T)^{-1}\bar{\varphi}(s)dB^H(s) \\ & - & L^*U^*(T,t)\int_0^T(\lambda I + \Gamma_0^T)^{-1}U(T,s)f(s,x_s)ds \\ & - & L^*U^*(T,t)\int_0^T(\lambda I + \Gamma_0^T)^{-1}U(T,s)\sigma(s)dB^H(s) \\ & - & L^*U^*(T,t)\sum_{0<t_k<t}(\lambda I + \Gamma_0^T)^{-1}U(t,t_k)I_k(x(t_k^-)),\ t \in [0,T]. \end{array} \tag{3.1}$$

Our first result in this paper is based on the following fixed-point theorem.

Theorem 3.1 (Schaefer's fixed point theorem)**.** *Let $\mathcal{V}$ be a Banach space, and $\Psi : \mathcal{V} \longrightarrow \mathcal{V}$ be a completely continuous operator. Then, either*
- *Ψ has a fixed point or*
- *the set $\Theta = \{x \in \mathcal{V} : x = \beta\Psi(x), 0 < \beta < 1\}$ is unbounded.*

Theorem 3.2. *Suppose that ($\mathcal{H}$.1) – ($\mathcal{H}$.4) hold. Then, the control system 1.1 has a mild solution.*

Proof Consider the set

$$S_T = \{x \in PC : x(s) = \varphi(s),\ \text{for}\ \ s \in [-\tau, 0]\}.$$

S_T is a closed subset of PC provided with the norm $\|.\|_{PC}$.

Substituting the control $u^\lambda(.)$ into the stochastic control system (2.6) yields a non-linear operator Π^λ on S_T given by

$$\begin{array}{rl} \Pi^\lambda(x)(t) & := U(t,0)\varphi(0) + \int_0^t U(t,s)f(s,x(s-\rho(s)))ds \\ & + \int_0^t U(t,s)Lu^\lambda(s,x_s)ds + \int_0^t U(t,s)\sigma(s)dB^H(s) \\ & + \sum_{0<t_k<t} U(t,t_k)I_k(x(t_k^-)), \quad \text{if}\ \ t \in [0,T]. \end{array} \tag{3.2}$$

We will show that Π^λ has a fixed point. This fixed point is a mild solution of (1.1). To this end, we will utilize Schaefer's fixed point theorem on the operator Π^λ. We aim to prove that the operator Π^λ is completely continuous and the set $\{x \in S_T : x = \lambda\Pi(x), 0 < \lambda < 1)\}$ is bounded.

For better readability, we break the proof into sequence of steps.
Step 1. Π^λ is completely continuous.
Claim 1. Π^λ is continuous.
Let x^n be a sequence such that $x^n \longrightarrow x$ in S_T. By $(\mathcal{H}.1), (\mathcal{H}.2), (\mathcal{H}.4)$, and the Hölder inequality, we get for $t \in [0,T]$

$$\begin{aligned}
\mathbb{E}\|(\Pi^\lambda x^n)(t) - (\Pi^\lambda x)(t)\|^2 &\leq 3\mathbb{E}\|\int_0^t U(t,s)L[u^\lambda(s,x^n) - u^\lambda(s,x)]ds\|^2 \\
&\quad +3\mathbb{E}\|\int_0^t U(t,s)[f(s,x_s^n) - f(s,x_s)]ds\|^2 \\
&\quad +3m\sum_{k=1}^m \mathbb{E}\|U(t,t_k)(I_k(x^n(t_k^-)) - I_k(x(t_k^-)))\|^2 \\
&\leq 3M^2M_L^2T\int_0^T \mathbb{E}\|u^\lambda(s,x^n) - u^\lambda(s,x)\|^2 ds \\
&\quad +3M^2T\int_0^T \mathbb{E}\|f(s,x_s^n) - f(s,x_s)\|^2 ds \\
&\quad +3mM^2\sum_{k=1}^m \mathbb{E}\|(I_k(x^n(t_k^-)) - I_k(x(t_k^-)))\|^2 \\
&\leq 3M^2M_L^2T\int_0^T \mathbb{E}\|u^\lambda(s,x^n) - u^\lambda(s,x)\|^2 ds \\
&\quad +3M^2C_1^2T\int_0^T \|x_s^n - x_s\|^2 ds \\
&\quad +3mM^2(\sum_{k=1}^m M_k)\sup_{0<t<T}\|x_t^n - x_t\|^2,
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{E}\|u_{x^n}^\lambda(s) - u_x^\lambda(s)\|^2 &\leq 2\frac{M^2M_L^2}{\lambda^2}\{\mathbb{E}\|\int_0^t U(t,s)[f(s,x_s^n) - f(s,x_s)]ds\|^2 \\
&\quad +m\sum_{k=1}^m \mathbb{E}\|U(t,t_k)(I_k(x^n(t_k^-)) - I_k(x(t_k^-)))\|^2\} \\
&\leq 2\frac{M^4M_L^2}{\lambda^2}\{C_1^2T\int_0^T \|x_s^n - x_s\|^2 ds \\
&\quad +m(\sum_{k=1}^m M_k)\sup_{0<t<T}\|x_t^n - x_t\|^2\}. \qquad (3.3)
\end{aligned}$$

Observe that $\mathbb{E}\|(\Pi^\lambda x^n)(t) - (\Pi^\lambda x)(t)\| \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore, Π^λ is continuous.
Claim 2. Now, we prove that Π^λ maps S_k bounded sets into equicontinuous sets, where

$$S_k = \{x \in S_T : \|x\|^2 \leq k\}, \qquad \text{for some } k \geq 0.$$

Note that $S_k \subseteq S_T$ is a bounded closed convex set.

Let $0 < t < T$ and $|h|$ be sufficiently small. Then for any fixed $x \in S_k$, we have

$$\begin{aligned}
\mathbb{E}\|\Pi^\lambda(x)(t &+ h) - \Pi^\lambda(x)(t)\|^2 \\
&\leq 5\mathbb{E}\|(U(h+t,0) - U(t,0))(\varphi(0))\|^2 \\
&+ 5\mathbb{E}\|\int_0^{t+h} U(t+h,s)f(s,x(s-\rho(s)))ds - \int_0^t U(t,s)f(s,x(s-\rho(s)))ds\|^2 \\
&+ 5\mathbb{E}\|\int_0^{t+h} U(t+h,s)\sigma(s)dB^H(s) - \int_0^t U(t,s)\sigma(s)dB^H(s)\|^2 \\
&+ 5\mathbb{E}\|\int_0^{t+h} U(t+h,s)Lu^\lambda(s,x_s)ds - \int_0^t U(t,s)Lu^\lambda(s,x_s)ds\|^2 \\
&+ 5\mathbb{E}\|\sum_{0<t_k<t+h} U(t+h,t_k)I_k(x(t_k^-)) - \sum_{0<t_k<t} U(t,t_k)I_k(x(t_k^-))\|^2 \\
&= 5\sum_{0\leq i\leq 5} \mathbb{E}\|I_i(t+h) - I_i(t)\|^2.
\end{aligned}$$

According to Definition 2.1, we get that

$$\lim_{h\longrightarrow 0}(U(t+h,0) - U(t,0))(\varphi(0)) = 0.$$

By $(\mathcal{H}.1)$, it holds that

$$\|(U(t+h,0) - U(t,0))(\varphi(0)))\| \leq Me^{-\beta t}(e^{-\beta h}+1)\|\varphi(0)\| \in L^2(\Omega).$$

Then, we conclude by the Lebesgue dominated theorem that

$$\lim_{h\longrightarrow 0}\mathbb{E}\|I_0(t+h) - I_0(t)\|^2 = 0.$$

For the term $I_1(h)$, we suppose $h > 0$ (similar calculus for $h < 0$). We have

$$\begin{aligned}
\|I_1(t+h) - I_1(t)\| &\leq \left\|\int_0^t (U(t+h,s) - U(t,s))f(s,x(s-\rho(s)))ds\right\| \\
&\quad + \left\|\int_t^{t+h} (U(t,s)f(s,x(s-\rho(s)))ds\right\| \\
&\leq I_{11}(h) + I_{12}(h).
\end{aligned}$$

By Hölder's inequality, we have

$$\mathbb{E}\|I_{11}(h)\| \leq t\mathbb{E}\int_0^t \|U(t+h,s) - U(t,s))f(s,x(s-\rho(s)))\|^2 ds.$$

Again exploiting properties of Definition 2.1, we obtain

$$\lim_{h\longrightarrow 0}(U(t+h,s) - U(t,s))f(s,x(s-\rho(s))) = 0,$$

and

$$\|U(t+h,s) - U(t,s))f(s,x(s-\rho(s)))\| \leq Me^{-\beta(t-s)}(e^{-\beta h}+1)\|f(s,x(s-\rho(s))\| \in \mathbb{L}^1([0,T],\, ds).$$

Then we conclude by the Lebesgue dominated theorem that

$$\lim_{h\longrightarrow 0}\mathbb{E}\|I_{11}(h)\|^2 = 0.$$

On the other hand, by $(\mathcal{H}.1)$, $(\mathcal{H}.2)$, and the Hölder's inequality, we have

$$\mathbb{E}\|I_{12}(h)\| \leq \frac{M^2C_2(1-e^{-2\beta h})}{2\beta}\int_t^{t+h}(1+\mathbb{E}\|x(s-\rho(s))\|^2)ds.$$

Thus

$$\lim_{h\to 0} I_{12}(h) = 0.$$

Now, for the term $I_2(h)$, we have

$$\begin{aligned}\|I_2(t+h)-I_2(t)\| &\leq \|\int_0^t(U(t+h,s)-U(t,s)\sigma(s)dB^H(s)\| \\ &+ \|\int_t^{t+h}U(t+h,s)\sigma(s)dB^H(s)\| \\ &\leq I_{21}(h)+I_{22}(h).\end{aligned}$$

By Lemma 2.2, we get that

$$E\|I_{21}(h)\|^2 \leq 2Ht^{2H-1}\int_0^t\|[U(t+h,s)-U(t,s)]\sigma(s)\|^2_{\mathcal{L}_2^0}ds.$$

Since

$$\lim_{h\to 0}\|[U(t+h,s)-U(t,s)]\sigma(s)\|^2_{\mathcal{L}_2^0} = 0$$

and

$$\|(U(t+h,s)-U(t,s)\sigma(s)\|_{\mathcal{L}_2^0} \leq MC_3e^{-\beta(t-s)}e^{-\beta h+1} \in \mathbb{L}^1([0,T],\, ds),$$

we conclude, by the dominated convergence theorem that,

$$\lim_{h\to 0}\mathbb{E}\|I_{21}(h)\|^2 = 0.$$

Again by Lemma 2.2, we get that

$$\mathbb{E}\|I_{22}(h)\|^2 \leq \frac{2Ht^{2H-1}C_3M^2(1-e^{-2\beta h})}{2\beta}.$$

Thus,

$$\lim_{h\to 0}\mathbb{E}|I_{22}(h)|^2 = 0.$$

For the term I_3, we have

$$\begin{aligned}\mathbb{E}\|I_3(h)\|^2 &\leq 2\mathbb{E}\|\int_t^{t+h}U(t+h,\nu)Lu^\lambda(s,x_s)ds)\|^2 \\ &\quad +2\mathbb{E}\|\int_0^t(U(t+h,\nu)-U(t,\nu))Lu^\lambda(s,x_s)ds)\|^2 \\ &\leq 2[\mathbb{E}\|I_{3,1}(h)\|^2+\mathbb{E}\|I_{3,2}(h)\|^2].\end{aligned}$$

Let's first deal with $I_{3,1}(h)$. It follows from the conditions $(\mathcal{H}.1)-(\mathcal{H}.3)$ that

$$\begin{aligned}\mathbb{E}\|I_{3,1}(h)\|^2 &\leq 5\frac{M^4M_L^4}{\lambda^2}\int_t^{t+h}\{\|E\bar{x}_T\|^2+M^2\mathbb{E}\|\varphi(0)\|^2+2HT^{2H-1}\int_0^T E\|\bar{\varphi}(s)\|^2_{L_0^2}ds\\ &+ M^2TC_2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)+2M^2HT^{2H-1}\int_0^T\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds+mM^2\sum_{k=1}^m\widetilde{M_k}\}d\nu.\end{aligned}$$

which shows that

$$\lim_{h\to 0}\mathbb{E}||I_{3,1}(h)||^2=0.$$

In a similar way, we have

$$\begin{aligned}&\mathbb{E}\|I_{3,2}(h)\|^2\\ \leq\ & 5\frac{M^2M_L^4}{\lambda^2}\int_0^t\|U(t+h,\nu)-U(t,\nu)\|^2\{\|E\bar{x}_T\|^2+M^2\mathbb{E}\|\varphi(0)\|^2+2HT^{2H-1}\int_0^T E\|\bar{\varphi}(s)\|^2_{L_0^2}ds\\ +\ & M^2TC_2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)+2M^2HT^{2H-1}\int_0^T\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds+mM^2\sum_{k=1}^m\widetilde{M_k}\}d\nu.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}&\|U(t+h,\nu)-U(t,\nu)\|^2\{\|E\bar{x}_T\|^2+M^2\mathbb{E}\|\varphi(0)\|^2+2HT^{2H-1}\int_0^T E\|\bar{\varphi}(s)\|^2_{L_0^2}ds\\ +\ & M^2TC_2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)+2M^2HT^{2H-1}\int_0^T\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds+mM^2\sum_{k=1}^m\widetilde{M_k}\}\\ \leq\ & 4M^2\{\|E\bar{x}_T\|^2+M^2\mathbb{E}\|\varphi(0)\|^2+2HT^{2H-1}\int_0^T E\|\bar{\varphi}(s)\|^2_{L_0^2}ds\\ +\ & M^2TC_2(1+\sup_{s\in[-\tau,T]}\mathbb{E}\|x(s)\|^2)+2M^2HT^{2H-1}\int_0^T\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds+mM^2\sum_{k=1}^m\widetilde{M_k}\}\\ &\in\mathbb{L}^1([0,T],\,ds),\end{aligned}$$

By dominated convergence theorem, we get that

$$\lim_{h\to 0}\mathbb{E}||I_{3,2}(h)||^2=0.$$

On the other hand, for a sufficiently small positive h, it is easy to see that

$$\begin{aligned}I_4 &\leq \mathbb{E}\|\sum_{0<t_k<t+h}(U(t+h,t_k)-U(t,t_k))I_k(z(t_k^-))\|^2\\ &\leq m\sum_{k=1}^m\sup_{0<t<T}\|(U(t+h,t_k)-U(t,t_k))\|^2\mathbb{E}\|I_k(z(t_k^-))\|^2\\ &\leq m\sum_{k=1}^m\widetilde{M_k}\sup_{0<t<T}\|(U(t+h,t_k)-U(t,t_k))\|^2. \qquad (3.4)\end{aligned}$$

So from the above arguments, we obtain that $\lim_{h\to 0}\mathbb{E}\|\Pi^\lambda(x)(t+h)-\Pi^\lambda(x)(t)\|^2=0$. Hence, we conclude that $\Pi^\lambda(S_k)$ is a equicontinuous set.

Claim 3. $(\Pi^\lambda S_k)(t)$ is a precompact set in X.
Given $x \in S_k$, and let $0 < t \leq T$ be fixed, choose $\delta \in (0, t)$ and define

$$\begin{aligned}
\Pi^{\lambda,\delta}(x)(t) &:= U(t,0)\varphi(0) + \int_0^{t-\delta} U(t,s)f(s,x_s)ds + \int_0^{t-\delta} U(t,s)Lu_x^\lambda(s)ds \\
&+ \int_0^{t-\delta} U(t,s)\sigma(s)dB^H(s) + \sum_{0<t_k<t} U(t,t_k)I_k(x(t_k^-))) \\
&= U(t,t-\delta)U(t-\delta,0)\varphi(0) + U(t,t-\delta)\int_0^{t-\delta} U(t-\delta,s)f(s,x_s)ds \\
&+U(t,t-\delta)\int_0^{t-\delta} U(t-\delta,s)Lu_x^\lambda(s)ds + U(t,t-\delta)\int_0^{t-\delta} U(t-\delta,s)\sigma(s)dB^H(s) \\
&+U(t,t-\delta)\sum_{0<t_k<t} U(t-\delta,t_k)I_k(x(t_k^-))).
\end{aligned}$$

From the compactness of $U(t, t-\delta)$ $(0 < \delta < t)$, we infer that the set $V^\delta(t) = \{(\Pi^{\lambda,\delta}x)(t) : x \in S_k\}$ is relative compact in X for every δ, $0 < \delta < t$. Moreover, for every $x \in S_k$,

$$\begin{aligned}
\mathbb{E}\|\Pi^\lambda(x)(t) - \Pi^{\lambda,\delta}(x)(t)\|^2 &\leq 3\mathbb{E}\|\int_0^t U(t,s)f(s,x_s)ds - \int_0^{t-\delta} U(t,s)f(s,x_s)ds\|^2 \\
&+3\mathbb{E}\|\int_0^t U(t,s)Lu_x^\lambda(s)ds - \int_0^{t-\delta} U(t,s)Lu_x^\lambda(s)ds\|^2 \\
&+3\mathbb{E}\|\int_0^t U(t,s)\sigma(s)dB^H(s) - \int_0^{t-\delta} U(t,s)\sigma(s)dB^H(s)\|^2 \\
&\leq 3\mathbb{E}\|\int_{t-\delta}^t U(t,s)f(s,x_s)ds\|^2 \\
&+3\mathbb{E}\|\int_{t-\delta}^t U(t,s)Lu_x^\lambda(s)ds\|^2 \\
&+3\mathbb{E}\|\int_{t-\delta}^t U(t,s)\sigma(s)dB^H(s)\|^2 \\
&\leq \{\frac{M^2}{\beta^2}(1-e^{\beta t})^2(\|f(s,x_s)\|^2 + \|Lu_x^\lambda(s)\|^2 + 2Ht^{2H-1}C_3^2)\}\delta^2.
\end{aligned} \tag{3.5}$$

It follows that

$$\mathbb{E}\|\Pi^\lambda(x)(t) - \Pi^{\lambda,\delta}(x)(t)\|^2 \to 0, \text{ as } \delta \to 0.$$

Therefore, there are precompact sets arbitrarily close to the set $V(t) = \{(\Pi^\lambda x)(t) : x \in S_k\}$. Hence, the set $V(t)$ is also precompact in X.

By virtue of Arzela-Ascoli theorem, we get that $\Pi^\lambda(S_k)$ is relatively compact. Consequently, we conclude that Π^λ is completely continuous.

Step 2. The set $\Theta = \{x(t) \in S_T : x(t) = \beta\Pi^\lambda(x(t)), 0 < \beta < 1\}$ is bounded. Let

$x(t) \in \Theta$. Observe that:

$$\begin{aligned} x(t) = \beta(\Pi^\lambda x)(t) &= \beta U(t,0)\varphi(0) + \beta\int_0^t U(t,s)f(s,x_s)ds + \beta\int_0^t U(t,s)Lu_x^\lambda(s)ds \\ &\quad +\beta\int_0^t U(t,s)\sigma(s)dB^H(s) + \beta\sum_{0<t_k<t} U(t,t_k)I_k(x(t_k^-))). \end{aligned} \quad (3.6)$$

Then, it follows that

$$\begin{aligned} \mathbb{E}\|x(t)\|^2 &\leq 5\beta\mathbb{E}\|U(t,0)\varphi(0)\|^2 + 5\beta\mathbb{E}\|\int_0^t U(t,s)f(s,x_s)ds\|^2 + 5\beta\mathbb{E}\|\int_0^t U(t,s)Lu_x^\lambda(s)ds\|^2 \\ &\quad +5\beta\mathbb{E}\|\int_0^t U(t,s)\sigma(s)dB^H(s)\|^2 + 5\beta\mathbb{E}\|\sum_{0<t_k<t} U(t,t_k)I_k(x(t_k^-)))\|^2 \\ &\leq 5\beta M^2\mathbb{E}\|\varphi(0)\|^2 + 5\beta M^2\int_0^T \mathbb{E}\|f(s,x_s)\|^2 ds + 5\beta M^2 M_L^2\int_0^T \mathbb{E}\|u_x^\lambda(s)\|^2 ds \\ &\quad +5\beta M^2 2HT^{2H-1}\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2^0} ds + 5\beta m\sum_{k=1}^m \|U(t,t_k)\|^2\mathbb{E}\|I_k(x(t_k^-)))\|^2 \\ &\leq 5\beta M^2\mathbb{E}\|\varphi(0)\|^2 + 5\beta M^2 C_2\int_0^T (1+\mathbb{E}\|x_s\|^2)ds + 5\beta M^2 M_L^2\int_0^T \mathbb{E}\|u^\lambda(s,x)\|^2 ds \\ &\quad +5\beta M^2 2HT^{2H}C_3^2 + 5\beta m M^2\sum_{k=1}^m \widetilde{M}_k \\ &\leq 5\beta M^2\mathbb{E}\|\varphi(0)\|^2 + 5\beta M^2 C_2\{T + 2M^2\mathbb{E}\|\varphi(0)\|^2 + 2\int_0^T \mathbb{E}\|x_s\|^2 ds\} \\ &\quad +5\beta M^2 M_L^2\int_0^T \mathbb{E}\|u^\lambda(s,x)\|^2 ds + 5\beta M^2 2HT^{2H}C_3^2 + 5\beta m M^2\sum_{k=1}^m \widetilde{M}_k, \end{aligned} \quad (3.7)$$

where

$$\begin{aligned}
\mathbb{E}\|u^\lambda(t,x)\|^2 &\leq 6\frac{M^2M_L^2}{\lambda^2}\{\|E\bar{x}_T\|^2 + M^2\mathbb{E}\|\varphi(0)\|^2 + 2HT^{2H-1}\int_0^T E\|\bar{\varphi}(s)\|^2_{L_0^2}ds \\
&+M^2C_2\int_0^T (1+\mathbb{E}\|x_s\|^2)ds + 2M^2HT^{2H-1}\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2^0}ds + mM^2\sum_{k=1}^m \widetilde{M_k}\} \\
&\leq 6\frac{M^2M_L^2}{\lambda^2}\{\|E\bar{x}_T\|^2 + M^2\mathbb{E}\|\varphi(0)\|^2 + 2HT^{2H-1}\int_0^T E\|\bar{\varphi}(s)\|^2_{L_0^2}ds \\
&+M^2C_2\int_0^T (1+\mathbb{E}\|x_s\|^2)ds + 2M^2HT^{2H}C_3^2 + mM^2\sum_{k=1}^m \widetilde{M_k}\} \\
&\leq 6\frac{M^2M_L^2}{\lambda^2}\{\|E\bar{x}_T\|^2 + M^2\mathbb{E}\|\varphi(0)\|^2 + 2HT^{2H-1}\int_0^T E\|\bar{\varphi}(s)\|^2_{L_0^2}ds \\
&+M^2C_2\{T + 2M^2\mathbb{E}\|\varphi(0)\|^2 + 2\int_0^T \mathbb{E}\|x_s\|^2ds\} + 2M^2HT^{2H}C_3^2 \\
&+mM^2\sum_{k=1}^m \widetilde{M_k}\} \qquad (3.8)
\end{aligned}$$

Using (3.8) and (3.7) we obtain

$$\mathbb{E}\|x(t)\|^2 \leq \mu + \nu\int_0^T \mathbb{E}\|x_s\|^2ds, \qquad (3.9)$$

where

$$\begin{aligned}
\mu &= 5\beta M^2\mathbb{E}\|\varphi(0)\|^2 + 5\beta M^2C_2\{T + 2M^2\mathbb{E}\|\varphi(0)\|^2\} \\
&+5\beta M^2M_L^2 6\frac{M^2M_L^2}{\lambda^2}T\{\|E\bar{x}_T\|^2 + M^2\mathbb{E}\|\varphi(0)\|^2 \\
&+2HT^{2H-1}\int_0^T E\|\bar{\varphi}(s)\|^2_{L_0^2}ds + M^2C_2\{T + 2M^2\mathbb{E}\|\varphi(0)\|^2\} \\
&+2M^2HT^{2H}C_3^2 + mM^2\sum_{k=1}^m \widetilde{M_k}\} + 5\beta mM^2\sum_{k=1}^m \widetilde{M_k}. \qquad (3.10)
\end{aligned}$$

and

$$\nu = 8\beta M^2C_2 + 5\beta M^2M_L^2T12\frac{M^2M_L^2}{\lambda^2}M^2C_2. \qquad (3.11)$$

Hence, by using Gronwall's Lemma, we deduce that Θ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that Π^λ has a fixed point. This completes the proof of the theorem.

4. Approximate Controllability Result

In this section, we present our main result on approximate controllability of system (1.1). To do this, we also need the following assumption:

$(\mathcal{H}.4)$ the operator $\lambda(\lambda I + \Gamma_0^T)^{-1}$ tends strongly to zero as $\lambda \to 0^+$.

The second result in this paper is given in the next theorem.

Theorem 4.1. *Assume that* $(\mathcal{H}.1)$- $(\mathcal{H}.5)$ *are satisfied. If the functions* f *is uniformly bounded, then the system (1.1) is approximately controllable on* $[-\tau, T]$.

Proof Let x_λ be a fixed point of Π^λ . By using the stochastic Fubini theorem, it can easily be seen that

$$
\begin{aligned}
x_\lambda(T) &= \bar{x}_T - \mathbb{E}\bar{x} + U(T,0)\varphi(0) + \int_0^T U(T,t)LL^*U^*(T,t)(\lambda I + \Gamma_0^T)^{-1}\{E\bar{x}_T - U(T,0)\varphi(0)\}dt \\
&- \int_0^T \bar{\varphi}(s)dB^H(s) + \int_0^T U(T,t)LL^*U^*(T,t)\int_0^T (\lambda I + \Gamma_0^T)^{-1}\bar{\varphi}(s)dB^H(s)dt \\
&+ \int_0^T U(T,s)f(s,x_{s,\lambda})ds \\
&- \int_0^T U(T,t)LL^*U^*(T,t)\int_0^T (\lambda I + \Gamma_0^T)^{-1}U(T,s)f(s,x_{s,\lambda})dsdt \\
&+ \int_0^T U(T,s)\sigma(s)dB^H(s) \\
&- \int_0^T U(T,t)LL^*U^*(T,t)\int_0^T (\lambda I + \Gamma_0^T)^{-1}U(T,s)\sigma(s)dB^H(s)dt \\
&+ \sum_{k=1}^m \{U(t,t_k)I_k(x_\lambda(t_k^-)) \\
&- \int_0^T U(T,s)LL^*U(T,s)(\lambda I + \Gamma_0^T)^{-1}U(t,t_k)I_k(x_\lambda(t_k^-))ds\} \\
&= \bar{x_T} - (\lambda I + \Gamma_0^T)(\lambda I + \Gamma_0^T)^{-1}\{\mathbb{E}\bar{x} - U(T,0)\varphi(0)\} + \Gamma(\lambda I + \Gamma_0^T)^{-1}\{E\bar{x}_T - U(T,0)\varphi(0)\} \\
&-(\lambda I + \Gamma_0^T)(\lambda I + \Gamma_0^T)^{-1}\int_0^T \bar{\varphi}(s)dB^H(s) + \Gamma\int_0^T (\lambda I + \Gamma_0^T)^{-1}\bar{\varphi}(s)dB^H(s) \\
&+(\lambda I + \Gamma_0^T)(\lambda I + \Gamma_0^T)^{-1}\int_0^T U(T,s)f(s,x_{s,\lambda})ds - \Gamma\int_0^T (\lambda I + \Gamma_0^T)^{-1}U(T,s)f(s,x_{s,\lambda})ds \\
&+(\lambda I + \Gamma_0^T)(\lambda I + \Gamma_0^T)^{-1}\int_0^T \sigma(s)dB^H(s) - \Gamma\int_0^T (\lambda I + \Gamma_0^T)^{-1}\sigma(s)dB^H(s) \\
&+ \sum_{k=1}^m \{(\lambda I + \Gamma_0^T)(\lambda I + \Gamma_0^T)^{-1}U(t,t_k)I_k(x_\lambda(t_k^-)) - \Gamma(\lambda I + \Gamma_0^T)^{-1}U(t,t_k)I_k(x_\lambda(t_k^-))\} \\
&= \bar{x}_T - \lambda(\lambda I + \Gamma_0^T)^{-1}\{\mathbb{E}\bar{x}_T - U(T,0)\varphi(0)\} \\
&-\lambda\int_0^T (\lambda I + \Gamma_0^T)^{-1}\bar{\varphi}(s)dB^H(s) \\
&+\lambda\int_0^T (\lambda I + \Gamma_0^T)^{-1}U(T,s)f(s,x_{s,\lambda})ds \\
&+\lambda\int_0^T (\lambda I + \Gamma_0^T)^{-1}U(T,s)\sigma(s)dB^H(s) \\
&+\lambda\sum_{k=1}^m (\lambda I + \Gamma_0^T)^{-1}U(t,t_k)I_k(x_\lambda(t_k^-)),\ t \in [0,T]. \qquad (4.1)
\end{aligned}
$$

It follows from the assumption on f that there exists $\bar{D} > 0$ such that

$$\|f(s, x_\lambda(s))\|^2 \leq \bar{D} \tag{4.2}$$

for all $(s, \omega) \in [0, T] \times \Omega$. Then, there is a subsequence (which we still denote by $f(s, x_\lambda(s))$) which converges weakly to, say, $f(s)$ in X. The compactness of $U(t, s)$, $0 < s < t$ implies that

$$U(T, s)f(s, x_\lambda(s)) \longrightarrow U(T, s)f(s). \tag{4.3}$$

From the equation (4.1), we get

$$\begin{aligned}
&E\|x_\lambda(T) - \bar{x}_T\|^2 \leq 5E\|\lambda(\lambda I + \Gamma_0^T)^{-1}\{E\bar{x}_T - U(T, 0)\varphi(0)\}\|^2 \\
&+5E\| \int_0^T \lambda(\lambda I + \Gamma_0^T)^{-1}\bar{\varphi}(s)dB^H(s)\|^2 \\
&+5E\| \int_0^T \lambda(\lambda I + \Gamma_0^T)^{-1}U(T, s)f(s, x_{s,\lambda})ds\|^2 \\
&+5E\| \int_0^T \lambda(\lambda I + \Gamma_0^T)^{-1}U(T, s)\sigma(s)dB^H(s)\|^2 \\
&+5m\sum_{k=1}^{m} E\|\lambda(\lambda I + \Gamma_0^T)^{-1}U(t, t_k)I_k(x_\lambda(t_k^-))\|^2, \ t \in [0, T].
\end{aligned}$$

On the other hand, by $(\mathcal{H}.5)$, the operator $\lambda(\lambda I + \Gamma)^{-1} \to 0$ strongly as $\lambda \to 0^+$ for all $0 \leq s \leq T$, and, moreover, $\|\lambda(\lambda I + \Gamma)^{-1}\| \leq 1$. Thus, by the Lebesgue dominated convergence theorem and the compactness of $U(t, s)$, we obtain that $E\|x_\lambda(T) - \bar{x}_T\| \longrightarrow 0$ as $\lambda \to 0^+$. This shows the approximate controllability of (1.1). The proof is completed.

5. An illustrative Example

Semilinear systems have gained popularity in recent years as a result of their successful applications in practical fields such as physics, chemical technology, bioengineering, and electrical networks. We consider the stochastic partial functional differential equation with finite delays τ_1 $(0 \leq \tau_1 \leq \tau < \infty)$ shown below:

$$\begin{cases}
dv(t, \zeta) = [\frac{\partial^2}{\partial^2\zeta}v(t, \zeta) + b(t, \zeta)v(t, \zeta) + f_1(t, v(t - \tau_1, \zeta)) + Lu(t, \xi)]dt \\
\qquad\qquad +\sigma(t)dB^H(t), \ 0 \leq t \leq T, \ 0 \leq \zeta \leq \pi, \\
\Delta v(t_k, \xi) = v(t_k^+, \xi) - v(t_k^-, \xi) = \int_0^{t_k} \alpha_k(t_k^- - s)v(s, \xi)ds, \quad k = 1, 2, ..., m; \\
v(t, 0) = v(t, \pi) = 0, 0 \leq t \leq T, \\
v(t, \zeta) = \varphi(t, \zeta), t \in [-\tau, 0], \ 0 \leq \zeta \leq \pi.
\end{cases} \tag{5.1}$$

where B^H is a fractional Brownian motion, $b(t, \zeta)$ is a continuous function and is uniformly Hölder continuous in t, $f_1 : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous function.
To study this system, we consider the space $X = L^2([0, \pi])$ and the operator $A : D(A) \subset X \longrightarrow X$ given by $Ay = y''$ with

$$D(A) = \{y \in X : y'' \in X, \quad y(0)) = y(\pi) = 0\}.$$

It is well known that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ on X. Furthermore, A has discrete spectrum with eigenvalues $-n^2$, $n\in\mathbb{N}$ and the corresponding normalized eigenfunctions given by

$$e_n := \sqrt{\frac{2}{\pi}}\sin nx,\ n = 1, 2, ...$$

In addition $(e_n)_{n\in\mathbb{N}}$ is a complete orthonormal basis in X and

$$T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} < x, e_n > e_n,$$

for $x \in X$ and $t \geq 0$.
Now, we define an operator $A^{'}(t) : D(A) \subset X \longrightarrow X$ by

$$A^{'}(t)x(\zeta) = Ax(\zeta) + b(t, \zeta)x(\zeta).$$

By assuming that $b(.,.)$ is continuous and that $b(t,\zeta) \leq -\gamma$ $(\gamma > 0)$ for every $t \in \mathbb{R}$, $\zeta \in [0, \pi]$, it follows that the system

$$\begin{cases} v'(t) & = A^{'}(t)v(t), \quad t \geq s, \\ v(s) & = x \in X, \end{cases}$$

has an associated evolution family given by

$$U(t,s)x(\zeta) = \left[T(t-s)\exp(\int_s^t b(\tau,\zeta))d\tau)x\right](\zeta).$$

From this expression, it follows that $U(t,s)$ is a compact linear operator and that for every $s,t \in [0,T]$ with $t > s$

$$\|U(t,s)\| \leq e^{-(\gamma+1)(t-s)}.$$

Thus, $\mathcal{H}_1$ is true.
To rewrite the initial-boundary value problem (5.1) in the abstract form we assume the following:

$i)$ We define an infinite dimensional space

$$U = \{u = \sum_{n=2}^{\infty} u_n e_n(\theta) | \sum_{n=2}^{\infty} u_n^2 < \infty\}$$

with a norm defined by $\|u\| = (\sum_{n=2}^{\infty} u_n^2)^{\frac{1}{2}}$ and a linear continuous mapping L from U to X as follows:

$$Lu = 2u_2 e_1(\theta) + \sum_{n=2}^{\infty} u_n e_n(\theta)$$

$ii)$ The substitution operator $f : [0,T] \times X \longrightarrow X$ defined by $f(t,v)(.) = f_1(t, v(.))$ is continuous and we impose suitable conditions on f_1 to verify assumption $\mathcal{H}_2$.

$iii)$ The functions $\alpha_k : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions and $\widetilde{M_k} = \int_0^{t_k} \alpha_k(s)^2 ds$ where $k = 1, 2, ..., m$ are bounded.

If we put

$$\begin{cases} x(t)(\zeta) = x(t,\zeta),\ t \in [0,T],\ \zeta \in [0,\pi] \\ x(t,\zeta) = \varphi(t,\zeta),\ t \in [-\tau,0],\ \zeta \in [0,\pi], \end{cases} \tag{5.2}$$

then, the problem (5.1) can be written in the abstract form

$$\begin{cases} d[x(t)] = [A'(t)x(t) + f(t, x(t-\rho(t))]dt + Lu(t) + \sigma(t)dB^H(t),\ 0 \le t \le T, \\ \Delta v(t_k,\xi) = v(t_k^+,\xi) - v(t_k^-,\xi) = \int_0^{t_k} \alpha_k(t_k^- - s)v(s,\xi)ds, \quad k = 1,2,...,m; \\ x(t) = \varphi(t),\ -\tau \le t \le 0. \end{cases}$$

On the other hand, because of the compactness of $U(t,s)$ generated by $A'(t)$, the associated linear system of (5.1) is not exactly controllable but it is approximately controllable (see [12]). Therefore, we can conclude that the stochastic control system (5.1) is approximately controllable on $[0,T]$.

Furthermore, if we assume that the initial data $\varphi = \{\varphi(t) : -\tau \le t \le 0\}$ satisfies $\varphi \in \mathcal{C}([-\tau,0], \mathbb{L}^2(\Omega, X))$, thus all the assumptions of Theorem 4.1 are fulfilled. Therefore, we conclude that the system (5.1) is approximate controllable on $[-\tau, T]$.

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