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MULTIVALUED ANISOTROPIC PROBLEM WITH ROBIN BOUNDARY CONDITION INVOLVING DIFFUSE RADON MEASURE DATA AND VARIABLE EXPONENTS

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Abstract. In this paper we study a nonlinear anisotropic elliptic problem under Robin type boundary condition governed by a general anisotropic operator with variable exponents and diffuse Radon measure data which does not charge the sets of zero $p(.)$-capacity. We prove an existence and uniqueness result of entropy or renormalized solution.

1. Introduction and preliminaries

Let Ω be an open bounded domain of $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$. In the last years, the study of mathematical problems in the anisotropic variable exponent Sobolev space $W^{1,\overrightarrow{p}(.)}(\Omega)$ has received considerable attention of many researchers due to the fact that they arise in many applications as the reaction-diffusion systems, modelling of propagation of epidemic disease (see [2]), elastic mechanics, electrorheological fluids or image restoration [1, 12, 14, 28].
The goal of this paper is to establish the existence and uniqueness of entropy solution for the following nonlinear multivalued elliptic anisotropic problem.

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, \frac{\partial u}{\partial x_i}) + \beta(u) \ \ni \ \mu \ \text{ in } \ \Omega \\ \\ \sum_{i=1}^{N} a_i(x, \frac{\partial u}{\partial x_i}).\eta_i = |u|^{r(x)-2}u \qquad \text{on } \ \partial\Omega, \end{cases} \tag{1.1}$$

where β is a maximal monotone graph on $\mathbb{R}$ such that $0 \in \beta(0)$, μ a bounded Radon diffuse measure, $|\mu|(\Omega)$(the total variation of μ) a bounded positive measure on Ω and $\overrightarrow{\eta} = (\eta_1, ..., \eta_N)$ the outward unit normal to $\partial\Omega$.
We set $\overline{dom(\beta)} = [m, \ M]$ with $m \leq 0 \leq M$ and denote by

$$p_M(x) := \max(p_1(x), ..., p_N(x)) \ \text{ and } \ p_m(x) := \min(p_1(x), ..., p_N(x)).$$

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In the classical Lebesgue and Sobolev spaces with constant exponent, many authors have studied problems with a maximal monotone graph and measure data (see [4, 5, 6, 13, 15, 20]). These problems have been extended to the Sobolev spaces with variable exponent in the context of isotropic operators (see [25, 26, 27]. In this paper, we extend the study of problems with maximal monotone graph and measure data to the Sobolev spaces with variable exponent in the context of anisotropic operators.
This paper is focused on the anisotropic elliptic strongly nonlinear equation with variable exponent in which the $\overrightarrow{p}(.)$-Laplacian is general. Note that there are many previous works treating problems with the same operator, to give some examples, we refer the reader to the papers [7, 9, 18, 21].
In this paper, we rely our ideas on the decomposition theorem of measure done by the authors in [26], and following them we prove the existence and uniqueness of entropy solution of the nonlinear multivalued elliptic anisotropic problem (1.1).
We denote by $\mathcal{L}^N$ the N-dimensional Lebesgue measure of $\mathbb{R}^N$ and by $\mathcal{M}_b(X)$ the space of bounded Radon measures in X, equipped with its standard norm $||.||_{\mathcal{M}_b(X)}$. Given $\mu \in \mathcal{M}_b(X)$, we say that μ is diffuse with respect to the capacity $W^{1,p(.)}(X)$ ($p(.)$-capacity for short) if $\mu(A) = 0$, for every set A such that $Cap_{p(.)}(A, X) = 0$, where the Sobolev $p(.)$-capacity of A with respect to X is defined by

$$Cap_{p(.)}(A, X) = \inf_{u \in S_{p(.)}(A)} \int_X \left(|u|^{p(x)} + |\nabla u|^{p(x)} \right) dx,$$

with

$$S_{p(.)}(A) = \{u \in W_0^{1,p(.)}(X) : u \geq 1 \text{ in an open set containing } A \text{ and } u \geq 0 \text{ in } X\}.$$

In the case $S_{p(.)}(A) = \emptyset$, we set $Cap_{p(.)}(A, X) = +\infty$.
The set of bounded Radon diffuse measure in the variable exponent setting is denoted by $\mathcal{M}_b^{p(.)}(X)$.
Note that, since we are dealing with the Robin boundary condition, we cannot work with the common space $W_0^{1,\overrightarrow{p}(.)}(\Omega)$. However, the common space is $W^{1,\overrightarrow{p}(.)}(\Omega)$, so we cannot use directly the argument of decomposition of measure, since the second part of the measure is in $W^{-1,p'_m(.)}(\Omega)$ (the dual of $W_0^{1,p_m(.)}(\Omega)$). To overcome this difficulty, we use the same ideas as authors in [26]. We consider a smooth domain Ω in order to work with the space $W_0^{1,\tilde{p}_m(.)}(U_\Omega)$, where $\tilde{p}_m(.) : U_\Omega \longrightarrow (1, \infty)$ is a continuous function such that $\tilde{p}_m(x) = p_m(x)$ for all $x \in \overline{\Omega}$, and return after to the space $W^{1,p_m(.)}(\Omega)$. More precisely, Ω is assumed to be a bounded domain in $\mathbb{R}^N$ with a boundary $\partial\Omega$ of class C^1. Then, Ω is an extension domain (see [10]), so we can fix an open bounded subset U_Ω of $\mathbb{R}^N$ such that $\overline{\Omega} \subset U_\Omega$, and there exists a bounded linear operator

$$E : W^{1,p_m(.)}(\Omega) \to W_0^{1,\tilde{p}_m(.)}(U_\Omega),$$

for which

i): $E(u) = u$ a.e. in Ω for each $u \in W^{1,p_m(.)}(\Omega)$,

ii): $||E(u)||_{W_0^{1,\tilde{p}_m(.)}(U_\Omega)} \leq C||u||_{W^{1,p_m(.)}(\Omega)}$, where C is a constant depending only on Ω.

We define

$$\mathfrak{M}_b^{p_m(.)}(\Omega) := \{\mu \in \mathcal{M}_b^{\tilde{p}_m(.)}(U_\Omega) : \mu \text{ is concentrated on } \Omega\}.$$

This definition is independent of the open set U_Ω. Note that for $u \in W^{1,p_m(.)}(\Omega) \cap L^\infty(\Omega)$ and $\mu \in \mathfrak{M}_b^{p_m(.)}(\Omega)$, we have

$$\langle \mu, E(u) \rangle = \int_\Omega u d\mu.$$

On the other hand, as μ is diffuse, there exist (see [25, 26]) $f \in L^1(U_\Omega)$ and $F \in (L^{\tilde{p}'_m(.)}(U_\Omega))^N$ such that $\mu = f - div(F)$ in $\mathcal{D}'(U_\Omega)$. Therefore, we can also write

$$\langle \mu, E(u) \rangle = \int_{U_\Omega} fE(u)dx + \int_{U_\Omega} F.\nabla E(u)dx.$$

Before presenting our main result, we first give the following hypotheses.
Let $\overrightarrow{p}(.) = (p_1(.), ..., p_N(.))$ be such that for any $i = 1, ..., N$, $p_i(.) : \overline{\Omega} \longrightarrow \mathbb{R}$ is a continuous function with

$$1 < p_i^- := \inf_{x \in \overline{\Omega}} p_i(x) \leq p_i^+ := \sup_{x \in \overline{\Omega}} p_i(x) < \infty. \tag{1.2}$$

For any $i = 1, ..., N$, let operator $a_i : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function (i.e. $a_i(x, \xi)$ is continuous in ξ for a.e. $x \in \Omega$ and measurable in x for every $\xi \in \mathbb{R}$) satisfying:
• There exists a positive constant C_1 such that

$$|a_i(x, \xi)| \leq C_1 \left(j_i(x) + |\xi|^{p_i(x)-1} \right), \tag{1.3}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}$, where j_i is a non-negative function in $L^{p'_i(.)}(\Omega)$, with $\frac{1}{p_i(x)} + \frac{1}{p'_i(x)} = 1$;
• For $\xi,\ \eta\ \in \mathbb{R}$ with $\xi \neq \eta$ and for every $x \in \Omega$, there exists a positive constant C_2 such that

$$(a_i(x, \xi) - a_i(x, \eta))(\xi - \eta) \geq \begin{cases} C_2|\xi - \eta|^{p_i(x)} & \text{if } |\xi - \eta| \geq 1 \\ \\ C_2|\xi - \eta|^{p_i^-} & \text{if } |\xi - \eta| < 1 \end{cases} \tag{1.4}$$

and,
• There exists a positive constant C_3 such that

$$a_i(x, \xi).\xi \geq C_3|\xi|^{p_i(x)}, \tag{1.5}$$

for $\xi \in \mathbb{R}$ and almost every $x \in \Omega$.
The hypotheses on a_i are classical in the study of nonlinear PDEs (see [3, 7, 18]). Throughout this paper, we assume that

$$\frac{\overline{p}(N-1)}{N(\overline{p}-1)} < p_i^- < \frac{\overline{p}(N-1)}{N-\overline{p}}, \quad \frac{p_i^+ - p_i^- - 1}{p_i^-} < \frac{\overline{p}-N}{\overline{p}(N-1)} \tag{1.6}$$

and

$$\sum_{i=1}^{N} \frac{1}{p_i^-} > 1, \tag{1.7}$$

where $\dfrac{N}{\overline{p}} = \displaystyle\sum_{i=1}^{N} \frac{1}{p_i^-}$, and for all $x \in \partial\Omega$,

$$p^{\partial}(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N \\ \\ +\infty & \text{if } p(x) \geq N. \end{cases}$$

We make the following assumption

$$r \in C(\overline{\Omega}) \text{ with } 1 < r^- \leq r^+ < \min\{p_1^{\partial}(x), ..., p_N^{\partial}(x)\}. \tag{1.8}$$

A prototype example that is covered by our assumption is the following anisotropic $\overrightarrow{p}(.)$-harmonic problem: set

$$a_i(x,\xi) = |\xi|^{p_i(x)-2}\xi, \text{ where } p_i(x) \geq 2 \text{ for any } i = 1, ..., N.$$

Then, we obtain the problem

$$\begin{cases} \beta(u) - \displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial u}{\partial x_i}\right) \ni \mu \text{ in } \Omega \\ \\ \displaystyle\sum_{i=1}^{N} \left|\frac{\partial}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial u}{\partial x_i}\eta_i = -|u|^{r(x)-2}u \quad \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

which, in the particular case when $p_i = p$ for any $i = 1, ..., N$, is the p-Laplace problem.

For any $l_0 > 0$, we consider a function h_0 such that.

(i) $h_0 \in C_c^1(\mathbb{R})$, $h_0(r) \geq 0$, for all $r \in \mathbb{R}$,
(ii) $h_0(r) = 1$ if $|r| \leq l_0$ and $h_0(r) = 0$ if $|r| \geq l_0 + 1$.

If γ is a maximal monotone operator defined on $\mathbb{R}$, we denote by γ_0 the main section of γ; i.e.,

$$\gamma_0(s) = \begin{cases} \text{minimal absolute value of } \gamma(s) & \text{if } \gamma(s) \neq \emptyset \\ +\infty & \text{if } [s, +\infty) \cap D(\gamma) = \emptyset \\ -\infty & \text{if } (-\infty, s] \cap D(\gamma) = \emptyset. \end{cases}$$

We give a useful convergence result (see [25]).

Lemma 1.1. *Let $(\beta_n)_{n\geq 1}$ be a sequence of maximal monotone graphs such that $\beta_n \to \beta$ in the sense of the graph (for $(x,y) \in \beta$, there exists $(x_n, y_n) \in \beta_n$ such that $x_n \to x$ and $y_n \to y$). We consider two sequences $(z_n)_{n\geq 1} \subset L^1(\Omega)$ and $(w_n)_{n\geq 1} \subset L^1(\Omega)$.*
We suppose that: $\forall n \geq 1, w_n \in \beta_n(z_n)$, $(w_n)_{n\geq 1}$ is bounded in $L^1(\Omega)$ and $z_n \to z$ in $L^1(\Omega)$. Then,

$$z \in dom(\beta).$$

The rest of the paper is organized as follows. In Section 2, we introduce some fundamental preliminary results which are useful in this work. Then, we study the existence and uniqueness of entropy or renormalized solution in Section 3. We recall in this section some definitions and basic properties of anisotropic Lebesgue and Sobolev spaces with variable exponents.

Set

$$C_+(\overline{\Omega}) = \left\{ p \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} p(x) > 1 \right\}.$$

For any $p \in C_+(\overline{\Omega})$, the variable exponent Lebesgue space is defined by

$$L^{p(.)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ a measurable real valued function such that } \int_\Omega |u|^{p(x)} dx < \infty \right\},$$

endowed with the so-called Luxemburg norm

$$|u|_{p(.)} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}.$$

The $p(.)$-modular of the space $L^{p(.)}(\Omega)$ is the mapping $\rho_{p(.)} : L^{p(.)}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)} dx.$$

For any $u \in L^{p(.)}(\Omega)$, the following inequality (see [16], [17]) will be used later.

$$\min \left\{ |u|^{p^-}_{p(.)};\ |u|^{p^+}_{p(.)} \right\} \le \rho_{p(.)}(u) \le \max \left\{ |u|^{p^-}_{p(.)};\ |u|^{p^+}_{p(.)} \right\}. \tag{1.10}$$

For any $u \in L^{p(.)}(\Omega)$ and $v \in L^{q(.)}(\Omega)$, with $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$ for any $x \in \Omega$, we have the Holder type inequality

$$\left| \int_\Omega uv dx \right| \le \left(\frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(.)} |v|_{q(.)}. \tag{1.11}$$

If Ω is bounded and $p,\ q \in C_+(\overline{\Omega})$ such that $p(x) \le q(x)$ for any $x \in \Omega$, then the embedding $L^{p(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$ is continuous (see [23], Theorem 2.8).

Herein, we need the following anisotropic Sobolev space with variable exponent.

$$W^{1,\vec{p}(.)}(\Omega) := \left\{ u \in L^{p_M(.)}(\Omega) :\ \frac{\partial u}{\partial x_i} \in L^{p_i(.)}(\Omega),\ i = 1, ..., N \right\},$$

which is a separable and reflexive Banach space (see [24]) under the norm

$$||u||_{\vec{p}(.)} = |u|_{p_M(.)} + \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|_{p_i(.)}.$$

We need the following embedding and trace results.

Theorem 1.2. *([16], Corollary 2.1). Let $\Omega \subset \mathbb{R}^N (N \geq 3)$ be a bounded open set and for all $i = 1, ..., N, p_i \in L^\infty(\Omega), p_i(x) \geq 1$ a.e. $x \in \Omega$. Then, for any $q \in L^\infty(\Omega)$ with $q(x) \geq 1$ a.e. $x \in \Omega$ such that*

$$ess \inf_{x\in\Omega}(p_M(x) - q(x)) > 0,$$

we have the compact embedding

$$W^{1,\overrightarrow{p}(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega). \tag{1.12}$$

Theorem 1.3. *([9] , Theorem 6). Let $\Omega \subset \mathbb{R}^N (N \geq 3)$ be a bounded open set with smooth boundary and let $\overrightarrow{p}(.) \in (C_+(\overline{\Omega}))^N, r \in C(\overline{\Omega})$ satisfy the condition*

$$1 \leq r(x) < \min\{p_1^\partial(x), ..., p_N^\partial(x)\}, \quad \forall x \in \partial\Omega. \tag{1.13}$$

Then, there exists a compact boundary trace embedding

$$W^{1,\overrightarrow{p}(.)}(\Omega) \hookrightarrow L^{r(.)}(\partial\Omega).$$

In particular,

$$W^{1,\overrightarrow{p}(.)}(\Omega) \hookrightarrow L^1(\partial\Omega).$$

We introduce now the numbers

$$q = \frac{N(\overline{p}-1)}{N-1} \text{ and } q^* = \frac{N(\overline{p}-1)}{N-\overline{p}} = \frac{Nq}{N-q}.$$

The following result is due to Troisi (see [30]).

Theorem 1.4. *Let $p_1, ..., p_N \in [1, \infty)$; $g \in W^{1,(p_1,...,p_N)}(\Omega)$ and*

$$\begin{cases} q = (\overline{p})^* & \text{if } (\overline{p})^* < N, \\ q \in [1, \infty) & \text{if } (\overline{p})^* \geq N. \end{cases}$$

Then, there exists a constant $C_4 > 0$ depending on $N, p_1, ..., p_N$ if $\overline{p} < N$ and also on q and $meas(\Omega)$ if $\overline{p} \geq N$ such that

$$||g||_{L^q(\Omega)} \leq C_4 \prod_{i=1}^{N} \left[||g||_{L^{p_M}(\Omega)} + \left\| \frac{\partial g}{\partial x_i} \right\|_{L^{p_i}(\Omega)} \right]^{\frac{1}{N}}. \tag{1.14}$$

In this paper, we will use the Marcinkiewicz space $\mathcal{M}^q(\Omega)(1 < q < +\infty)$ as the set of measurable function $g : \Omega \longrightarrow \mathbb{R}$ for which the distribution

$$\lambda_g(k) := meas(\{x \in \Omega : |g(x)| > k\}), \ k \geq 0 \tag{1.15}$$

satisfies an estimate of the form

$$\lambda_g(k) \leq Ck^{-q}, \quad \text{for some finite constant } C > 0. \tag{1.16}$$

We will use the following pseudo norm in $\mathcal{M}^q(\Omega)$.

$$||g||_{\mathcal{M}^q(\Omega)} := \inf\{C > 0 : \lambda_g(k) \leq Ck^{-q}, \ \forall k > 0\}. \tag{1.17}$$

Finally, we will use through the paper, the truncation function T_k, $(k > 0)$ defined by

$$T_k(s) = \max\{-k, \min\{k; s\}\}. \tag{1.18}$$

It is clear that $\lim_{k\to+\infty} T_k(s) = s$ and $|T_k(s)| = \min\{|s|; k\}$.
For any $v \in W^{1,\vec{p}(.)}(\Omega)$, we use v instead of $v|_{\partial\Omega}$ for the trace of v on $\partial\Omega$.
Set $\mathcal{T}^{1,\vec{p}(.)}(\Omega)$ as the set of the measurable functions $u : \Omega \longrightarrow \mathbb{R}$ such that for any $k > 0$, $T_k(u) \in W^{1,\vec{p}(.)}(\Omega)$. We define the space $\mathcal{T}^{1,\vec{p}(.)}_{tr}(\Omega)$ as the set of functions $u \in \mathcal{T}^{1,\vec{p}(.)}(\Omega)$ such that there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,\vec{p}(.)}(\Omega)$ satisfying

i): $u_n \longrightarrow u$ a.e. in Ω as $n \to +\infty$,
ii): $\frac{\partial T_k(u_n)}{\partial x_i} \longrightarrow \frac{\partial T_k(u)}{\partial x_i}$ in $L^1(\Omega)$, for all $k > 0$ as $n \to +\infty$,
iii): there exists a measurable function v on $\partial\Omega$ such that $u_n \longrightarrow v$ a.e. on $\partial\Omega$ as $n \to +\infty$.

We need the following lemma proved in [7].

Lemma 1.5. *Let g be a nonnegative function in $W^{1,\vec{p}(.)}(\Omega)$. Assume $\overline{p} < N$ and there exists a constant $C > 0$ such that*

$$\int_\Omega |T_k(g)|^{p^-_M} dx + \sum_{i=1}^N \int_{\{|g|\le k\}} \left|\frac{\partial g}{\partial x_i}\right|^{p_i^-} dx \le C(k+1), \tag{1.19}$$

for every $k > 0$.
Then, there exists a constant D, depending on C such that

$$||g||_{\mathcal{M}^{q^*}(\Omega)} \le D,$$

where $q^* = \dfrac{N(\overline{p}-1)}{N-\overline{p}}$.

2. Main results

We first define the concept of solution to the problem (1.1).

Definition 2.1. For $\mu \in \mathfrak{M}_b^{p_m(.)}(\Omega)$, a renormalized solution of the problem (1.1) is a triplet $(u,\ w,\ \nu) \in \mathcal{T}^{1,\vec{p}(.)}_{tr}(\Omega) \times L^1(\Omega) \times \mathcal{M}_b^{p_m(.)}(\Omega)$ such that

(i) $u \in dom(\beta)\mathcal{L}^N$- a.e. in Ω, $|u|^{r(x)-2}u \in L^1(\Omega)$;
(ii) $w \in \beta(u)\mathcal{L}^N$- a.e. in Ω;
(iii) $\nu \perp \mathcal{L}^N$, ν^+ is concentrated on $[u = M]$, ν^- is concentrated on $[u = m]$;
(iv) for any $\xi \in W^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$,

$$\sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial \xi}{\partial x_i} dx + \int_\Omega w\xi dx + \int_\Omega \xi d\nu + \int_{\partial\Omega} |u|^{r(x)-2}u\xi d\sigma = \int_\Omega \xi d\mu. \tag{2.1}$$

Moreover,

$$\lim_{k\to+\infty} \int_{[k\le|u|\le k+1]} \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx = 0. \tag{2.2}$$

Theorem 2.2. *Assume that (1.2)-(1.8) hold true; for any $\mu \in \mathfrak{M}_b^{p_m(.)}(\Omega)$, the Robin boundary problem (1.1) admit at least one renormalized solution (u, w, ν).*

The connexion between our notion of solution and the entropic formulation is given in the following theorem (see [25]).

Theorem 2.3. *If (u,w) is a solution of (1.1) in the sense of Theorem (2.2), then (u,w) is a solution in the following sense : for any $\xi \in W^{1,\vec{p}(.)}(\Omega)\cap L^\infty(\Omega)$ such that $\xi \in dom\beta$,*

$$\sum_{i=1}^{N}\int_\Omega a_i\left(x,\frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k(u-\xi)dx+\int_\Omega wT_k(u-\xi)dx+\int_{\partial\Omega}|u|^{r(x)-2}uT_k(u-\xi)d\sigma$$
$$\leq \int_\Omega T_k(u-\xi)d\mu, \tag{2.3}$$

for any $k>0$.

The uniqueness of solution is given in the following theorem (see [27]).

Theorem 2.4. *Let (u_1,w_1,ν_1) and (u_2,w_2,ν_2) be two solutions of (1.1). Then,*

$$\begin{cases} u_1 = u_2 \ a.e. \ \ in\ \Omega,\\ u_1 = u_2 \ a.e. \ \ on\ \partial\Omega,\\ w_1 = w_2 \ a.e. \ \ in\ \Omega,\\ \nu_1 = \nu_2. \end{cases}$$

Proof of Theorem 2.2

For any $n\in\mathbb{N}^*$, we consider the Yosida regularization β_n of β given by

$$\beta_n = n(I-(I+\frac{1}{n}\beta)^{-1}).$$

Thanks to [11], there exists a non negative, convex and l.s.c. function j defined on $\mathbb{R}$ such that

$$\beta = \partial j.$$

To regularise β, we consider

$$j_n(s)=\min_{r\in\mathbb{R}}\left\{\frac{n}{2}|s-r|^2+j(r)\right\},\ \forall\ s\in\mathbb{R},\ \forall n>0.$$

By Proposition 2.11 in [11] we have

$$\begin{cases} dom(\beta)\subset dom(j)\subset\overline{dom(j)}=\overline{dom(\beta)},\\ j_n(s)=\dfrac{1}{2n}|\beta_n(s)|^2+j(J_n) \text{ where } J_n:=(I+\frac{1}{n}\beta)^{-1},\\ j_n \text{ is a convex, Frechet-differentiable function and } \beta_n=\partial j_n,\\ j_n\uparrow j \text{ as } n\downarrow\infty. \end{cases}$$

Moreover, for any $n>0$, β_n is a nondecreasing and Lipschitz-continuous function. Since $\mu\in\mathcal{M}_b^{\tilde{p}_m(.)}(U_\Omega)$, recall that $\mu=f-div(F)$ in $\mathcal{D}'(U_\Omega)$ with $f\in L^1(U_\Omega)$ and $F\in(L^{\tilde{p}'_m(.)}(U_\Omega))^N$ where U_Ω is the open subset of $\mathbb{R}^N$ which extends Ω via the operator E.
To regularize μ, for any $n>0$ and $x\in U_\Omega$, we define the function

$$f_n(x)=T_n(f(x))\chi_\Omega(x).$$

Let $(F_n)_{n>1} \subset C_0^\infty(U_\Omega)$ be a sequence such that $F_n \longrightarrow F$ strongly in $(L^{\tilde{p}'_m(.)}(U_\Omega))^N$. For any $n > 0$, we set $\tilde{F}_n = \chi_\Omega F_n$ and $\mu_n = f_n - div(\tilde{F}_n)$. For any $n > 0$, one has:

- $\mu_n \in \mathfrak{M}_b^{p_m(.)}(\Omega)$, $\mu_n \rightharpoonup \mu$ in $\mathcal{M}_b(U_\Omega)$ and $\mu_n \in L^\infty(\Omega)$.
- $(f_n)_{n>0}$ is a sequence of bounded functions which converges to $f \in L^1(\Omega)$.

We have the following lemma (see [26], Lemma 4.1).

Lemma 2.5. *The Yosida regularization β_n is a surjective operator.*

Now, we consider the following approximating scheme problem.

$$P(\beta_n, \mu_n)\begin{cases} -\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, \frac{\partial u_n}{\partial x_i}) + \beta_n(u_n) + \frac{1}{n}|u_n|^{P_M(x)-2}u_n = \mu_n \ \text{ in } \ \Omega, \\ \sum_{i=1}^N a_i(x, \frac{\partial u_n}{\partial x_i}).\eta_i = -|u_n|^{r(x)-2}u_n \qquad \text{on } \ \partial\Omega, \end{cases} \tag{2.4}$$

for any $n > 0$.

Theorem 2.6. *The problem (2.4) admits at least one weak solution u_n in the sense that $u_n \in W^{1,\vec{p}(.)}(\Omega)$, $\beta_n(u_n) \in L^\infty(\Omega)$ and*

$$\sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial u_n}{\partial x_i}\right)\frac{\partial \xi}{\partial x_i}dx + \int_\Omega \beta_n(u_n)\xi dx + \frac{1}{n}\int_\Omega |u_n|^{P_M(x)-2}u_n\xi dx + \int_{\partial\Omega}|u_n|^{r(x)-2}u_n\xi d\sigma = \int_\Omega \mu_n\xi dx, \tag{2.5}$$

for all $\xi \in W^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$.

Proof. We define the reflexive space

$$E = W^{1,\vec{p}(.)}(\Omega) \times L^{p_M(.)}(\partial\Omega).$$

Let X_0 be the subspace of E defined by

$$X_0 = \{(u, v) \in E : v = \tau(u)\},$$

where $\tau(u)$ is the trace of $u \in \mathcal{T}_{tr}^{1,\overrightarrow{p(.)}}(\Omega)$ in the usual sense, since $u \in W^{1,\vec{p}(.)}(\Omega)$. In the sequel, we will identify an element $(u, v) \in X_0$ with its representative $u \in W^{1,\vec{p}(.)}(\Omega)$.

Since b is a surjective, continuous and nondecreasing function with $b(0) = 0$ and $\Upsilon \in L^\infty(\Omega)$, for any $n > 0$, the following problem

$$P(T_n(b), \Upsilon)\begin{cases} -\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, \frac{\partial u}{\partial x_i}) + T_n(b(u)) + \frac{1}{n}|u|^{P_M(x)-2}u = \Upsilon \ \text{ in } \ \Omega \\ \sum_{i=1}^N a_i(x, \frac{\partial u_n}{\partial x_i}).\eta_i = T_n(-|u|^{r(x)-2}u) \qquad \text{on } \ \partial\Omega \end{cases}$$

admits at least one solution $u_n \in W^{1,\vec{p}(.)}(\Omega)$ such that $\forall\, \xi \in W^{1,\vec{p}(.)}(\Omega)\cap L^\infty(\Omega)$,

$$\sum_{i=1}^{N}\int_\Omega a_i\left(x,\frac{\partial u_n}{\partial x_i}\right)\frac{\partial \xi}{\partial x_i}dx+\int_\Omega T_n(b(u_n))\xi dx+\frac{1}{n}\int_\Omega |u_n|^{P_M(x)-2}u_n\xi dx$$

$$+\int_{\partial\Omega} T_n(|u_n|^{r(x)-2}u_n)\xi d\sigma=\int_\Omega \Upsilon\xi dx. \tag{2.6}$$

Furthermore,

$$\forall n>||\Upsilon||_\infty,\quad |b(u_n)|\le ||\Upsilon||_\infty \text{ a.e. in } \Omega. \tag{2.7}$$

Indeed, We define the operator A_n as follows.

$$\langle A_n u,\xi\rangle = \langle Au,\xi\rangle+\int_\Omega T_n(b(u))\xi dx+\int_{\partial\Omega} T_n(|u|^{p_M(x)-2}u)\xi d\sigma,\ \forall\, u,\xi \in W^{1,\vec{p}(.)}(\Omega),$$

where

$$\langle Au,v\rangle = \int_\Omega \sum_{i=1}^{N} a_i\left(x,\frac{\partial u}{\partial x_i}\right)\frac{\partial \xi}{\partial x_i}dx+\frac{1}{n}\int_\Omega |u|^{P_M(x)-2}u\xi dx.$$

According to [18], the operator A_n is of type M, bounded and coercive from $X_0\subset E$ into its dual $X_0'\subset E'$. Thus, A_n is surjective (see [29], Corollary 2.2). By setting $F=\displaystyle\int_\Omega \xi\Upsilon dx$, we have $F\in E'\subset X_0'$. Then, we can deduce the existence of a function $u_n\in X_0$ such that

$$\langle A_n(u_n),\xi\rangle = \langle F,\xi\rangle,\quad \text{for all } \xi\in X_0.$$

Namely, u_n is a weak solution of problem $P(T_n(b),\Upsilon)$.
Let us fix $n>||\Upsilon||_\infty+1$; we get the existence of a solution of the problem $P(b,\Upsilon)$. The proof of (2.7) is detailed in [8]. Then, by setting $b=\beta_n$ and $\mu_n=\Upsilon$, we get the proof of the Theorem 2.6. □

We need the following a priori estimates.

Proposition 2.7. *Let u_n be a weak solution of (2.4). Then, there exists a positive constant $C(\mu,\Omega)$ such that for any $k>0$,*

$$\sum_{i=1}^{N}\int_\Omega \left|\frac{\partial}{\partial x_i}T_k(u_n)\right|^{p_i(x)}dx\le kC(\mu,\Omega), \tag{2.8}$$

$$\int_\Omega \beta_n(u_n)T_k(u_n)dx\le kC(\mu,\Omega), \tag{2.9}$$

$$\frac{1}{n}\int_\Omega |u_n|^{P_M(x)-2}u_nT_k(u_n)\le kC(\mu,\Omega), \tag{2.10}$$

$$\||u_n|^{r(x)-2}u_n\|_{L^1(\partial\Omega)}=\||u_n|^{r(x)-1}\|_{L^1(\partial\Omega)}\le C(\mu,\Omega) \tag{2.11}$$

and

$$\|\beta_n(u_n)\|_1\le C(\mu,\Omega). \tag{2.12}$$

Proof. We start by the proof of (2.8). By Taking $\xi = T_k(u_n)$ as a test function in (2.5), we get

$$\sum_{i=1}^{N}\int_\Omega a_i\left(x,\frac{\partial u_n}{\partial x_i}\right)\frac{\partial T_k(u_n)}{\partial x_i}dx+\int_\Omega \beta_n(u_n)T_k(u_n)dx+\frac{1}{n}\int_\Omega |u_n|^{P_M(x)-2}u_nT_k(u_n)dx$$
$$+\int_{\partial\Omega}|u_n|^{r(x)-2}u_nT_k(u_n)d\sigma=\int_\Omega T_k(u_n)d\mu_n. \tag{2.13}$$

On the other hand, we have

$$\int_\Omega T_k(u_n)d\mu_n\le\left|\int_\Omega T_k(u_n)d\mu_n\right|\le kC(\mu,\Omega). \tag{2.14}$$

Then, we use (1.5) and (2.14) in (2.13) to obtain

$$\sum_{i=1}^{N}\int_{\{|u_n|\le k\}}\left|\frac{\partial}{\partial x_i}T_k(u_n)\right|^{p_i(x)}dx+\int_\Omega \beta_n(u_n)T_k(u_n)dx+\int_{\partial\Omega}|u_n|^{r(x)-2}u_nT_k(u_n)d\sigma$$
$$+\frac{1}{n}\int_\Omega |u_n|^{P_M(x)-2}u_nT_k(u_n)dx\le kC(\mu,\Omega). \tag{2.15}$$

Since T_k, $\beta_n, s\mapsto |s|^{r(.)-2}s$ are nondecreasing and $\beta_n(0)=T_k(0)=0$, all the integrals in (2.15) are nonnegative. Therefore, we deduce from (2.15) that

$$\sum_{i=1}^{N}\int_\Omega\left|\frac{\partial}{\partial x_i}T_k(u_n)\right|^{p_i(x)}dx\le kC(\mu,\Omega),$$
$$\int_\Omega \beta_n(u_n)T_k(u_n)dx\le kC(\mu,\Omega),$$
$$\frac{1}{n}\int_\Omega |u_n|^{P_M(x)-2}u_nT_k(u_n)\le kC(\mu,\Omega),$$
$$\int_{\partial\Omega}|u_n|^{r(x)-2}u_nT_k(u_n)d\sigma\le kC(\mu,\Omega). \tag{2.16}$$

Dividing (2.16) by k, passing to the limit as $k\to 0$ and using the Lebesgue dominated convergence Theorem, we obtain (2.11).
We end by proving (2.12). We deduce from (2.9) that

$$\int_\Omega \beta_n(u_n)\frac{1}{k}T_k(u_n)dx\le C(\mu,\Omega).$$

Then, passing to limit as $k\longrightarrow+\infty$, we get

$$\int_\Omega \beta_n(u_n)sign_0(u_n)dx\le C(\mu,\Omega)$$

Hence,

$$\int_\Omega |\beta_n(u_n)|dx\le C(\mu,\Omega).$$

□

Proposition 2.8. *Let u_n be a weak solution of (2.4). Then, for all $k>0$,*

$$\int_\Omega |T_k(u_n)|^{p_M^-} d\sigma + \sum_{i=1}^N \int_{\{|u_n|\le k\}} \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i^-} dx \le C_4(1+k), \tag{2.17}$$

$$\int_{\partial\Omega} |T_k(u_n)|^{r^-} d\sigma \le C_5(k+1). \tag{2.18}$$

Proof. We have

$$\begin{aligned}
\sum_{i=1}^N \int_{\{|u_n|\le k\}} \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i^-} dx &= \sum_{i=1}^N \int_{\{|u_n|\le k, |\frac{\partial u_n}{\partial x_i}|\le 1\}} \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i^-} dx + \sum_{i=1}^N \int_{\{|u_n|\le k, |\frac{\partial u_n}{\partial x_i}|\ge 1\}} \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i^-} dx \\
&\le N.meas(\Omega) + \sum_{i=1}^N \int_{\{|u_n|\le k, |\frac{\partial u_n}{\partial x_i}|\ge 1\}} \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} dx \\
&\le N.meas(\Omega) + \sum_{i=1}^N \int_{\{|u_n|\le k\}} \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} dx \\
&\le N.meas + kC(\Omega,\mu).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\int_\Omega |T_k(u_n)|^{p_M^-} dx &= \int_{\{|T_k(u_n)|\le 1\}} |T_k(u_n)|^{p_M^-} dx + \int_{\{|T_k(u_n)|>1\}} |T_k(u_n)|^{p_M^-} dx \\
&\le meas(\Omega) + \int_\Omega |T_k(u_n)|^{p_M^-} dx \\
&\le meas(\Omega) + \int_\Omega k^{p_M^-} dx \\
&\le meas(\Omega)(1+k^{p_M^-}).
\end{aligned}$$

By adding the two last inequalities, we get

$$\begin{aligned}
\int_\Omega |T_k(u_n)|^{p_M^-} dx + \sum_{i=1}^N \int_{\{|u_n|\le k\}} \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i^-} dx &\le (meas(\Omega)(N+1+k^{p_M^-}) + kC(\mu,\Omega) \\
&\le C_4(1+k),
\end{aligned}$$

where $C_4 = \max\{(meas(\Omega)(N+1+k^{p_M^-}), C(\mu,\Omega)\}$.
For the inequality (2.18), according to (2.16), we deduce the following

$$\begin{aligned}
\int_{\partial\Omega} |T_k(u_n)|^{r^-} dx &= \int_{\partial\Omega\cap\{|T_k(u_n)|\le 1\}} |T_k(u_n)|^{r^-} d\sigma + \int_{\partial\Omega\cap\{|T_k(u_n)|>1\}} |T_k(u_n)|^{r^-} d\sigma \\
&\le meas(\partial\Omega) + \int_{\partial\Omega} |T_k(u_n)|^{r(x)} d\sigma \\
&\le meas(\partial\Omega) + \int_{\partial\Omega} |u_n|^{r(x)-2} u_n T_k(u_n) d\sigma \\
&\le meas(\partial\Omega) + kC(\mu,\Omega) \\
&\le C_5(1+k)
\end{aligned}$$

where $C_5 = \max\{meas(\partial\Omega), C(\mu,\Omega)\}$. □

We have the following result (see [19, 21]).

Lemma 2.9. *Let u_n be a weak solution of (2.4). Then, there exists a constant D which depends on μ and Ω such that*

$$meas\{|u_n| > k\} \leq \frac{D}{\min(\beta_n(k), |\beta_n(-k)|)}, \forall k > 0 \tag{2.19}$$

and a constant D' which depends on μ and Ω such that

$$meas\left\{\left|\frac{\partial u_n}{\partial x_i}\right| > k\right\} \leq \frac{D'}{k^{\frac{1}{(\overline{p}_M)'}}}, \ \forall k \geq 1. \tag{2.20}$$

Proposition 2.10. *For any $k > 0$, the sequences $(\beta_n(u_n))_{n>0}$ and $(\beta_n(T_k(u_n)))_{n>0}$ are uniformly bounded in $L^1(\Omega)$.*

For the proof of the Proposition 2.10, we refer the to ([27], Proposition 3.9).
We also have the following lemma (see [7]).

Lemma 2.11. *For any $k > 0$, there exists some positive constants C_6, C_7 such that.*

(i) $||u_n||_{\mathcal{M}^{q^*}(\Omega)} \leq C_6$;

(ii) $\left\|\frac{\partial u_n}{\partial x_i}\right\|_{\mathcal{M}^{p_i^- \frac{q}{p}}(\Omega)} \leq C_7, \quad \forall i = 1, ..., N.$

2.1. **Convergence results.** We have the following results (see [7], [22]).

Lemma 2.12. *For $i = 1, ..., N$, as $n \to +\infty$, we have*

$$a_i\left(x, \frac{\partial u_n}{\partial x_i}\right) \longrightarrow a_i\left(x, \frac{\partial u}{\partial x_i}\right) \text{ in } L^1(\Omega) \text{ a.e. } x \in \Omega. \tag{2.21}$$

Proposition 2.13. *There exists $u \in W^{1,\vec{p}(.)}(\Omega) \subset \mathcal{T}^{1,\vec{p}(.)}(\Omega)$ such that $u \in dom(\beta)$ a.e. in Ω and*

$$u_n \longrightarrow u \text{ in measure and a.e. in } \Omega \text{ as } n \longrightarrow +\infty. \tag{2.22}$$

Proof. For the proof of (2.22), we refer to [7] (see also [22]).
As for $k > 0$, T_k is continuous, then $T_k(u_n) \to T_k(u)$ a.e. in Ω. Finally, using Lemma 1.1 we deduce that for all $k > 0$, $T_k(u) \in dom(\beta)$ a.e. in Ω. Since $T_k(u) \in dom(\beta)$, we get $u \in dom(\beta)$ a.e. in Ω and as $dom(\beta)$ is bounded, then $u \in W^{1,\vec{p}(.)}(\Omega)$. □

Proposition 2.14. *Let $u_n \in W^{1,\vec{p}(.)}(\Omega)$ be a weak solution of (2.4). Then,*

(i) *for all $i = 1, ..., N$, $\frac{\partial u_n}{\partial x_i}$ converges in measure to the weak partial gradient of u;*

(ii) *for all* $i = 1, ..., N$ *and* $k > 0$, $a_i\left(x, \frac{\partial}{\partial x_i} T_k(u_n)\right)$ *converges to* $a_i\left(x, \frac{\partial}{\partial x_i} T_k(u)\right)$ *in* $L^1(\Omega)$ *strongly and in* $L^{p_i'(.)}(\Omega)$ *weakly.*

(iii) u_n *converges to some function* v *a.e. on* $\partial\Omega$.

(iv) *For any* $h \in C_c^1(\mathbb{R})$, $|u_n|^{r(x)-2}u_n h(u_n) \to |u|^{r(x)-2}uh(u)$ *in* $L^1(\partial\Omega)$.

(v) *For* $i = 1, ..., N$, $a_i\left(x, \frac{\partial u_n}{\partial x_i}\right)\frac{\partial u_n}{\partial x_i} \longrightarrow a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial u}{\partial x_i}$ *in* $L^1(\Omega)$ *and a.e.* $x \in \Omega$.

Proof. • For the proof of (i), (ii) and (iii) we refer to [7] (see also [22]).
• For the proof of (iv) see ([27], Proposition 3.14).
• We end by proving (v).
The continuity of $a_i(x, \xi)$ with respect to $\xi \in \mathbb{R}$ gives us

$$a_i\left(x, \frac{\partial u_n}{\partial x_i}\right) \longrightarrow a_i\left(x, \frac{\partial u}{\partial x_i}\right) \text{ a.e. } x \in \Omega.$$

Therefore,

$$a_i\left(x, \frac{\partial u_n}{\partial x_i}\right)\frac{\partial u_n}{\partial x_i} \longrightarrow a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial u}{\partial x_i} \text{ a.e. } x \in \Omega.$$

Setting $y_n = a_i\left(x, \frac{\partial u_n}{\partial x_i}\right)\frac{\partial u_n}{\partial x_i}$ and $y = a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial u}{\partial x_i}$, for $i = 1, ..., N$, we have

$$\begin{cases} y_n \geq 0, y_n \to y \text{ a.e. in } \Omega, \ y \in L^1(\Omega), \\ \\ \displaystyle\int_\Omega y_n dx \to \int_\Omega y dx \end{cases}$$

and as $\displaystyle\int_\Omega |y_n - y| dx = 2\int_\Omega (y - y_n)^+ dx + \int_\Omega (y_n - y) dx$ and $(y - y_n)^+ \leq y$, it follows by using Lebesgue dominated convergence Theorem, that

$$\lim_{n \to +\infty} \int_\Omega |y_n - y| dx = 0,$$

which means that

$$a_i\left(x, \frac{\partial u_n}{\partial x_i}\right)\frac{\partial u_n}{\partial x_i} \longrightarrow a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial u}{\partial x_i} \text{ in strongly in } L^1(\Omega).$$

□

For the proof of the following result we refer to the paper [25].

Lemma 2.15. *For any* $h \in C_c^1(\mathbb{R})$ *and* $\varphi \in W^{1,\overrightarrow{p}(.)}(\Omega) \cap L^\infty(\Omega)$, *for any* $i = 1, ..., N$,

$$\frac{\partial}{\partial x_i}(h(u_n)\varphi) \longrightarrow \frac{\partial}{\partial x_i}(h(u)\varphi) \text{ strongly in } L^1(\Omega) \text{ as } n \to +\infty.$$

Lemma 2.16. *For any* $h \in C_c^1(\mathbb{R})$ *and* $\xi \in W^{1,\overrightarrow{p}(.)}(\Omega) \cap L^\infty(\Omega)$,

$$\lim_{n \to +\infty} \int_\Omega h(u_n)\xi d\mu_n = \int_\Omega h(u)\xi d\mu \tag{2.23}$$

and

$$\lim_{n\to+\infty}\frac{1}{n}\int_\Omega |u_n|^{p_M(x)-2}u_n h(u_n)\xi dx = 0. \tag{2.24}$$

Proof. • For the proof of (2.23) we refer to [27].
• For (2.24) we have

$$\begin{aligned}\left|\frac{1}{n}\int_\Omega |u_n|^{p_M(x)-2}u_n h(u_n)\xi dx\right| &\le \frac{1}{n}\int_\Omega |u_n|^{p_M(x)-1}|h(u_n)||\xi| dx\\ &\le \frac{1}{n}||\xi||_\infty \int_\Omega |u_n|^{p_M(x)-1}|h(u_n)| dx\\ &\le \frac{1}{n}C(h,||\xi||_\infty)\int_\Omega |u_n|^{p_M(x)-1}dx \longrightarrow 0\end{aligned}$$

Since $|u_n|^{p_M(x)-1}\in L^{p'_M(x)}(\Omega)$, (2.24) follows as $n\to+\infty$. □

Lemma 2.17. *(See [27]) For any $h\in C^1_c(\mathbb{R})$ and $\xi\in W^{1,\vec{p}(.)}(\Omega)\cap L^\infty(\Omega)$,*

$$\lim_{n\to+\infty}\int_{\partial\Omega}|u_n|^{r(x)-2}u_n h_k(u_n)\xi d\sigma = \int_{\partial\Omega}|u|^{r(x)-2}u h_k(u)\xi d\sigma. \tag{2.25}$$

Since, for any $k>0$, $(h_k(u_n)\beta_n)_{n>0}$ is bounded in $L^1(\Omega)$, there exists $z_k\in\mathcal{M}_b(\Omega)$ such that

$$h_k(u_n)\beta_n(u_n)\rightharpoonup z_k \text{ in } \mathcal{M}_b(\Omega) \text{ as } n\to 0.$$

Moreover, for any $\xi\in W^{1,\vec{p}(.)}(\Omega)\cap L^\infty(\Omega)$, we have

$$\int_\Omega \xi dz_k = \int_\Omega \xi h_k(u)d\mu - \int_{\partial\Omega}|u|^{r(x)-2}u h_k(u)\xi d\sigma - \int_\Omega\sum_{i=1}^N a_i(x,\frac{\partial u}{\partial x_i})\frac{\partial}{\partial x_i}[h_k(u)\xi]dx,$$

which implies that $z_k\in\mathcal{M}_b^{p_m(.)}(\Omega)$ and, for any $k\le l$,

$$z_k = z_l \text{ on } [|T_k(u)|<k].$$

Let us consider the Radon measure defined by

$$\begin{cases} z = z_k, \text{ on } [|T_k(u)|<k] \text{ for } k\in\mathbb{N}^*,\\ z = 0 \text{ on } \bigcap\limits_{k\in\mathbb{N}^*}[|T_k(u)|=k]. \end{cases} \tag{2.26}$$

For any $h\in\mathcal{C}_c(\mathbb{R})$, $h(u)\in L^\infty(\Omega,d|z|)$ and

$$\int_\Omega h(u)\xi dz = -\int_\Omega\sum_{i=1}^N a_i(x,\frac{\partial u}{\partial x_i})\frac{\partial}{\partial x_i}[h(u)\xi]dx - \int_{\partial\Omega}|u|^{r(x)-2}u h(u)\xi d\sigma + \int_\Omega h(u)\xi d\mu,$$

for any $\xi \in W^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$. Indeed, let $k_0 > 0$ be such that $supp(h) \subseteq [-k_0, k_0]$,

$$\begin{aligned}
\int_\Omega h(u)\xi dz &= \int_\Omega h(u)\xi dz_{k_0} \\
&= -\lim_{n\to+\infty}\int_\Omega \sum_{i=1}^N a_i(x, \frac{\partial u_n}{\partial x_i})\frac{\partial}{\partial x_i}[h(u_n)\xi]dx - \lim_{n\to+\infty}\frac{1}{n}\int_\Omega |u_n|^{p_M(x)-2}u_n h(u_n)\xi dx \\
&- \lim_{n\to+\infty}\int_{\partial\Omega} |u_n|^{r(x)-2}u_n h(u_n)\xi d\sigma + \lim_{n\to+\infty}\int_\Omega h(u_n)\xi d\mu_n \\
&= -\lim_{n\to+\infty}\int_\Omega \sum_{i=1}^N a_i(x, \frac{\partial T_k(u_n)}{\partial x_i})\frac{\partial}{\partial x_i}[h(u_n)\xi]dx - \lim_{n\to+\infty}\frac{1}{n}\int_\Omega |u_n|^{p_M(x)-2}u_n h(u_n)\xi dx \\
&- \lim_{n\to\infty}\int_{\partial\Omega} |u_n|^{r(x)-2}u_n h(u_n)\xi d\sigma + \lim_{n\to+\infty}\int_\Omega h(u_n)\xi d\mu_n \\
&= -\int_\Omega \sum_{i=1}^N a_i(x, \frac{\partial u}{\partial x_i})\frac{\partial}{\partial x_i}[h(u)\xi]dx - \int_{\partial\Omega} |u|^{r(x)-2}u h(u)\xi d\sigma + \int_\Omega h(u)\xi d\mu.
\end{aligned}$$

Moereover, we have the following lemma (see [27], Lemma 3.18).

Lemma 2.18. *The Radon-Nikodym decomposition of the measure z given by (2.26) with respect to $\mathcal{L}^N$,*

$$z = w\mathcal{L}^N + \nu \quad with,\ \nu \perp \mathcal{L}^N$$

satisfies the following properties.

$$\begin{cases}
w \in \beta(u)\mathcal{L}^N - a.e.\ in\ \Omega,\ w \in\ L^1(\Omega),\ \nu\ \in \mathcal{M}_b^{p_m(.)}(\Omega), \\
\nu^+ \ is\ concentrated\ on\ \ [u = M], \\
\nu^- \ is\ concentrated\ on\ \ [u = m].
\end{cases}$$

To finish the proof of the Theorem 2.2, we consider $\xi \in W^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$ and $h \in C_c^\infty(\Omega)$. Then, we take $h(u_n)\xi$ as test function in (2.5). We get

$$\sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial u_n}{\partial x_i}\right)\frac{\partial}{\partial x_i}[h(u_n)\xi]dx + \int_\Omega \beta_n(u_n)h(u_n)\xi dx + \frac{1}{n}\int_\Omega |u_n|^{P_M(x)-2}u_n h(u_n)\xi dx$$

$$+ \int_{\partial\Omega} |u_n|^{r(x)-2}u_n h(u_n)\xi d\sigma = \int_\Omega h(u_n)\xi d\mu_n. \tag{2.27}$$

The first term of (2.27) can be written as

$$\sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial u_n}{\partial x_i}\right)\frac{\partial}{\partial x_i}[h(u_n)\xi]dx = \sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial T_{l_0+1}(u_n)}{\partial x_i}\right)\frac{\partial}{\partial x_i}[h_0(u_n)\xi]dx,$$

for some $l_0 > 0$ so that, by Proposition 2.14 and Lemma 2.15, we have

$$\begin{aligned}\lim_{n\to+\infty}\sum_{i=1}^{N}\int_\Omega a_i\Big(x,\frac{\partial u_n}{\partial x_i}\Big)\frac{\partial}{\partial x_i}[h(u_n)\xi]dx &= \lim_{n\to+\infty}\sum_{i=1}^{N}\int_\Omega a_i\Big(x,\frac{\partial T_{l_0+1}(u_n)}{\partial x_i}\Big)\frac{\partial}{\partial x_i}[h_0(u_n)\xi]dx\\ &= \sum_{i=1}^{N}\int_\Omega a_i\Big(x,\frac{\partial T_{l_0+1}(u}{\partial x_i}\Big)\frac{\partial}{\partial x_i}[h_0(u)\xi]dx\\ &= \sum_{i=1}^{N}\int_\Omega a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial}{\partial x_i}[h(u)\xi]dx.\end{aligned}$$

Thanks to the convergences of Lemmas 2.15, 2.16, 2.17 and Proposition 2.14, we obtain from (2.27),

$$\begin{aligned}\lim_{n\to+\infty}\int_\Omega \beta_n(u_n)h(u_n)\xi dx &= -\sum_{i=1}^{N}\int_\Omega a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial}{\partial x_i}[h(u)\xi]dx - \int_{\partial\Omega}|u|^{r(x)-2}uh(u)\xi d\sigma\\ &\quad+\int_\Omega h(u)\xi d\mu\\ &= \int_\Omega h(u)\xi dz\\ &= \int_\Omega h(u)w\xi dx + \int_\Omega h(u)\xi d\nu.\end{aligned}$$

Letting n go to $+\infty$ in (2.27), it follows that

$$\begin{aligned}\sum_{i=1}^{N}\int_\Omega a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial}{\partial x_i}[h(u)\xi]dx+\int_\Omega h(u)w\xi dx+\int_\Omega h(u)\xi d\nu+\int_{\partial\Omega}|u|^{r(x)-2}uh(u)\xi d\sigma\\ = \int_\Omega h(u)\xi d\mu. \qquad (2.28)\end{aligned}$$

In (2.28), we take $h \in C_c^1(\mathbb{R})$ such that $[m, M] \subset supp(h) \subset [-l, l]$ and $h(s) = 1$ for all $s \in [m, M]$. As $u \in \mathrm{dom}\beta$, we get $h(u) = 1$ and it yields that (u, w) is a solution of the problem (1.1).

To end the proof of Theorem 2.2, we prove (2.2). We take $\xi = T_1(u_n - T_k(u_n))$ as a test function in (2.5) to get

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i\Big(x,\frac{\partial u_n}{\partial x_i}\Big)\frac{\partial}{\partial x_i}(T_1(u_n - T_k(u_n)))dx + \int_\Omega \beta_n(u_n)T_1(u_n - T_k(u_n))dx\\ &+\frac{1}{n}\int_\Omega |u_n|^{P_M(x)-2}u_nT_1(u_n - T_k(u_n))dx + \int_{\partial\Omega}|u_n|^{r(x)-2}u_nT_1(u_n - T_k(u_n))d\sigma\\ &= \int_\Omega T_1(u_n - T_k(u_n))d\mu_n. \qquad (2.29)\end{aligned}$$

Since

$$\int_\Omega \beta_n(u_n)T_1(u_n - T_k(u_n))dx + \frac{1}{n}\int_\Omega |u_n|^{P_M(x)-2}u_nT_1(u_n - T_k(u_n))dx$$

$$+\int_{\partial\Omega}|u_n|^{r(x)-2}u_nT_1(u_n-T_k(u_n))d\sigma\geq 0$$

and $\frac{\partial}{\partial x_i}(T_1(u_n-T_k(u_n)))=\frac{\partial u_n}{\partial x_i}\chi_{[k<|u_n|<k+1]}$, we get from (2.29),

$$\sum_{i=1}^{N}\int_{[k<|u_n|<k+1]}a_i\left(x,\frac{\partial u_n}{\partial x_i}\right)\frac{\partial u_n}{\partial x_i}dx\leq\int_{\Omega}T_1(u_n-T_k(u_n))d\mu_n. \tag{2.30}$$

We have (see [25])

$$\lim_{k\to+\infty}\lim_{n\to+\infty}\int_{\Omega}T_1(u_n-T_k(u_n))d\mu_n=0.$$

Then, using (1.5), and letting $n\to+\infty$, $k\to+\infty$ respectively in (2.30), we obtain

$$\lim_{k\to+\infty}\lim_{n\to+\infty}\frac{1}{C}\sum_{i=1}^{N}\int_{[k<|u_n|<k+1]}\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)}dx=\lim_{k\to+\infty}\frac{1}{C}\sum_{i=1}^{N}\int_{[k<|u|<k+1]}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}dx\leq 0. \tag{2.31}$$

Therefore, we get (2.2).

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