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SOME QUESTIONS ON THE DISTRIBUTION OF PRIME GAPS AND SOME FORMULAS INVOLVING PRIME GAPS

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ABSTRACT. In this article we present some questions on the distribution of prime gaps. Apparently the solutions of these questions are very difficult. We also obtain some formulas involving prime gaps. Our results are valid for more general integer sequences.

1. Introduction and Some Questions on the Distribution of Prime Gaps

Let p_n be the n-th prime. The n-th prime gap is $d_n = p_{n+1} - p_n$. In this article we present some questions on the distribution of prime gaps, apparently the solutions of these questions are very difficult. We also obtain some formulas involving prime gaps. Our results are valid for more general integer sequences. For example, we prove the following result.

Let $\epsilon > 0$ an arbitrary fixed small number and $M > 1 + \epsilon$. Let $D_M(x)$ be the number of primes p_i not exceeding x such that $d_i = p_{i+1} - p_i < M \log p_i$. Then $D_M(x) \geq \left(1 - \frac{1+\epsilon}{M}\right) \pi(x)$. Note that if M is large then $1 - \frac{1+\epsilon}{M}$ is near of 1. $\pi(x)$ denotes (as usual) the prime counting function.

There are in the literature many papers on primes and prime gaps and in general studies of numbers in small intervals, for example [8], and also about prime factors in composite numbers, for example [9] [10].

There are consecutive repeated prime gaps. That is, $d_{i-1} \neq d_i = d_{i+1} = \cdots = d_{i+k-1} \neq d_{i+k}$, where the number of consecutive prime gaps is $k \geq 2$. For example, the first repeated prime gaps are

$$d_1 \neq d_2 = d_3 = 2 \neq d_4.$$

$$d_{14} \neq d_{15} = d_{16} = 6 \neq d_{17}.$$

$$d_{35} \neq d_{36} = d_{37} = 6 \neq d_{38}.$$

There are interesting questions. For example, is k bounded in the sequence of primes or not? That is, there are arbitrarily large number of repeated prime gaps? On the other hand, if we eliminate the consecutive repeated prime gaps in

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the first n prime gaps, is the sum of the rest of gaps asymptotically equal to p_n or not?

Another question on prime gaps is the following:

Given $\epsilon > 0$, the number $G(n)$ of prime gaps d_i such that

$$(1-\epsilon)\log p_i < d_i < (1+\epsilon)\log p_i \qquad (1 \leq i \leq n) \tag{1.1}$$

is asymptotically equal to n, that is, $G(n) \sim n$, independently of $\epsilon > 0$. Is this establishment true or not?

We use the notation $l_i = \log p_{i+1} - \log p_i$ for the i-th logarithmic prime gap. Below, we shall prove the equation $p_i \sim \frac{d_i}{l_i}$. By use of this equation we can to prove that (1.1) is equivalent to the following establishment:

Given $\epsilon > 0$, the number $H(n)$ of logarithmic prime gaps l_i such that

$$(1-\epsilon)\frac{1}{i} < l_i < (1+\epsilon)\frac{1}{i} \qquad (1 \leq i \leq n) \tag{1.2}$$

is asymptotically equal to n, that is, $H(n) \sim n$, independently of $\epsilon > 0$.

We do not respond these questions in this article, we think they are very difficult, we simply establish them.

Equation $p_i \sim \frac{d_i}{l_i}$ is an immediate consequence of the mean value theorem applied to the function $\log x$. We have (mean value theorem)

$$\frac{1}{p_{i+1}}\left(p_{i+1} - p_i\right) < \log p_{i+1} - \log p_i < \frac{1}{p_i}\left(p_{i+1} - p_i\right), \tag{1.3}$$

from here we obtain

$$p_i \sim \frac{d_i}{l_i} \tag{1.4}$$

and from equation (1.4) and the prime number theorem [6] $p_i \sim i \log i$ we obtain the equation

$$\frac{d_i}{\log i} \sim \frac{l_i}{\frac{1}{i}}. \tag{1.5}$$

From equation (1.3) we obtain

$$p_i\left(\log p_{i+1} - \log p_i\right) < p_{i+1} - p_i < p_{i+1}\left(\log p_{i+1} - \log p_i\right),$$

that is

$$0 < d_i - p_i l_i < d_i l_i < \frac{d_i^2}{p_i}.$$

Therefore

$$d_i - p_i l_i = O\left(\frac{d_i^2}{p_i}\right),$$

that is

$$d_i = p_i l_i + O\left(\frac{d_i^2}{p_i}\right). \tag{1.6}$$

Equation (1.6) gives

$$l_i = \frac{d_i}{p_i} + O\left(\frac{d_i^2}{p_i^2}\right) \tag{1.7}$$

and equations (1.6) and (1.4) give

$$p_i = \frac{d_i}{l_i} + O\left(d_i\right) \tag{1.8}$$

If prime numbers satisfy $d_n = O(p_n^\theta)$, where $0 < \theta < \frac{1}{2}$, or the Cramér's conjecture, namely $d_n = O\left(\log^2 p_n\right)$, then $\frac{d_i^2}{p_i} \to 0$ in equation (1.6) and consequently

$$d_i = p_i l_i + o(1). \tag{1.9}$$

Note that the series $\sum_{i=1}^{\infty} \frac{d_i^2}{p_i}$ diverges since the series $\sum_{i=1}^{\infty} \frac{1}{p_i}$ diverges. We do not know if $\frac{d_i^2}{p_i} \to 0$. If $\frac{d_i^2}{p_i} \to 0$ then we obtain that $d_n = o(\sqrt{p_n})$, this is a result stronger that it obtained by use of the Riemann hypothesis, namely $d_n = O\left(\sqrt{p_n}\log p_n\right)$. Note that by equation (1.4) we have

$$\frac{d_i^2}{p_i} \sim p_i l_i^2 \sim d_i l_i.$$

Consequently also the series $\sum_{i=1}^{\infty} p_i l_i^2$ and $\sum_{i=1}^{\infty} d_i l_i$ are divergent.

Note that equation (1.3) and the prime number theorem $p_i \sim i \log i$ give

$$c_1 \frac{1}{i \log i} < \frac{1}{p_{i+1}} < \frac{1}{p_{i+1}}\left(p_{i+1} - p_i\right) < \log p_{i+1} - \log p_i = l_i, \tag{1.10}$$

where c_1 is a positive constant.

On the other hand, equation (1.3) implies that the Cramér's conjecture is equivalent to the following establishment for l_i,

$$l_i = \log p_{i+1} - \log p_i = O\left(\frac{\log^2 p_i}{p_i}\right) = O\left(\frac{\log p_i}{i}\right) = O\left(\frac{\log i}{i}\right). \tag{1.11}$$

Note that the i-th logarithmic prime gap $l_i \to 0$ as $i \to \infty$, since $\frac{p_{i+1}}{p_i} \to 1$ as $i \to \infty$.

Note that there exist asymptotic formulas for p_n, as for example, (see below)

$$p_n = \sum_{i=1}^{n-1} \log p_i + O\left(\frac{n}{\log^m n}\right), \tag{1.12}$$

for all nonnegative integer m. We have $d_n = O\left(p_n^{\theta+\epsilon}\right)$, where $\theta = \frac{6}{11}$ and $\epsilon > 0$ [5]. Therefore equation (1.8) gives

$$\begin{aligned} p_{n+2} &= p_{n+1} + d_{n+1} = p_n + d_n + d_{n+1} \\ &= \frac{d_n}{l_n} + O\left(d_n\right) + d_n + d_{n+1} = \frac{d_n}{l_n} + O(p_n^{\theta+\epsilon}), \end{aligned}$$

that is

$$p_{n+2} = \frac{p_{n+1} - p_n}{\log p_{n+1} - \log p_n} + O(p_n^{\theta+\epsilon}). \tag{1.13}$$

If the Riemann hypothesis is true then $d_n = O\left(\sqrt{p_n}\log p_n\right)$ and consequently $O(p_n^{\theta+\epsilon})$ can be replaced by $O\left(\sqrt{p_n}\log p_n\right)$ in equation (1.13). If the Cramér's conjecture is true then $d_n = O\left(\log^2 p_n\right)$ and consequently $O(p_n^{\theta+\epsilon})$ can be replaced by $O\left(\log^2 p_n\right)$ in equation (1.13).

In the literature the i-th gap $d_i = p_{i+1} - p_i$ is compared with $\log p_i \sim \log i$ [5], since (prime number theorem [6]) $\sum_{p_i \leq x} \log p_i \sim x$ and consequently $\sum_{i=1}^{n} \log p_i \sim p_n$. It is well-known the following limits (see [17] and [4]):

$$\limsup \frac{d_i}{\log p_i} = \infty, \tag{1.14}$$

$$\liminf \frac{d_i}{\log p_i} = 0. \tag{1.15}$$

Now, by analogy, we shall compare the i-th logarithmic prime gap $l_i = \log p_{i+1} - \log p_i$ with $\frac{\log p_i}{p_i} \sim \frac{1}{i}$, since $\sum_{p_i \leq x} \frac{\log p_i}{p_i} \sim \log x$ [6] and consequently $\sum_{i=1}^{n} \frac{\log p_i}{p_i} \sim \log p_n$. We also have

$$\limsup \frac{l_i}{\frac{\log p_i}{p_i}} = \infty, \tag{1.16}$$

$$\liminf \frac{l_i}{\frac{\log p_i}{p_i}} = 0. \tag{1.17}$$

Limits (1.16) and (1.17) are an immediate consequence of equation (1.5) and limits (1.14) and (1.15).

We have the following theorem.

Theorem 1.1. *The following asymptotic formula holds*

$$\sum_{i=1}^{n} \frac{d_i}{l_i} = \frac{n^2}{2}\log n + \frac{n^2}{2}\log\log n - \frac{3}{4}n^2 + \sum_{j=1}^{m} \frac{(-1)^{j-1} n^2 Q_j(\log\log n)}{\log^j n} + o\left(\frac{n^2}{\log^m n}\right), \tag{1.18}$$

where the m are nonnegative integers and the $Q_j(x)$ are polynomials (see below, in the proof). In particular, if $m = 0$ then we obtain

$$\sum_{i=1}^{n} \frac{d_i}{l_i} = \frac{n^2}{2}\log n + \frac{n^2}{2}\log\log n - \frac{3}{4}n^2 + o(n^2).$$

The following asymptotic formula holds

$$\sum_{i=1}^{n} \frac{l_i}{d_i} = \log\log n + c + o(1), \tag{1.19}$$

where c is a constant.

Proof. We have (see (1.8))

$$\frac{d_i}{l_i} = p_i + O\left(d_i\right). \tag{1.20}$$

On the other hand, it is well-known the asymptotic formula [11]

$$\sum_{i=1}^{n} p_i = \frac{n^2}{2}\log n + \frac{n^2}{2}\log\log n - \frac{3}{4}n^2 + \sum_{j=1}^{m}\frac{(-1)^{j-1}n^2 Q_j(\log\log n)}{\log^j n}$$
$$+ \quad o\left(\frac{n^2}{\log^m n}\right), \tag{1.21}$$

where the polynomials $Q_j(x)$ have rational coefficients, they are of degree j and leading coefficient $\frac{1}{2j}$. The m are nonnegative integers. If $m = 0$ then the formula becomes

$$\sum_{i=1}^{n} p_i = \frac{n^2}{2}\log n + \frac{n^2}{2}\log\log n - \frac{3}{4}n^2 + o(n^2).$$

Now, we have (see (1.20))

$$\sum_{i=1}^{n}\frac{d_i}{l_i} = \sum_{i=1}^{n} p_i + O\left(\sum_{i=1}^{n} d_i\right) = \sum_{i=1}^{n} p_i + O\left(n\log n\right). \tag{1.22}$$

Therefore equations (1.22) and (1.21) give equation (1.18) and we observe that $\sum_{i=1}^{n}\frac{d_i}{l_i}$ has the same asymptotic expansion as $\sum_{i=1}^{n} p_i$.

We have (see (1.7))

$$\frac{l_i}{d_i} = \frac{1}{p_i} + O\left(\frac{d_i}{p_i^2}\right).$$

Therefore

$$\sum_{i=1}^{n}\frac{l_i}{d_i} = \sum_{i=1}^{n}\frac{1}{p_i} + \sum_{i=1}^{n} O\left(\frac{d_i}{p_i^2}\right).$$

It is well-known the formula $\sum_{p_i\le x}\frac{1}{p_i} = \log\log x + M + o(1)$, where M is Mertens's constant [6]. If we put $x = p_n$ then we obtain

$$\sum_{i=1}^{n}\frac{1}{p_i} = \log\log p_n + M + o(1) = \log\log n + M + o(1).$$

On the other hand the series $\sum_{i=1}^{\infty}\frac{d_i}{p_i^2}$ converges, since it is well-known that $d_i = O\left(p_i^{\theta+\epsilon}\right)$, where $\theta = \frac{6}{11}$. Consequently the series $\sum_{i=1}^{\infty} O\left(\frac{d_i}{p_i^2}\right)$ converges absolutely, since

$$\sum_{i=1}^{n}\left|O\left(\frac{d_i}{p_i^2}\right)\right| \le N\sum_{i=1}^{\infty}\frac{d_i}{p_i^2},$$

where $|O(1)| < N$. The theorem is proved. ∎

There is for the mean gap d_n in the sequence of primes, namely $\log p_n$, an asymptotic expansion. The first terms are [2]

$$\log p_n = \log n + \log\log n + o(1),$$

$$\log p_n = \log n + \log\log n + \frac{\log\log n - 1}{\log n} + o\left(\frac{1}{\log n}\right).$$

There is for the mean logarithmic gap l_n in the sequence of primes, namely $\frac{\log p_n}{p_n}$, an asymptotic expansion. The first terms are [15]

$$\frac{\log p_n}{p_n} = \frac{1}{n} + \frac{1}{n\log n} + o\left(\frac{1}{n\log n}\right),$$

$$\frac{\log p_n}{p_n} = \frac{1}{n} + \frac{1}{n\log n} - \frac{\log\log n - 2}{n\log^2 n} + o\left(\frac{1}{n\log^2 n}\right).$$

2. Some Formulas Involving Prime Gaps

Let us consider a strictly increasing sequence A_n of positive integers. We define the n-th gap (as for primes) $d_n = A_{n+1} - A_n$. We suppose that $\frac{A_{n+1}}{A_n} \to 1$ as $n \to \infty$, this condition is equivalent to $\frac{d_n}{A_n} \to 0$ as $n \to \infty$. We also suppose that $\frac{d_n}{A_n} < 1$ $(n \geq 1)$.

Clearly the sequence p_n of prime numbers satisfies these conditions, since (prime number theorem) $p_n \sim n\log n$ and (Bertrand's postulate [6]) $p_{n+1} < 2p_n$ $(n \geq 1)$, that is, $d_n = p_{n+1} - p_n < p_n$.

Theorem 2.1. *Suppose that the series*

$$\sum_{n=1}^{\infty} \frac{d_n^2}{A_n^2} \tag{2.1}$$

converges. Then

$$\sum_{n=1}^{\infty} \frac{d_n^2}{A_n^2} > \sum_{n=1}^{\infty} \frac{d_n^3}{A_n^3} > \sum_{n=1}^{\infty} \frac{d_n^4}{A_n^4} > \sum_{n=1}^{\infty} \frac{d_n^5}{A_n^5} > \cdots \tag{2.2}$$

and

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} \frac{d_n^k}{A_n^k} = 0. \tag{2.3}$$

The series

$$\sum_{k=2}^{\infty} (-1)^k \left(\sum_{n=1}^{\infty} \frac{d_n^k}{A_n^k} \right) \tag{2.4}$$

is a Leibniz series that converges absolutely.

Proof. Inequality (2.2) is an immediate consequence of the convergence of (2.1) and the inequality $\frac{d_n}{A_n} < 1$ $(n \geq 1)$.

Let $a < 1$ the greatest value of the sequence $\frac{d_n}{A_n} < 1$ $(n \geq 1)$. Then

$$\sum_{n=1}^{\infty} \frac{d_n^k}{A_n^k} = \sum_{n=1}^{\infty} \frac{d_n^2}{A_n^2} \frac{d_n^{k-2}}{A_n^{k-2}} \leq \sum_{n=1}^{\infty} \frac{d_n^2}{A_n^2} a^{k-2} = a^{k-2} \sum_{n=1}^{\infty} \frac{d_n^2}{A_n^2} \to 0 \quad (k \to \infty).$$

This proves limit (2.3).

Therefore the series (2.4) is a Leibniz series. This series converges absolutely, since (geometric series)

$$\sum_{k=2}^{\infty} \left(\sum_{n=1}^{\infty} \frac{d_n^k}{A_n^k} \right) = \sum_{n=1}^{\infty} \frac{d_n^2}{A_n^2} \frac{1}{1 - \frac{d_n}{A_n}}$$

and the last series converges since (2.1) converges by hypothesis and $\frac{d_n}{A_n} \to 0$. The theorem is proved. □

Lemma 2.2. *Suppose that* $\sum_{i=1}^{\infty} a_i$ *and* $\sum_{i=1}^{\infty} b_i$ *are two series of positive terms such that* $\frac{a_i}{b_i} \to c$. *Then if* $\sum_{i=1}^{\infty} b_i$ *is divergent the following limit holds* $\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} \to c$.

Proof. See [18]. □

Theorem 2.3. *The following asymptotic formula holds*

$$\sum_{i=1}^{n} \frac{d_i}{A_i} \sim \log A_n. \tag{2.5}$$

Proof. We have

$$A_i = A_{i-1} + d_{i-1} = A_{i-1} \left(1 + \frac{d_{i-1}}{A_{i-1}} \right) \qquad (i \geq 2).$$

Therefore

$$A_n = A_1 \left(1 + \frac{d_1}{A_1} \right) \left(1 + \frac{d_2}{A_2} \right) \cdots \left(1 + \frac{d_{n-1}}{A_{n-1}} \right). \tag{2.6}$$

Equation (2.6) gives

$$\log A_n = \log A_1 + \log \left(1 + \frac{d_1}{A_1} \right) + \log \left(1 + \frac{d_2}{A_2} \right) + \cdots + \log \left(1 + \frac{d_{n-1}}{A_{n-1}} \right). \tag{2.7}$$

Now, $\log(1 + x) \sim x$ as $x \to 0$. Therefore Lemma 2.2 gives

$$\log A_n \sim \log \left(1 + \frac{d_1}{A_1} \right) + \log \left(1 + \frac{d_2}{A_2} \right) + \cdots + \log \left(1 + \frac{d_{n-1}}{A_{n-1}} \right) \sim \sum_{i=1}^{n} \frac{d_i}{A_i},$$

that is, equation (2.5). The theorem is proved. □

In the following theorem we give a sufficient condition such that series (2.1) converges.

Theorem 2.4. *Suppose that $d_n < cA_n^{\theta}$ $(n \geq 1)$, where $c > 0$ and $0 < \theta < 1$. Then the series*

$$\sum_{n=1}^{\infty} \frac{d_n^2}{A_n^2}$$

converges.

Proof. We have (Theorem 2.3)

$$\sum_{i=1}^{n} \frac{d_i^2}{A_i} \leq \sum_{i=1}^{n} \frac{d_i}{A_i} cA_i^{\theta} < cA_n^{\theta} \sum_{i=1}^{n} \frac{d_i}{A_i} < c_1 A_n^{\theta} \log A_n,$$

that is,

$$\sum_{A_i \leq A_n} \frac{d_i^2}{A_i} < c_1 A_n^{\theta} \log A_n.$$

Consequently

$$\sum_{A_i \leq x} \frac{d_i^2}{A_i} < c_1 x^{\theta} \log x.$$

Hence Abel summation [6] gives

$$\sum_{A_i \leq x} \frac{d_i^2}{A_i^2} = \left(\sum_{A_i \leq x} \frac{d_i^2}{A_i} \right) \frac{1}{x} + \int_{A_1}^{x} \frac{1}{t^2} \left(\sum_{A_i \leq t} \frac{d_i^2}{A_i} \right) dt \leq \frac{c_1 x^{\theta} \log x}{x}$$

$$+ \int_{A_1}^{x} \frac{c_1 t^{\theta} \log t}{t^2} \, dt \leq 1 + c_1 \int_{A_1}^{\infty} \frac{\log t}{t^{2-\theta}} \, dt.$$

The theorem is proved. □

Remark 2.5. As it is well-known $d_n < cp_n^{\theta+\epsilon}$ $(n \geq 1)$, where $c > 0$, $\theta = \frac{6}{11}$ and $\epsilon > 0$ [5]. Therefore if $A_n = p_n$ is the sequence of primes then the series $\sum_{i=1}^{\infty} \frac{d_i^2}{p_i^2}$ converges. Therefore Theorem 2.4 is true for the sequence of primes. Note that (see (1.4)) we have $\frac{d_i^2}{p_i^2} \sim l_i^2$ consequently the series $\sum_{i=1}^{\infty} l_i^2$ also converges.

In the following theorem we give a more precise formula than (2.5) if the series (2.1) converges. Therefore this theorem is true for the sequence $A_n = p_n$ of primes.

Theorem 2.6. *Suppose that the series (2.1) converges. The following asymptotic formula holds*

$$\sum_{i=1}^{n} \frac{d_i}{A_i} = \log A_n + A + o(1), \tag{2.8}$$

where the constant A is

$$A = -\log A_1 + \sum_{k=2}^{\infty} (-1)^k \frac{1}{k} \left(\sum_{n=1}^{\infty} \frac{d_n^k}{A_n^k} \right) \tag{2.9}$$

Proof. Note that by Theorem 2.1, A is a constant, since the series (2.9) is a Leibniz series absolutely convergent.

We have the well-known logarithmic power series ($|x| < 1$) [18]

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \tag{2.10}$$

Hence, equations (2.7) and (2.10) give

$$\begin{aligned} & \log A_n = \log A_1 + \log\left(1 + \frac{d_1}{A_1}\right) + \log\left(1 + \frac{d_2}{A_2}\right) + \cdots \\ + & \log\left(1 + \frac{d_{n-1}}{A_{n-1}}\right) = \log A_1 + \sum_{i=1}^{n-1} \frac{d_i}{A_i} - \frac{1}{2}\sum_{i=1}^{n-1} \frac{d_i^2}{A_i^2} + \frac{1}{3}\sum_{i=1}^{n-1} \frac{d_i^3}{A_i^3} - \cdots \\ = & \sum_{i=1}^{n} \frac{d_i}{A_i} - A + o(1), \end{aligned}$$

that is, equation (2.8). Since

$$\begin{aligned} & \left|(A + \log A_1) - \left(\frac{1}{2}\sum_{i=1}^{n-1} \frac{d_i^2}{A_i^2} - \frac{1}{3}\sum_{i=1}^{n-1} \frac{d_i^3}{A_i^3} + \cdots\right)\right| \\ \leq & \sum_{i=n}^{\infty} \frac{d_i^2}{A_i^2} + \sum_{i=n}^{\infty} \frac{d_i^3}{A_i^3} + \sum_{i=n}^{\infty} \frac{d_i^4}{A_i^4} + \cdots = \sum_{i=n}^{\infty} \frac{d_i^2}{A_i^2} \frac{1}{1 - \frac{d_i}{A_i}} < \epsilon \quad (n \geq n_\epsilon) \end{aligned}$$

and consequently

$$\lim_{n\to\infty} \left(\frac{1}{2}\sum_{i=1}^{n-1} \frac{d_i^2}{A_i^2} - \frac{1}{3}\sum_{i=1}^{n-1} \frac{d_i^3}{A_i^3} + \cdots\right) = A + \log A_1,$$

since $\epsilon > 0$ can be arbitrarily small. The theorem is proved. □

Remark 2.7. It is well-known the formula for primes [6]

$$\sum_{p \leq x} \frac{\log p}{p} = \log x + E + o(1), \tag{2.11}$$

where E is a constant whose value is $E = -\gamma - \sum_p \frac{\log p}{p(p-1)}$ and γ is Euler constant [3]. Equation (2.11) gives ($x = p_n$)

$$\sum_{i=1}^{n} \frac{\log p_i}{p_i} = \log p_n + E + o(1). \tag{2.12}$$

Compare with equation (2.8) ($A_n = p_n$). Consequently equations (2.8) and (2.12) give

$$\sum_{i=1}^{n} \frac{d_i - \log p_i}{p_i} = A - E + o(1)$$

and then

$$\sum_{i=1}^{\infty} \frac{d_i - \log p_i}{p_i} = \sum_{i=1}^{\infty} \frac{\log p_i}{p_i}\left(\frac{d_i}{\log p_i} - 1\right) = A - E. \tag{2.13}$$

Note that the series $\sum_{i=1}^{\infty}\frac{d_i-\log p_i}{p_i}$ is of variable sign, since some times $d_i-\log p_i>0$ and some times $d_i-\log p_i<0$. Since, as it is well-known, $\limsup\frac{d_n}{\log p_n}=\infty$ and $\liminf\frac{d_n}{\log p_n}=0$ [17] [4].

Theorem 2.8. *Let k be a positive integer. Suppose that the series (2.1) converges. The following formulas hold (for example, for primes, $A_n=p_n$).*

$$\sum_{i=1}^{n}\left(\frac{A_{i+1}}{A_i}\right)^k=n+k\log A_n+C_k+o(1), \tag{2.14}$$

where C_k are constants defined below (in the proof)

$$\sum_{i=1}^{n}\frac{A_{i+1}^k-A_i^k}{A_i^k}=k\log A_n+C_k+o(1). \tag{2.15}$$

$$\sum_{i=1}^{n}\frac{d_i}{A_{i+1}}=\log A_n+B+o(1), \tag{2.16}$$

where B is a constant defined below (in the proof). Compare with equation (2.8).

$$\sum_{i=1}^{n}\left(\frac{A_i}{A_{i+1}}\right)^k=n-k\log A_n+D_k+o(1), \tag{2.17}$$

where D_k are constants defined below (in the proof)

$$\sum_{i=1}^{n}\frac{A_{i+1}^k-A_i^k}{A_{i+1}^k}=k\log A_n-D_k+o(1). \tag{2.18}$$

Proof. We have the identity

$$\frac{A_{i+1}}{A_i}=1+\frac{d_i}{A_i}.$$

Therefore (binomial formula)

$$\left(\frac{A_{i+1}}{A_i}\right)^k=\left(1+\frac{d_i}{A_i}\right)^k=\sum_{j=0}^{k}\binom{k}{j}\left(\frac{d_i}{A_i}\right)^j$$

and consequently (see Theorem 2.1 and equation (2.8))

$$\begin{aligned}\sum_{i=1}^{n}\left(\frac{A_{i+1}}{A_i}\right)^k&=\sum_{i=1}^{n}\left(1+\frac{d_i}{A_i}\right)^k=\sum_{j=0}^{k}\binom{k}{j}\sum_{i=1}^{n}\left(\frac{d_i}{A_i}\right)^j=n\\ &+\ k\left(\log A_n+A+o(1)\right)+\sum_{j=2}^{k}\binom{k}{j}\left(\sum_{i=1}^{\infty}\left(\frac{d_i}{A_i}\right)^j+o(1)\right)\\ &=\ n+k\log A_n+C_k+o(1),\end{aligned}$$

that is, equation (2.14). Therefore equation (2.14) is proved. Equation (2.15) is an immediate consequence of equation (2.14).

Now, we prove equation (2.16). We have

$$\sum_{i=1}^{n}\left(\frac{d_i}{A_i}-\frac{d_i}{A_{i+1}}\right)=\sum_{i=1}^{n}\frac{d_i^2}{A_iA_{i+1}}<\sum_{i=1}^{n}\frac{d_i^2}{A_i^2}.$$

Therefore the series $\sum_{i=1}^{\infty}\frac{d_i^2}{A_iA_{i+1}}$ converges, since the series (2.1) converges. Now, equation (2.16) is an immediate consequence of equation (2.8).

Equation (2.17) can be proved in the same form as equation (2.14) from the identity

$$\frac{A_i}{A_{i+1}}=1-\frac{d_i}{A_{i+1}}.$$

Therefore equation (2.17) is proved. Equation (2.18) is an immediate consequence of equation (2.17). The theorem is proved. □

Now, we establish a general theorem. In particular this theorem is valid for primes, $A_n=p_n$.

Theorem 2.9. *The following asymptotic formulas hold*

$$(\log A_{n+1}-\log A_n)\sim\frac{d_n}{A_n}, \tag{2.19}$$

$$d_n\sim A_n(\log A_{n+1}-\log A_n), \tag{2.20}$$

$$A_n\sim\frac{d_n}{\log A_{n+1}-\log A_n}=\frac{A_{n+1}-A_n}{\log A_{n+1}-\log A_n}, \tag{2.21}$$

$$A_n\sim\sum_{i=1}^{n}A_i(\log A_{i+1}-\log A_i), \tag{2.22}$$

$$\sum_{i=1}^{n}A_i(\log A_{i+1}-\log A_i)=A_n+O\left(\sum_{i=1}^{n}\frac{d_i^2}{A_i}\right). \tag{2.23}$$

Proof. We have $\log(1+x)\sim x$ $(x\to 0)$ and hence we find that

$$\log A_{n+1}-\log A_n=\log\left(\frac{A_{n+1}}{A_n}\right)=\log\left(1+\frac{d_n}{A_n}\right)\sim\frac{d_n}{A_n}.$$

This proves equation (2.19). Equations (2.20) and (2.21) are immediate consequences of equation (2.19).

Equation (2.22) is an immediate consequence of equation (2.20) and Lemma 2.2.

Now, we shall prove equation (2.23). The logarithmic power series (2.10) gives

$$\log A_{n+1}-\log A_n=\frac{d_n}{A_n}-\frac{1}{2}\frac{d_n^2}{A_n^2}+\frac{1}{3}\frac{d_n^3}{A_n^3}-\frac{1}{4}\frac{d_n^4}{A_n^4}+\cdots \tag{2.24}$$

Note that

$$\left|-\frac{1}{2}\frac{d_n^2}{A_n^2}+\frac{1}{3}\frac{d_n^3}{A_n^3}-\frac{1}{4}\frac{d_n^4}{A_n^4}+\cdots\right|\le\frac{d_n^2}{A_n^2}\frac{1}{1-\frac{d_n}{A_n}}\le M\frac{d_n^2}{A_n^2}, \tag{2.25}$$

where M is a upper bound for the sequence $\frac{1}{1-\frac{d_n}{A_n}}$.

Equation (2.24) and equation (2.25) give

$$\log A_{n+1} - \log A_n = \frac{d_n}{A_n} + O\left(\frac{d_n^2}{A_n^2}\right),$$

that is,

$$A_n (\log A_{n+1} - \log A_n) = d_n + O\left(\frac{d_n^2}{A_n}\right).$$

Summing we obtain (2.23). The theorem is proved. □

Remark 2.10. In this remark we establish some observations about Theorem 2.9 and primes ($A_n = p_n$).

If the Riemann hypothesis is true is well-known the formula

$$d_n = O\left(p_n^{\frac{1}{2}} \log p_n\right). \tag{2.26}$$

On the other hand, the Cramér's conjecture establish,

$$d_n = O\left(\log^2 p_n\right). \tag{2.27}$$

Therefore if Cramér's conjecture is true then equation (1.6) becomes

$$p_n (\log p_{n+1} - \log p_n) = d_n + O\left(\frac{\log^4 n}{n \log n}\right) = d_n + o(1),$$

, that is,

$$d_n = p_n (\log p_{n+1} - \log p_n) + o(1). \tag{2.28}$$

It is well-known that if $\theta = \frac{6}{11}$ and $\epsilon > 0$, then $d_n < c p_n^{\theta+\epsilon}$ $(n \geq 1)$, where $c > 0$. Consequently we obtain

$$\sum_{i=1}^{n} \frac{d_i^2}{p_i} < \sum_{i=1}^{n} \frac{d_i}{p_i} c p_i^{\theta} < c p_n^{\theta} \sum_{i=1}^{n} \frac{d_i}{p_i} < c_1 p_n^{\theta} \log p_n$$

and hence equation (2.23) becomes

$$\sum_{i=1}^{n} p_i (\log p_{i+1} - \log p_i) = p_n + O\left(p_n^{\theta} \log p_n\right). \tag{2.29}$$

If the Riemann hypothesis is true (see (2.26)) equation (2.29) becomes

$$\sum_{i=1}^{n} p_i (\log p_{i+1} - \log p_i) = p_n + O\left(p_n^{\frac{1}{2}} \log^2 p_n\right). \tag{2.30}$$

If the Cramér's conjecture is true (see (2.27)) equation (2.29) becomes

$$\sum_{i=1}^{n} p_i (\log p_{i+1} - \log p_i) = p_n + O\left(\log^3 p_n\right). \tag{2.31}$$

It is well-known the equation [19]

$$\vartheta(x) = \sum_{p \leq x} \log p = x + O\left(\frac{x}{e^{c\sqrt{\log x}}}\right), \tag{2.32}$$

where c is a positive constant. If we put $x = p_n$ then (2.32) becomes

$$\sum_{i=1}^{n} \log p_i = p_n + O\left(\frac{p_n}{e^{c\sqrt{\log p_n}}}\right) = p_n + O\left(\frac{n}{\log^m n}\right). \tag{2.33}$$

If the Riemann hypothesis is true then equation (2.32) becomes [7]

$$\vartheta(x) = \sum_{p \le x} \log p = x + O\left(\sqrt{x}\log^2 x\right). \tag{2.34}$$

If we put $x = p_n$ then (2.34) becomes

$$\sum_{i=1}^{n} \log p_i = p_n + O\left(\sqrt{p_n}\log^2 p_n\right). \tag{2.35}$$

Note that if the Cramér's conjecture is true then the error term in equation (2.31) is much better that the error term in equation (2.35) if the Riemann hypothesis is true.

It is well-known that $\sum_{i=1}^{n} \log p_i$ has the same asymptotic Cipolla's expansion than p_n [12]. Equation (2.29) implies that also $\sum_{i=1}^{n} p_i(\log p_{i+1} - \log p_i)$ has the same asymptotic Cipolla's expansion than p_n. The Cipolla's asymptotic expansion for p_n is [2]

$$p_n = n\log n + n\log\log n - n + \cdots + o\left(\frac{n}{\log^m n}\right), \tag{2.36}$$

where m is a arbitrary fixed nonnegative integer. For example if $m = 0$ then the first Cipolla's formula is $p_n = n\log n + n\log\log n - n + o(n)$

Equation (2.31) can be improved if the Cramér's conjecture is true. If we use (2.24) and proceed as in Theorem 2.6 we find that

$$\sum_{i=1}^{n-1} p_i(\log p_{i+1} - \log p_i) = p_n - \frac{1}{2}\sum_{i=1}^{n-1}\frac{d_i^2}{p_i} + C + o(1), \tag{2.37}$$

where C is a constant. Note that the series $\sum_{i=1}^{\infty}\frac{d_i^2}{p_i}$ diverges, since $\sum_{i=1}^{\infty}\frac{1}{p_i}$ diverges. However the sum $\sum_{i=1}^{n-1}\frac{d_i^2}{p_i}$ increase slowly, since $\sum_{i=1}^{n-1}\frac{d_i^2}{p_i} = \sum_{i=1}^{n-1}\frac{d_i}{p_i}d_i < c_1\log^3 p_n$, where c_1 is a positive constant.

If we proceed as in Theorem 2.6, from equation (2.24) and equation (2.44) we obtain

$$\sum_{i=1}^{n}\frac{l_i}{d_i} = \sum_{i=1}^{n}\frac{\log p_{i+1} - \log p_i}{p_{i+1} - p_i} = \log\log p_n + D + o(1), \tag{2.38}$$

where D is a constant. Note that the series $\sum_{i=1}^{\infty}\frac{d_i^k}{p_i^{k+1}}$ $(k \ge 1)$converges, since $p_n < c_\epsilon p_n^{\theta+\epsilon}$, for $\theta = \frac{6}{11}$.

Equation (2.38) also gives

$$\sum_{i=1}^{\infty}\left(\frac{l_i}{d_i} - \frac{1}{p_i}\right) = \sum_{i=1}^{\infty}\left(\frac{\log p_{i+1} - \log p_i}{p_{i+1} - p_i} - \frac{1}{p_i}\right) = F_1, \tag{2.39}$$

where F_1 is a constant. Remember (see (1.4)) that we have $\frac{d_i}{l_i} \sim p_i$ and consequently $\frac{l_i}{d_i} \sim \frac{1}{p_i}$.

We have $d_i \sim p_i l_i$. From equation (2.24) we obtain

$$\sum_{i=1}^{n} \frac{p_i l_i}{d_i} = n - \frac{1}{2} \log p_n + F_2 + o(1), \tag{2.40}$$

where F_2 is a constant.

Note that equation (2.8) can be written in the form

$$\sum_{p_i \leq x} \frac{d_i}{p_i} = \log x - (\log x - \log p_n) + A + o(1) = \log x + A + o(1), \tag{2.41}$$

since $(\log p_{n+1} - \log p_n) \to 0$.

Abel summation gives

$$\sum_{p_i \leq x} \frac{d_i}{p_i \log p_i} = \log \log x + F_3 + o(1), \tag{2.42}$$

where F_3 is a constant, that is,

$$\sum_{i=1}^{n} \frac{d_i}{p_i \log p_i} = \log \log p_n + F_3 + o(1). \tag{2.43}$$

From equation (2.43) and the equation (see Theorem 1.1)

$$\sum_{i=1}^{n} \frac{1}{p_i} = \log \log p_n + M + o(1) \tag{2.44}$$

we find that the following series of terms with variable sign

$$\sum_{i=1}^{\infty} \frac{d_i - \log p_i}{p_i \log p_i} = \sum_{i=1}^{\infty} \frac{\frac{d_i}{\log p_i} - 1}{p_i} \tag{2.45}$$

converges.

Finally, we establish two questions:

Is the series $\sum_{i=1}^{\infty} \frac{1}{i d_i}$ convergent or divergent?

Is the series $\sum_{i=1}^{\infty} \frac{1}{p_i d_i}$ convergent or divergent?

Theorem 2.11. *Let us consider the integer sequence A_n, where $d_n = A_{n+1} - A_n$ and $l_n = \log A_{n+1} - \log A_n$. The following formula holds*

$$\sum_{i=1}^{n} \log A_i = \sum_{i=1}^{n} \log d_i + \sum_{i=1}^{n} (-\log l_i) + O\left(\log A_{n+1}\right). \tag{2.46}$$

Proof. If we apply the mean value theorem to the function $\log x$ then we obtain

$$\frac{1}{A_{i+1}} \left(A_{i+1} - A_i\right) < \log A_{i+1} - \log A_i < \frac{1}{A_i} \left(A_{i+1} - A_i\right),$$

that is,

$$\frac{1}{A_{i+1}} d_i < l_i < \frac{1}{A_i} d_i.$$

Therefore

$$\sum_{i=1}^{n} \log d_i - \sum_{i=1}^{n} \log A_{i+1} < \sum_{i=1}^{n} \log l_i < \sum_{i=1}^{n} \log d_i - \sum_{i=1}^{n} \log A_i.$$

Consequently we find the following two inequalities

$$\sum_{i=1}^{n} \log A_i < \sum_{i=1}^{n} \log d_i + \sum_{i=1}^{n} (-\log l_i),$$

$$\sum_{i=1}^{n} \log A_i > \sum_{i=1}^{n} \log d_i + \sum_{i=1}^{n} (-\log l_i) + \log A_1 - \log A_{n+1}.$$

Equation (2.46) is an immediate consequence of these two inequalities. The theorem is proved. □

Now, we apply Theorem 2.11 to prime numbers and obtain the following theorem.

Theorem 2.12. *The following formula holds*

$$p_n \sim \sum_{i=1}^{n} (-\log l_i) = \sum_{i=1}^{n} \left(-\log\log\left(\frac{p_{i+1}}{p_i}\right)\right) \sim \prod_{i=1}^{n} \frac{p_{i+1}}{p_i}. \tag{2.47}$$

Proof. If $A_n = p_n$ then equation (2.46) becomes

$$\sum_{i=1}^{n} \log p_i = \sum_{i=1}^{n} \log d_i + \sum_{i=1}^{n} (-\log l_i) + O\left(\log p_{n+1}\right). \tag{2.48}$$

Now, as it is well-known [6], we have (prime number theorem) $\sum_{p_i \le x} \log p_i = x + o(x)$. Therefore

$$\sum_{i=1}^{n} \log p_i = p_n + o(p_n). \tag{2.49}$$

Bertrand's postulate [6] $p_{i+1} < 2p_i$ $(i \ge 1)$ gives $d_i < p_i$ $(i \ge 1)$. On the other hand, it is well-known that in a set of 1 density we have $d_i < \log^2 i$ (see, for example, [14]). Hence

$$\begin{aligned} \sum_{i=1}^{n} \log d_i &\le \sum_{i=3}^{n} \log\left(\log^2 i\right) + o(n)\log p_n \\ &\le 2n\log\log n + o(n\log p_n) = o(n\log n) = o(p_n). \end{aligned} \tag{2.50}$$

Equations (2.48), (2.49) and (2.50) give equation (2.47). Note the asymptotic formula

$$\sum_{i=1}^{n} \left(-\log\log\left(\frac{p_{i+1}}{p_i}\right)\right) \sim \prod_{i=1}^{n} \frac{p_{i+1}}{p_i}.$$

The theorem is proved. □

In the following theorem we obtain a more precise formula than equation (2.46).

Theorem 2.13. *The following asymptotic formulas hold*

$$\sum_{i=1}^{n} \log A_i = \sum_{i=1}^{n} \log d_i + \sum_{i=1}^{n} (-\log l_i) - \frac{1}{2}\log A_n + C + o(1), \tag{2.51}$$

where C is a constant.

$$\left(\prod_{i=1}^{n} A_i\right) \sim \frac{e^C}{\sqrt{A_n}} \prod_{i=1}^{n} \frac{d_i}{l_i}. \tag{2.52}$$

Proof. We put

$$x_i = -\frac{1}{2}\frac{d_i}{A_i} + \frac{1}{3}\frac{d_i^2}{A_i^2} - \frac{1}{4}\frac{d_i^3}{A_i^3} + \frac{1}{5}\frac{d_i^4}{A_i^4} - \cdots \tag{2.53}$$

The series (2.53) is absolutely convergent since

$$\begin{aligned} |x_i| &= \left| -\frac{1}{2}\frac{d_i}{A_i} + \frac{1}{3}\frac{d_i^2}{A_i^2} - \frac{1}{4}\frac{d_i^3}{A_i^3} + \frac{1}{5}\frac{d_i^4}{A_i^4} - \cdots \right| \\ &\leq \frac{d_i}{A_i} + \frac{d_i^2}{A_i^2} + \frac{d_i^3}{A_i^3} + \frac{d_i^4}{A_i^4} + \cdots = \frac{d_i}{A_i}\frac{1}{1-\frac{d_i}{A_i}} < 1 \qquad (i \geq n_0). \end{aligned} \tag{2.54}$$

Note that we have proved also that $|x_i| \to 0$ as $i \to \infty$ and consequently there exists n_0 such that if $i \geq n_0$ then $|x_i| < 1$. Remember that $\frac{d_i}{A_i} \to 0$ as $i \to \infty$.

Now, equation (2.8) and the same proof as the proof of Theorem 2.6 gives

$$\sum_{i=n_0}^{n} x_i = -\frac{1}{2}\log A_n + c_1 + o(1), \tag{2.55}$$

where c_1 is a constant.

If $k \geq 2$ then the series $\sum_{i=n_0}^{\infty} x_i^k$ is absolutely convergent since (see (2.54))

$$\sum_{i=n_0}^{\infty} |x_i|^k \leq \sum_{i=n_0}^{\infty} \left(\frac{d_i}{A_i}\frac{1}{1-\frac{d_i}{A_i}}\right)^k = \sum_{i=n_0}^{\infty} \frac{d_i^k}{A_i^k}\left(\frac{1}{1-\frac{d_i}{A_i}}\right)^k, \tag{2.56}$$

where the last series converges since we assume that the series $\sum_{i=n_0}^{\infty} \frac{d_i^2}{A_i^2}$ converges (see Theorem 2.1).

Note that

$$\begin{aligned} &\frac{1}{2}\sum_{i=n_0}^{\infty} |x_i|^2 + \frac{1}{3}\sum_{i=n_0}^{\infty} |x_i|^3 + \frac{1}{4}\sum_{i=n_0}^{\infty} |x_i|^4 + \cdots \\ \leq &\sum_{i=n_0}^{\infty} \left(|x_i|^2 + |x_i|^3 + |x_i|^4 + \cdots\right) = \sum_{i=n_0}^{\infty} |x_i|^2 \frac{1}{1-|x_i|}, \end{aligned} \tag{2.57}$$

where the last series converges (see equation (2.56) with $k = 2$).

We recall the logarithmic series

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \qquad (|x| < 1).$$

We have

$$\log A_{i+1} - \log A_i = \log\left(1 + \frac{d_i}{A_i}\right) = \frac{d_i}{A_i} - \frac{1}{2}\frac{d_i^2}{A_i^2} + \frac{1}{3}\frac{d_i^3}{A_i^3} - \frac{1}{4}\frac{d_i^4}{A_i^4} + \cdots,$$

that is,

$$\log A_i + \log l_i = \log d_i + \log(1 + x_i). \tag{2.58}$$

Equation (2.58) and the logarithmic series give

$$\begin{aligned}
&\sum_{i=1}^{n} \log A_i = \sum_{i=1}^{n} \log d_i + \sum_{i=1}^{n} (-\log l_i) + \sum_{i=1}^{n} \log(1 + x_i) \\
= \;& \sum_{i=1}^{n} \log d_i + \sum_{i=1}^{n} (-\log l_i) + \sum_{i=1}^{n_0 - 1} \log(1 + x_i) \\
+ \;& \sum_{i=n_0}^{n} x_i - \frac{1}{2}\sum_{i=n_0}^{n} x_i^2 + \frac{1}{3}\sum_{i=n_0}^{n} x_i^3 - \frac{1}{4}\sum_{i=n_0}^{n} x_i^4 + \cdots \\
= \;& \sum_{i=1}^{n} \log d_i + \sum_{i=1}^{n} (-\log l_i) - \frac{1}{2}\log A_n + C + o(1),
\end{aligned}$$

that is, equation (2.51), where the constant C is

$$C = c_1 + \sum_{i=1}^{n_0 - 1} \log(1 + x_i) - \frac{1}{2}\sum_{i=n_0}^{\infty} x_i^2 + \frac{1}{3}\sum_{i=n_0}^{\infty} x_i^3 - \frac{1}{4}\sum_{i=n_0}^{\infty} x_i^4 + \cdots \tag{2.59}$$

Note that the series in equation (2.59), namely

$$-\frac{1}{2}\sum_{i=n_0}^{\infty} x_i^2 + \frac{1}{3}\sum_{i=n_0}^{\infty} x_i^3 - \frac{1}{4}\sum_{i=n_0}^{\infty} x_i^4 + \cdots \tag{2.60}$$

converges absolutely (see equation (2.57))

On the other hand, the absolute value of the difference between series (2.60) and the sum

$$-\frac{1}{2}\sum_{i=n_0}^{n} x_i^2 + \frac{1}{3}\sum_{i=n_0}^{n} x_i^3 - \frac{1}{4}\sum_{i=n_0}^{n} x_i^4 + \cdots$$

does not exceed

$$\begin{aligned}
&\frac{1}{2}\sum_{i=n}^{\infty} |x_i|^2 + \frac{1}{3}\sum_{i=n}^{\infty} |x_i|^3 + \frac{1}{4}\sum_{i=n}^{\infty} |x_i|^4 + \cdots \\
\leq \;& \sum_{i=n}^{\infty} \left(|x_i|^2 + |x_i|^3 + |x_i|^4 + \cdots\right) = \sum_{i=n}^{\infty} |x_i|^2 \frac{1}{1 - |x_i|} \to 0 \quad (n \to \infty)
\end{aligned}$$

see equation (2.57). The theorem is proved. □

Remark 2.14. If we apply the more precise Theorem 2.13 to prime numbers then we obtain

$$\sum_{i=1}^{n} \log p_i = \sum_{i=1}^{n} \log d_i + \sum_{i=1}^{n} (-\log l_i) - \frac{1}{2} \log p_n + C + o(1), \tag{2.61}$$

where C is a constant.

$$\left(\prod_{i=1}^{n} p_i \right) \sim \frac{e^C}{\sqrt{p_n}} \prod_{i=1}^{n} \frac{d_i}{l_i}.$$

The product $\prod_{i=1}^{n} p_i$ is called primorial.

It is well-known the formula [19]

$$\sum_{p_i \leq x} \log p_i = x + O\left(\frac{x}{e^{c\sqrt{\log x}}} \right), \tag{2.62}$$

where c is a positive constant. Equation (2.62) gives

$$\sum_{i=1}^{n} \log p_i = p_n + O\left(\frac{p_n}{e^{c\sqrt{\log p_n}}} \right) = p_n + o\left(\frac{n}{\log^m n} \right) \tag{2.63}$$

for all nonnegative integer m. Equation (2.63) gives by difference the weak formula

$$d_n = p_{n+1} - p_n = o\left(\frac{n}{\log^m n} \right)$$

for all nonnegative integer m.

We wish know what happen with the error term in equation (2.63) if we substitute $\log p_i$ by $\frac{p_i}{i} \sim \log p_i$ and $\frac{1}{i} \frac{d_i}{l_i} \sim \log p_i$ (see equation (1.4)). The first Cipolla's formula for p_n is $p_n = n \log n + n \log \log n - n + o(n)$ and for $\log p_n$ is $\log p_n = \log n + \log \log n + o(1)$. Consequently we obtain

$$\frac{p_i}{i} = \log p_i - 1 + o(1)$$

and therefore (see equation (2.63))

$$\sum_{i=1}^{n} \frac{p_i}{i} = \sum_{i=1}^{n} \log p_i - n + o(n) = p_n - n + o(n).$$

On the other hand, we have the formula (see equation (1.8)) $p_i = \frac{d_i}{l_i} + O(d_i)$ and consequently

$$\frac{p_i}{i} = \frac{1}{i} \frac{d_i}{l_i} + O\left(\frac{d_i}{i} \right) = \frac{1}{i} \frac{d_i}{l_i} + o(1).$$

Hence we obtain the formula

$$\sum_{i=1}^{n} \frac{1}{i} \frac{d_i}{l_i} = \sum_{i=1}^{n} \frac{p_i}{i} + o(n) = p_n - n + o(n).$$

Substituting (2.63) into (2.61) we obtain the equation

$$p_n = \sum_{i=1}^{n} \log d_i + \sum_{i=1}^{n} (-\log l_i) + o\left(\frac{n}{\log^m n}\right) = \sum_{i=1}^{n} \log\left(\frac{d_i}{l_i}\right) + o\left(\frac{n}{\log^m n}\right).$$

Observe then that the sum

$$\sum_{i=1}^{n} \log\left(\frac{d_i}{l_i}\right)$$

has the same Cipolla's asymptotic expansion as p_n, that is

$$\sum_{i=1}^{n} \log\left(\frac{d_i}{l_i}\right) = n\log n + n\log\log n - n + \cdots + o\left(\frac{n}{\log^m n}\right).$$

For example, since the first Cipolla's formula ($m = 0$) is $p_n = n\log n + n\log\log n - n + o(n)$, then we also have

$$\sum_{i=1}^{n} \log\left(\frac{d_i}{l_i}\right) = n\log n + n\log\log n - n + o(n).$$

Definition 2.15. Let us consider a continuous function $f(x) > 0$ defined in an interval $[a, \infty]$ such that $\lim_{x\to\infty} f(x) = \infty$ and with continuous derivative $f'(x) > 0$. The function $f(x)$ is of slow increase if and only if the following limit holds

$$\lim_{x\to\infty} \frac{f'(x)}{\frac{f(x)}{x}} = \lim_{x\to\infty} \frac{xf'(x)}{f(x)} = 0. \tag{2.64}$$

Typical functions of slow increase are $\log x$, $\log\log x$, $\frac{\log x}{(\log\log x)^k}$ (where k is a positive integer), etc.

The functions of slow increase are studied in [13]. A property of the functions of slow increase is the following limit

$$\lim_{x\to\infty} \frac{f(x)}{x^\beta} = 0$$

for all $\beta > 0$.

The product of functions of slow increase is again a function of slow increase, if $f(x)$ is a function of slow increase then $\lim_{x\to\infty} \frac{f(x+c)}{f(x)} = 1$, where c is a constant and if $f(x)$ is a function of slow increase then $\lim_{x\to\infty} \frac{f(cx)}{f(x)} = 1$, where $c > 0$ is a constant. (see [13])

Theorem 2.16. *Given the sequence A_n, suppose that $g(x)$ is a function of slow increase. Then we have*

$$\sum_{A_i \le x} \frac{d_i}{g(A_i)} \sim \frac{x}{g(x)}. \tag{2.65}$$

Proof. Since $\frac{d_n}{A_n} \to 0$ we obtain

$$\sum_{A_i \le x} d_i = x + o(x).$$

Therefore Abel summation and the definition of function of slow increase give

$$\sum_{A_i \le x} \frac{d_i}{g(A_i)} = (x + o(x)) \frac{1}{g(x)} + \int_{A_1}^{x} (t + o(t)) \frac{g'(t)}{g(t)^2} \, dt$$

$$= \frac{x}{g(x)} + o\left(\frac{x}{g(x)}\right) + \int_{A_1}^{x} o\left(\frac{1}{g(t)}\right) \, dt$$

$$= \frac{x}{g(x)} + o\left(\frac{x}{g(x)}\right) + o\left(\int_{A_1}^{x} \frac{1}{g(t)} \, dt\right)$$

$$= \frac{x}{g(x)} + o\left(\frac{x}{g(x)}\right),$$

since (L'Hospital's rule)

$$\int_{A_1}^{x} \frac{1}{g(t)} \, dt \sim \frac{x}{g(x)}.$$

The theorem is proved. □

Theorem 2.17. *Given the sequence A_n, suppose that $A_n \sim nf(n)$, where $f(x)$ is a function of slow increase, and suppose that $f(A_n) \sim f(n)$. If the counting function of the sequence is $A(x) = \sum_{A_i \le x} 1$ then $A(x) \sim \frac{x}{f(x)}$. For example, the sequence of primes $A_n = p_n$ satisfies these conditions with $f(x) = \log x$, in this case, $A(x) = \pi(x)$.*

Then if $k \ge 1$ is a positive integer we have

$$\sum_{A_i \le x} \frac{d_i}{f(i)^k} \sim \sum_{A_i \le x} \frac{d_i}{f(A_i)^k} \sim \frac{x}{f(x)^k} \tag{2.66}$$

and

$$\sum_{i=1}^{n} \frac{d_i}{f(i)^k} \sim \frac{n}{f(n)^{k-1}}. \tag{2.67}$$

In particular if $k = 1$ we obtain

$$\sum_{A_i \le x} \frac{d_i}{f(i)} \sim \sum_{A_i \le x} \frac{d_i}{f(A_i)} \sim A(x) \tag{2.68}$$

and

$$\sum_{i=1}^{n} \frac{d_i}{f(i)} \sim n. \tag{2.69}$$

We also have

$$\sum_{i=1}^{n} \frac{i d_i}{A(i)} \sim n. \tag{2.70}$$

Proof. We have $A_n \sim nf(n) \sim nf(A_n)$ and consequently $A(A_n) = n \sim \frac{A_n}{f(A_n)}$. Therefore if $x \in [A_n, A_{n+1})$ we have $A(x) = n$. The function $\frac{x}{f(x)}$ is strictly increasing since its derivative is positive (use equation (2.64)). Hence

$$1 \leftarrow \frac{n}{\frac{A_n}{f(A_n)}} \frac{A_n}{A_{n+1}} \frac{f(A_{n+1})}{f(A_n)} = \frac{n}{\frac{A_{n+1}}{f(A_{n+1})}} \le \frac{A(x)}{\frac{x}{f(x)}} \le \frac{n}{\frac{A_n}{f(A_n)}} \to 1,$$

since $f(A_n) \sim f(n)$. In definitive, we have proved that $A(x) \sim \frac{x}{f(x)}$.

Equation (2.66) is an immediate consequence of equation (2.65), Lemma 2.2 and the fact that $f(x)^k$ is also a function of slow increase. If we put $x = A_n$ in equation (2.66) we obtain equation (2.67). Equation (2.70) is an immediate consequence of equation (2.69) and Lemma 2.2, since $\frac{id_i}{if(i)} \sim \frac{id_i}{A_i}$. The theorem is proved. □

It is well-known [16] that the counting function $\pi(x)$ of the sequence p_n of primes has the following asymptotic expansion.

$$\pi(x) = \left(\sum_{h=1}^{m} \frac{(h-1)!x}{\log^h x} \right) + o\left(\frac{x}{\log^m x} \right) \quad (m \ge 1). \tag{2.71}$$

In the following theorem we obtain, for the sequence of primes, a more precise result that the former theorem.

Theorem 2.18. *Let us consider the sequence p_n of primes, where (as usual) $\pi(x)$ is its counting function. We have*

$$\sum_{p_i \le x} \frac{d_i}{\log p_i} = \pi(x) + O\left(\frac{x}{\log^m x} \right) \tag{2.72}$$

for all positive integer m.

We have

$$\sum_{p_i \le x} \frac{d_i}{\log p_i} = \left(\sum_{h=1}^{m} \frac{(h-1)!x}{\log^h x} \right) + o\left(\frac{x}{\log^m x} \right). \tag{2.73}$$

We have

$$\sum_{i=1}^{n} \frac{d_i}{\log p_i} = n + O\left(\frac{n}{\log^m n} \right) \tag{2.74}$$

for all positive integer m.

We have

$$\sum_{i=1}^{n} \frac{id_i}{p_i} = n + \frac{n}{\log n} + o\left(\frac{n}{\log n} \right). \tag{2.75}$$

Proof. We have $d_n < dp_n^{\theta+\epsilon}$ $(n \ge 1)$, where $d > 0$ is a constant and $\theta = \frac{6}{11}$. Therefore we obtain

$$\sum_{p_i \le x} d_i = x + O(x^{\theta+\epsilon}). \tag{2.76}$$

On the other hand we have

$$\vartheta(x) = \sum_{p_i \le x} \log p_i = x + O\left(\frac{x}{e^{c\sqrt{\log x}}} \right) = x + O\left(\frac{x}{\log^m x} \right) \tag{2.77}$$

for all positive integer m, where c is a positive constant.

Equations (2.76) and (2.77) give

$$\sum_{p_i \leq x} (d_i - \log p_i) = O\left(\frac{x}{\log^m x}\right) \tag{2.78}$$

for all positive integer m.

Equation (2.78) and Abel summation give

$$\sum_{p_i \leq x} \frac{d_i - \log p_i}{\log p_i} = O\left(\frac{x}{\log^{m+1} x}\right) \tag{2.79}$$

for all positive integer m.

Equation (2.79) gives

$$\sum_{p_i \leq x} \frac{d_i}{\log p_i} = \sum_{p_i \leq x} 1 + \sum_{p_i \leq x} \frac{d_i - \log p_i}{\log p_i} = \pi(x) + O\left(\frac{x}{\log^m x}\right) \tag{2.80}$$

for all positive integer m, that is, equation (2.72).

Equation (2.72) and equation (2.71) give equation (2.73).

Therefore the function $\sum_{p_i \leq x} \frac{d_i}{\log p_i}$ has the same asymptotic expansion as $\pi(x)$.

Equation (2.74) is an immediate consequence of equation (2.72) if we put $x = p_n$ and use the prime number theorem $p_n \sim n \log n$.

The following formulas are well-known [2]

$$p_i = i \log i + i \log \log i - i + o(i), \quad \log p_i = \log i + \log \log i + o(1) \tag{2.81}$$

We have (see equation (2.74) with $m = 1$, equation (2.81), Lemma 2.2 and equation (2.65))

$$\begin{aligned}
\sum_{i=1}^{n} \frac{id_i}{p_i} &= \sum_{i=1}^{n} \frac{d_i}{\log p_i} + \sum_{i=1}^{n} \left(\frac{id_i}{p_i} - \frac{id_i}{i \log p_i}\right) = n + o\left(\frac{n}{\log n}\right) \\
&+ \sum_{i=1}^{n} id_i \left(\frac{i \log p_i - p_i}{ip_i \log p_i}\right) = n + o\left(\frac{n}{\log n}\right) \\
&+ \sum_{i=1}^{n} (1 + o(1)) \frac{id_i}{p_i \log p_i} = n + o\left(\frac{n}{\log n}\right) + \sum_{i=1}^{n} (1 + o(1)) \frac{d_i}{\log^2 p_i} \\
&= n + o\left(\frac{n}{\log n}\right) + (1 + o(1)) \sum_{i=1}^{n} \frac{d_i}{\log^2 p_i} \\
&= n + o\left(\frac{n}{\log n}\right) + (1 + o(1)) \frac{p_n}{\log^2 p_n} = n + \frac{n}{\log n} + o\left(\frac{n}{\log n}\right),
\end{aligned}$$

that is, equation (2.75). The theorem is proved. □

Remark 2.19. Note that a weak consequence of equation (2.72) is

$$\sum_{p_i \leq x} \frac{d_i}{\log p_i} \sim \pi(x) \tag{2.82}$$

and if $x = p_n$ then we obtain

$$\sum_{i=1}^{n} \frac{d_i}{\log i} \sim \sum_{i=1}^{n} \frac{d_i}{\log p_i} \sim n. \tag{2.83}$$

Now, Lemma 2.2 and equation $p_i l_i \sim d_i$ (see (1.4)) give

$$\sum_{p_i \le x} \frac{l_i}{\frac{\log p_i}{p_i}} = \sum_{p_i \le x} \frac{p_i l_i}{\log p_i} \sim \sum_{p_i \le x} \frac{d_i}{\log p_i} \sim \pi(x),$$

that is,

$$\sum_{p_i \le x} \frac{l_i}{\frac{\log p_i}{p_i}} \sim \pi(x). \tag{2.84}$$

Therefore

$$\sum_{i=1}^{n} \frac{l_i}{\frac{1}{i}} \sim \sum_{i=1}^{n} \frac{l_i}{\frac{\log p_i}{p_i}} \sim n. \tag{2.85}$$

From equations (2.82) and (2.84) we shall obtain, in the following theorem, some information about d_n and l_n.

Theorem 2.20. *1) Let $\epsilon > 0$ an arbitrary fixed small number and $M > 1 + \epsilon$. Let $D_M(x)$ be the number of primes p_i not exceeding x such that $d_i < M \log p_i$. Then $D_M(x) \ge \left(1 - \frac{1+\epsilon}{M}\right)\pi(x)$. Note that if M is large then $1 - \frac{1+\epsilon}{M}$ is near of 1.*

2) Let $\epsilon > 0$ an arbitrary fixed small number and $M > 1 + \epsilon$. Let $L_M(x)$ be the number of primes p_i not exceeding x such that $l_i < M \frac{\log p_i}{p_i}$. Then $L_M(x) \ge \left(1 - \frac{1+\epsilon}{M}\right)\pi(x)$. Note that if M is large then $1 - \frac{1+\epsilon}{M}$ is near of 1.

Proof. We shall prove 1). The proof of 2) is the same. Given $\epsilon > 0$, let $C_M(x)$ be the number of primes p_i not exceeding x such that $\frac{d_i}{\log p_i} \ge M > 1 + \epsilon$. Then from a certain value of x depending of ϵ denoted x_ϵ we have $C_M(x) \le \frac{1+\epsilon}{M}\pi(x)$ $(x \ge x_\epsilon)$. Suppose that this is not true, then there exist a sequence of values of $x \to \infty$ such that $C_M(x) > \frac{1+\epsilon}{M}\pi(x)$, but this is impossible since (see equation (2.82))

$$\pi(x) \sim \sum_{p_i \le x} \frac{d_i}{\log p_i} \ge M C_M(x) > M \frac{1+\epsilon}{M}\pi(x) = (1+\epsilon)\pi(x).$$

Therefore

$$C_M(x) \le \frac{1+\epsilon}{M}\pi(x) \qquad (x \ge x_\epsilon), \tag{2.86}$$

as we desired. Note that if M is large then $\frac{1+\epsilon}{M}$ is near of zero. Equation (2.86) gives

$$D_M(x) = \pi(x) - C_M(x) \ge \pi(x) - \frac{1+\epsilon}{M}\pi(x) = \left(1 - \frac{1+\epsilon}{M}\right)\pi(x) \quad (x \ge x_\epsilon).$$

The theorem is proved. $\square$

Theorem 2.21. *Let $\epsilon > 0$ an arbitrary fixed small number and $\alpha > 0$ a arbitrary small number . Let $D_\alpha(x)$ be the number of primes p_i not exceeding x such that $d_i < \alpha h(p_i) \log p_i$, where $h(x)$ is a function of slow increase (for example $\log x$, $\log\log x$, $\log\log\log x$, etc). Then $D_\alpha(x) \geq \left(1 - \frac{1+\epsilon}{\alpha h(x)}\right)\pi(x)$. A weaker consequence is $D_\alpha(x) \sim \pi(x)$. Therefore almost all primes not exceeding x satisfy $d_i < \alpha h(p_i) \log p_i$, where α can be arbitrarily small.*

Proof. Suppose that $g(x)$ is a function of slow increase. By equation (2.65) we have

$$\sum_{p_i \leq x} \frac{d_i}{g(p_i)} \sim \frac{x}{g(x)}. \tag{2.87}$$

If we put $g(x) = h(x)\log x$, where $h(x)$ is a function of slow increase, then Lemma 2.2 gives

$$\sum_{p_i \leq x} \frac{d_i}{h(p_i)\log p_i} \sim \frac{x}{h(x)\log x} \sim \frac{\pi(x)}{h(x)}. \tag{2.88}$$

Given $\epsilon > 0$, let $C_\alpha(x)$ be the number of primes p_i not exceeding x such that $\frac{d_i}{\log p_i h(p_i)} \geq \alpha$. Then from a certain value of x depending of ϵ denoted x_ϵ we have $C_\alpha(x) \leq \frac{1+\epsilon}{\alpha h(x)}\pi(x)$ $(x \geq x_\epsilon)$. Suppose that this is not true, then there exist a sequence of values of $x \to \infty$ such that $C_\alpha(x) > \frac{1+\epsilon}{\alpha h(x)}\pi(x)$, but this is impossible since (see equation (2.88))

$$\frac{\pi(x)}{h(x)} \sim \sum_{p_i \leq x} \frac{d_i}{h(p_i)\log p_i} \geq \alpha C_\alpha(x) > \alpha \frac{1+\epsilon}{\alpha h(x)}\pi(x) = (1+\epsilon)\frac{\pi(x)}{h(x)}.$$

Therefore

$$C_\alpha(x) \leq \frac{1+\epsilon}{\alpha h(x)}\pi(x) \qquad (x \geq x_\epsilon), \tag{2.89}$$

as we desired. Note that $C_\alpha(x) = o(\pi(x))$ is a weak consequence of (2.89). Equation (2.89) gives

$$D_\alpha(x) = \pi(x) - C_\alpha(x) \geq \pi(x) - \frac{1+\epsilon}{\alpha h(x)}\pi(x) = \left(1 - \frac{1+\epsilon}{\alpha h(x)}\right)\pi(x) \quad (x \geq x_\epsilon).$$

The theorem is proved. □

Corollary 2.22. *Let $\epsilon > 0$ an arbitrary fixed small number and $\alpha > 0$ a arbitrary small number . Let $D_\alpha(x)$ be the number of primes p_i not exceeding x such that $d_i < \alpha \log^2 p_i$. Then $D_\alpha(x) \geq \left(1 - \frac{1+\epsilon}{\alpha \log x}\right)\pi(x)$. A weaker consequence is $D_\alpha(x) \sim \pi(x)$. Therefore almost all primes not exceeding x satisfy $d_i < \alpha \log^2 p_i$, where α can be arbitrarily small.*

Proof. By equation (2.65) we have

$$\sum_{p_i \leq x} \frac{d_i}{\log^2 p_i} \sim \frac{x}{\log^2 x} \sim \frac{1}{\log x}\pi(x).$$

From here the proof is the same as the proof of Theorem 2.21. The corollary is proved. □

Theorem 2.23. *Let us consider the sequence A_n. We have*

$$\sum_{i=1}^{\infty} \frac{d_i}{A_i A_{i+1}} = \frac{1}{A_1}. \tag{2.90}$$

The following series converges

$$\sum_{i=1}^{\infty} \frac{d_i}{A_i^2}. \tag{2.91}$$

If $A_n = p_n$ then the following series converges

$$\sum_{i=1}^{\infty} \frac{\frac{d_i}{\log^2 p_i}}{i^2}. \tag{2.92}$$

We have

$$\sum_{i=1}^{n} \frac{d_i}{i \log^2 i} \sim \sum_{i=1}^{n} \frac{\frac{d_i}{\log^2 p_i}}{i} \sim \sum_{i=1}^{n} \frac{\frac{d_i}{\log p_i}}{i \log i} \sim \sum_{i=1}^{n} \frac{d_i}{p_i \log p_i} \sim \log\log n. \tag{2.93}$$

Proof. We have

$$\sum_{i=1}^{n} \frac{d_i}{A_i A_{i+1}} = \sum_{i=1}^{n} \left(\frac{1}{A_i} - \frac{1}{A_{i+1}} \right) = \frac{1}{A_1} - \frac{1}{A_{n+1}}.$$

This proves equation (2.90). Equation (2.91) is an immediate consequence of equation (2.90) since $\frac{A_{i+1}}{A_i} \to 1$. Equation (2.92) is an immediate consequence of equation (2.91) and the prime number theorem $p_i \sim i \log p_i$.

From $\sum_{p_i \le x} d_i = x + O(x^{\theta+\epsilon})$ ($\theta = \frac{6}{11}$) and Abel summation we find that $\sum_{p_i \le x} \frac{d_i}{p_i \log p_i} = \log\log x + c + o(1)$, where c is a constant. The theorem is proved. □

Now, we suppose $A_1 \neq 1$ and study the sequence $\log A_n$. For sake of simplicity we put $l_n = \log A_{n+1} - \log A_n$.

Theorem 2.24. *The following formulas hold*

$$l_n = (\log A_{n+1} - \log A_n) \to 0, \tag{2.94}$$

$$\frac{l_n}{\log A_n} = \frac{\log A_{n+1} - \log A_n}{\log A_n} \to 0, \tag{2.95}$$

$$\frac{\log A_{n+1}}{\log A_n} \to 1. \tag{2.96}$$

Proof. Since $\frac{A_{n+1}}{A_n} \to 1$ we obtain

$$l_n = (\log A_{n+1} - \log A_n) = \log\left(\frac{A_{n+1}}{A_n}\right) \to 0.$$

Equation (2.94) is proved. Equation (2.95) is an immediate consequence of equation (2.94).

Finally, we have

$$\frac{\log A_{n+1}}{\log A_n} = \left(1 + \frac{l_n}{\log A_n}\right) \to 1,$$

that is, equation (2.96). The theorem is proved. □

Theorem 2.25. *The following asymptotic formula holds*

$$\sum_{i=1}^{n} \frac{l_i}{\log A_i} = \sum_{i=1}^{n} \frac{\log A_{i+1} - \log A_i}{\log A_i} \sim \log\log A_n. \tag{2.97}$$

Proof. The proof is the same as the proof of Theorem 2.3 substituting d_i by l_i and A_i by $\log A_i$. The theorem is proved. □

Theorem 2.26. *The series*

$$\sum_{i=1}^{\infty} \frac{l_i^2}{\log^2 A_i} = \sum_{i=1}^{\infty} \frac{(\log A_{i+1} - \log A_i)^2}{\log^2 A_i} \tag{2.98}$$

converges.

Proof. Let M be an upper bound for the sequence l_n. Then by Theorem 2.25 we have

$$\sum_{A_i \leq A_n} \frac{l_i^2}{\log A_i} = \sum_{i=1}^{n} \frac{l_i^2}{\log A_i} = \sum_{i=1}^{n} \frac{l_i}{\log A_i} l_i \leq 2M \log\log A_n$$

and consequently

$$\sum_{A_i \leq x} \frac{l_i^2}{\log A_i} \leq 2M \log\log x \qquad (x \geq x_0).$$

Now, Abel summation gives

$$\begin{aligned}
\sum_{A_i \leq x} \frac{l_i^2}{\log^2 A_i} &= \left(\sum_{A_i \leq x} \frac{l_i^2}{\log A_i}\right) \frac{1}{\log x} + \int_{A_1}^{x} \left(\sum_{A_i \leq t} \frac{l_i^2}{\log A_i}\right) \frac{1}{t \log^2 t}\, dt \\
&\leq 1 + \int_{A_1}^{x_0} \left(\sum_{A_i \leq t} \frac{l_i^2}{\log A_i}\right) \frac{1}{t \log^2 t}\, dt + 2M \int_{x_0}^{\infty} \frac{\log\log t}{t \log^2 t}\, dt.
\end{aligned}$$

The theorem is proved. □

Theorem 2.27. *Suppose that* $\frac{l_n}{\log A_n} < 1$ $(n \geq 1)$*. Then the following asymptotic formulas hold*

$$\sum_{i=1}^{n} \frac{l_i}{\log A_i} = \log\log A_n + A' + o(1), \tag{2.99}$$

where A' *is a constant.*

$$\sum_{i=1}^{n} \left(\frac{\log A_{i+1}}{\log A_i}\right)^k = n + k \log\log A_n + C'_k + o(1), \tag{2.100}$$

where C'_k are constants

$$\sum_{i=1}^{n} \frac{\log^k A_{i+1} - \log^k A_i}{\log^k A_i} = k \log\log A_n + C'_k + o(1), \tag{2.101}$$

$$\sum_{i=1}^{n} \frac{l_i}{\log A_{i+1}} = \log\log A_n + B' + o(1), \tag{2.102}$$

where B' is a constant. Compare with equation (2.99).

$$\sum_{i=1}^{n} \left(\frac{\log A_i}{\log A_{i+1}} \right)^k = n - k \log\log A_n + D'_k + o(1), \tag{2.103}$$

where D'_k are constants

$$\sum_{i=1}^{n} \frac{\log^k A_{i+1} - \log^k A_i}{\log^k A_{i+1}} = k \log\log A_n - D'_k + o(1). \tag{2.104}$$

Proof. The proof is the same as the proof of Theorem 2.6 and Theorem 2.8 substituting d_i by l_i and A_i by $\log A_i$. The theorem is proved. □

Remark 2.28. The sequence $A_n = p_n$ of primes satisfies these theorems on $\log A_n$, since $\frac{l_n}{\log p_n} = \frac{\log p_{n+1} - \log p_n}{\log p_n} < 1$ $(n \geq 1)$, as it can be proved from Bertrand's postulate $p_{n+1} < 2p_n$ $(n \geq 1)$.

It is well-known the formula

$$\sum_{p_i \leq x} \frac{1}{p_i} = \log\log x + M + o(1),$$

where M is Mertens's constant. If we put $x = p_n$ then we obtain

$$\sum_{i=1}^{n} \frac{1}{p_i} = \sum_{i=1}^{n} \frac{\frac{\log p_i}{p_i}}{\log p_i} = \log\log p_n + M + o(1). \tag{2.105}$$

Compare equations (2.105) and (2.99). If we add both equations then we obtain

$$\sum_{i=1}^{\infty} \frac{l_i - \frac{\log p_i}{p_i}}{\log p_i} = A' - M. \tag{2.106}$$

Note that $\frac{\log p_i}{p_i} \sim \frac{1}{i}$.

Theorem 2.29. *Let us consider an integer sequence A_n. Let $A(x)$ be the counting function of the sequence and let $f(x)$ be a function of slow increase. Then*

$$A(x) \sim \frac{x}{f(x)} \Leftrightarrow \sum_{A_i \leq x} f(A_i) = x + o(x). \tag{2.107}$$

Furthermore

$$\sum_{i=1}^{n} \frac{f(A_i)}{A_i} \sim \log A_n. \tag{2.108}$$

Proof. If $A(x) \sim \frac{x}{f(x)}$ then Abel summation gives

$$\sum_{A_i \leq x} f(A_i) = \left(\frac{x}{f(x)}(1+o(1))\right) f(x) - \int_{A_1}^{x} \frac{t}{f(t)}(1+o(1)) f'(t)\ dt$$
$$= x + o(x) + \int_{A_1}^{x} o(1)\ dt = x + o(x) + o\left(\int_{A_1}^{x} dt\right) = x + o(x).$$

If $\sum_{A_i \leq x} f(A_i) = x + o(x)$ then Abel summation gives

$$\sum_{A_i \leq x} 1 = (x + o(x))\frac{1}{f(x)} + \int_{A_1}^{x} (t + o(t))\frac{f'(t)}{f(t)^2}\ dt$$
$$= \frac{x}{f(x)} + o\left(\frac{x}{f(x)}\right) + \int_{A_1}^{x} o(1)\frac{1}{f(t)}\ dt$$
$$= \frac{x}{f(x)} + o\left(\frac{x}{f(x)}\right) + o\left(\int_{A_1}^{x} \frac{1}{f(t)}\ dt\right) = \frac{x}{f(x)} + o\left(\frac{x}{f(x)}\right),$$

since (L'Hospital's rule) $\int_{A_1}^{x} \frac{1}{f(t)}\ dt \sim \frac{x}{f(x)}$.

Finally, Abel summation gives

$$\sum_{A_i \leq x} \frac{f(A_i)}{A_i} = (x + o(x))\frac{1}{x} + \int_{A_1}^{x} (t + o(t))\frac{1}{t^2}\ dt = \log x + o(\log x).$$

The theorem is proved. □

In the following theorem we use the notation well-known $x = \log_0 x$, $\log x = \log_1 x$, $\log\log x = \log_2 x$, $\log\log\log x = \log_3 x$, etc.

Theorem 2.30. *Let us consider an integer sequence A_n. Let $A(x)$ be the counting function of the sequence and let $f(x)$ be a function of slow increase. Suppose that $f(A_n) \sim f(n)$ and suppose that*

$$\sum_{A_i \leq x} \frac{f(A_i)}{A_i} = \log x + E_1 + o(1), \tag{2.109}$$

where E_1 is a constant. Therefore

$$\sum_{i=1}^{n} \frac{f(A_i)}{A_i} = \log A_n + E_1 + o(1). \tag{2.110}$$

Then $A(x) \sim \frac{x}{f(x)}$ and $A_n \sim nf(n)$.

Furthermore if $k \geq 1$ is a positive integer then equation (2.109) is generalized in the following way

$$\sum_{A_i \leq x} \frac{f(A_i)}{A_i \log A_i \log_2 A_i \cdots \log_k A_i} = \log_{k+1} x + E_{k+1} + o(1), \tag{2.111}$$

where E_{k+1} are constants, and consequently

$$\sum_{i=1}^{n} \frac{f(A_i)}{A_i \log A_i \log_2 A_i \cdots \log_k A_i} = \log_{k+1} A_n + E_{k+1} + o(1). \tag{2.112}$$

Also, we have if $k \geq 0$ (compare with equation (2.110) ($k = 0$) and equation (2.112) ($k \geq 1$))

$$\sum_{i=1}^{n} \frac{d_i}{A_i \log A_i \log_2 A_i \cdots \log_k A_i} \sim \log_{k+1} A_n. \tag{2.113}$$

If the condition $d_n = A_{n+1} - A_n = O\left(A_n^{\theta}\right)$, where $0 < \theta < 1$, holds then we have the stronger results (compare with equation (2.111) and equation (2.112))

$$\sum_{A_i \leq x} \frac{d_i}{A_i \log A_i \log_2 A_i \cdots \log_k A_i} = \log_{k+1} x + F_{k+1} + o(1), \tag{2.114}$$

$$\sum_{i=1}^{n} \frac{d_i}{A_i \log A_i \log_2 A_i \cdots \log_k A_i} = \log_{k+1} A_n + F_{k+1} + o(1), \tag{2.115}$$

where F_{k+1} are constants.

Proof. Abel summation gives

$$\sum_{A_i \leq x} f(A_i) = (\log x + E_1 + o(1))\, x - \int_{a_2}^{x} (\log t + E_1 + o(1))\ dt = x + o(x),$$

since $\int \log t\ dt = t \log t - t$. By Theorem 2.29 this implies that $A(x) \sim \frac{x}{f(x)}$ and consequently $n = A(A_n) \sim \frac{A_n}{f(A_n)} \sim \frac{A_n}{f(n)}$, that is $A_n \sim n f(n)$ and hence $A_{n+1} \sim A_n$.

Note that if $h \geq 1$ then

$$D(\log_h t) = \frac{1}{\log_{h-1} t} \frac{1}{\log_{h-2} t} \cdots \frac{1}{\log t} \frac{1}{t}.$$

Consequently we have

$$\log t \log_2 t \cdots \log_{h-1} t D(\log_h t) \log_{h+1} t \cdots \log_k t = \frac{1}{t} \log_{h+1} t \cdots \log_k t.$$

Hence

$$D\left(\log t \log_2 t \cdots \log_k t\right) = \frac{1}{t} \left(\sum_{h=1}^{k-1} \left(\log_{h+1} t \cdots \log_k t\right) + 1 \right)$$

and then

$$D\left(\frac{1}{\log t \log_2 t \cdots \log_k t}\right) = -\frac{\left(\sum_{h=1}^{k-1} \left(\log_{h+1} t \cdots \log_k t\right) + 1\right)}{t \log^2 t \log_2^2 t \cdots \log_k^2 t} \quad (k \geq 1) \tag{2.116}$$

Equation (2.116), (2.109) and Abel summation give

$$\begin{aligned} \sum_{A_i \leq x} \frac{f(A_i)}{A_i \log A_i \log_2 A_i \cdots \log_k A_i} &= (\log x + E_1 + o(1)) \frac{1}{\log x \log_2 x \cdots \log_k x} \\ + & \int_{a_k}^{x} (\log t + E_1 + o(1)) \frac{\left(\sum_{h=1}^{k-1} \left(\log_{h+1} t \cdots \log_k t\right) + 1\right)}{t \log^2 t \log_2^2 t \cdots \log_k^2 t}\ dt \\ = & \log_{k+1} x + E_{k+1} + o(1) \quad (k \geq 1), \end{aligned}$$

since the integrals $\int_{a_k}^{\infty} \frac{1}{t \log^2 t}\, dt$ and $\int_{a_k}^{\infty} \frac{1}{t \log t \log_2^2 t}\, dt$ are convergent. Equations (2.111) and (2.112) are proved.

Note that the mean value theorem gives

$$\frac{d_i}{A_{i+1} \log A_{i+1} \log_2 A_{i+1} \cdots \log_k A_{i+1}} < \log_{k+1} A_{i+1} - \log_{k+1} A_i$$
$$< \frac{d_i}{A_i \log A_i \log_2 A_i \cdots \log_k A_i},$$

that is, since $A_{i+1} \sim A_i$ and consequently $\log_h A_{i+1} \sim \log_h A_i$,

$$\log_{k+1} A_{i+1} - \log_{k+1} A_i \sim \frac{d_i}{A_i \log A_i \log_2 A_i \cdots \log_k A_i}. \tag{2.117}$$

Equation (2.117) and Lemma 2.2 give

$$\sum_{i=1}^{n} \frac{d_i}{A_i \log A_i \log_2 A_i \cdots \log_k A_i} \sim \log_{k+1} A_n \qquad (k \geq 0),$$

that is, equation (2.113). In particular if $k = 0$ then we obtain $\sum_{i=1}^{n} \frac{d_i}{A_i} \sim \log A_n$. Thus, a simple application of the mean value theorem relate the sum of real differences $\sum_{i=1}^{n} \frac{d_i}{A_i}$ with the logarithmic function $\log x$, this is not surprising. However, it is not obvious that $f(A_i)$ in the corresponding sum $\sum_{i=1}^{n} \frac{f(A_i)}{A_i}$ also is related with the logarithmic function $\log x$. The also simple proof of this situation was given in Theorem 2.29 where $f(x)$ is related with the counting function $A(x)$ of these integer sequences A_n in the form $A(x) \sim \frac{x}{f(x)}$ as in prime numbers, in primes $f(x) = \log x$ as it is well-known (prime number theorem).

If $d_n = O\left(A_n^{\theta}\right)$ then we have (see Theorem 2.4 and Theorem 2.6)

$$\sum_{i=1}^{n} \frac{d_i}{A_i} = \log A_n + F_1 + o(1)$$

and consequently if $x \in [A_n, A_{n+1})$ then

$$\sum_{A_i \leq x} \frac{d_i}{A_i} = \log x - (\log x - \log A_n) + F_1 + o(1),$$

where $0 \leq \log x - \log A_n \leq \log A_{n+1} - \log A_n = \log\left(\frac{A_{n+1}}{A_n}\right) = o(1)$. Therefore we obtain

$$\sum_{A_i \leq x} \frac{d_i}{A_i} = \log x + F_1 + o(1). \tag{2.118}$$

Equation (2.118) is analogous to equation (2.109). The proof of equations (2.114) and (2.115) by Abel summation from equation (2.118) is identical as the proof of equations (2.111) and (2.112) from equation (2.109). The theorem is proved. □

Remark 2.31. Compare equations (2.111) and (2.114). The unique difference between these equations is the replace of the real difference $d_i = A_{i+1} - A_i$ by $f(A_i)$ (if $A_i = p_i$ is the sequence of primes then$f(A_i) = \log A_i$) and the results of the sums are practically the same except that the constants can be different.

However $f(A_i)$ can be very different of the real difference $d_i = A_{i+1} - A_i$ (as it occur, for example, with primes).

Remark 2.32. If $A_n = p_n$ is the sequence of primes then $f(x) = \log x$ and equation (2.112) becomes

$$\sum_{i=1}^{n} \frac{\log p_i}{p_i \log p_i \log_2 p_i \cdots \log_k p_i} = \log_{k+1} p_n + E_{k+1} + o(1), \tag{2.119}$$

where $k \geq 0$. In particular for $k = 1$ becomes the Mertens's formula

$$\sum_{i=1}^{n} \frac{1}{p_i} = \log \log p_n + E_2 + o(1), \tag{2.120}$$

where E_2 is the Mertens's constant. Equation (2.119) is a generalization of the well-known equation (2.109) for primes. Namely $(k = 0)$

$$\sum_{i=1}^{n} \frac{\log p_i}{p_i} = \log p_n + E_1 + o(1). \tag{2.121}$$

In section 1 we compared $\frac{\log p_i}{p_i}$ with the real difference $l_i = \log p_{i+1} - \log p_i$ and we proved that

$$\liminf \frac{l_i}{\frac{\log p_i}{p_i}} = 0, \tag{2.122}$$

$$\limsup \frac{l_i}{\frac{\log p_i}{p_i}} = \infty. \tag{2.123}$$

Now, we put $l_i^{k+1} = \log_{k+1} p_{i+1} - \log_{k+1} p_i$ $(k \geq 0)$ (if $k = 0$ then $l_i^1 = l_i$) and by equation (2.119) we compare the real difference $l_i^{k+1} = \log_{k+1} p_{i+1} - \log_{k+1} p_i$ with

$$\frac{\log p_i}{p_i \log p_i \log_2 p_i \cdots \log_k p_i} \qquad (k \geq 0). \tag{2.124}$$

If $k = 0$ then we obtain the result of Section 1, $\frac{\log p_i}{p_i}$, if $k = 1$ then we obtain $\frac{1}{p_i}$, if $k = 2$ then we obtain $\frac{1}{p_i \log \log p_i}$, etc.

Equation (2.117) becomes for primes

$$l_i^{k+1} = \log_{k+1} p_{i+1} - \log_{k+1} p_i \sim \frac{d_i}{p_i \log p_i \log_2 p_i \cdots \log_k p_i} \qquad (k \geq 0), \tag{2.125}$$

that is,

$$\frac{l_i^{k+1}}{\frac{\log p_i}{p_i \log p_i \log_2 p_i \cdots \log_k p_i}} \sim \frac{d_i}{\log p_i} \qquad (k \geq 0). \tag{2.126}$$

Therefore we also obtain

$$\liminf \frac{l_i^{k+1}}{\frac{\log p_i}{p_i \log p_i \log_2 p_i \cdots \log_k p_i}} = 0 \qquad (k \geq 0), \tag{2.127}$$

$$\limsup \frac{l_i^{k+1}}{\frac{\log p_i}{p_i \log p_i \log_2 p_i \cdots \log_k p_i}} = \infty \qquad (k \geq 0). \tag{2.128}$$

Since we have (see (2.83)) $\sum_{i=1}^{n} \frac{d_i}{\log p_i} \sim n$, equation (2.126) and Lemma 2.2 give

$$\sum_{i=1}^{n} \frac{l_i^{k+1}}{\frac{\log p_i}{p_i \log p_i \log_2 p_i \cdots \log_k p_i}} \sim n \qquad (k \geq 0) \tag{2.129}$$

On the other hand, equation (2.126) implies that the Cramér's conjecture $d_n = O\left(\log^2 p_n\right)$ is equivalent to the following establishment for $k \geq 0$

$$l_n^{k+1} = \log_{k+1} p_{n+1} - \log_{k+1} p_n = O\left(\frac{\log^2 p_n}{p_n \log p_n \log_2 p_n \cdots \log_k p_n}\right). \tag{2.130}$$

To finish this section, we shall prove a generalization of a weak form of Theorem 2.30.

Theorem 2.33. *Let A_n be an integer sequence strictly increasing such that $A_{n+1} \sim A_n$. Let $F(x)$ be a positive function such that $F(x) \to \infty$ as $x \to \infty$ with positive continuous derivative $F'(x)$ increasing or decreasing such that $F(A_{n+1}) \sim F(A_n)$ and $F'(A_{n+1}) \sim F'(A_n)$. Suppose that*

$$\sum_{A_i \leq x} f(A_i) \sim F(x). \tag{2.131}$$

Then the following formulas hold

$$\sum_{A_i \leq x} \frac{f(A_i)}{F(A_i)} \sim \log F(x), \tag{2.132}$$

$$\sum_{A_i \leq x} \frac{f(A_i)}{F(A_i) \log F(A_i) \log_2 F(A_i) \cdots \log_k F(A_i)} \sim \log_{k+1} F(x) \quad (k \geq 1). \tag{2.133}$$

$$\sum_{A_i \leq x} F'(A_i) d_i \sim F(x). \tag{2.134}$$

$$\sum_{A_i \leq x} \frac{F'(A_i) d_i}{F(A_i)} \sim \log F(x), \tag{2.135}$$

$$\sum_{A_i \leq x} \frac{F'(A_i) d_i}{F(A_i) \log F(A_i) \log_2 F(A_i) \cdots \log_k F(A_i)} \sim \log_{k+1} F(x) \quad (k \geq 1). \tag{2.136}$$

Proof. Equation (2.131) and Abel summation give

$$\sum_{A_i \leq x} \frac{f(A_i)}{F(A_i)} = (F(x) + o(F(x))) \frac{1}{F(x)} + \int_{A_1}^{x} (F(t) + o(F(t))) \frac{F'(t)}{F(t)^2}\, dt$$
$$= \log F(x) + o(\log F(x)),$$

that is, equation (2.132). Now, equation (2.132) becomes equation (2.131) and consequently we obtain

$$\sum_{A_i \leq x} \frac{f(A_i)}{F(A_i) \log F(A_i)} \sim \log_2 F(x),$$

etc. In this form we prove equation (2.133).

The proof of equation (2.134) is an immediate consequence of the mean value theorem and Lemma 2.2. The theorem is proved. □

Remark 2.34. Note that if in Theorem 2.33 we put $F(x) = x$ then $F'(x) = 1$ and we obtain a weak form of Theorem 2.30.

3. Definition of Regular Sequences

To finish, in this section, for sake of completeness, we define simple integer sequences whose study is almost trivial. We call these simple integer sequences: regular sequences. The rest of the integer sequences we call irregular sequences. Primes are an irregular sequence. Are there irregular sequences almost regular?, that is, very similar to regular sequences. Are primes almost a regular sequence?

We are interested only in integer sequences with zero density. We denote an integer sequence A_n, we denote $d_n = A_{n+1} - A_n$ and we denote $A(x)$ the counting function of the integer sequence A_n.

A integer sequence is regular if and only if

Case 1) $A_n \sim cn^s$, where $c > 0$, $s > 1$ and $d_n \sim csn^{s-1}$

Case 2) $A_n \sim n^s f(n)$, where $s \geq 1$, $d_n \sim sn^{s-1} f(n)$ and $f(x)$ is a function of slow increase.

Lemma 3.1. *Let $\alpha > 0$. If $f(x)$ is a function of slow increase then*

$$\int_a^x \frac{t^\alpha}{f(t)}\, dt \sim \frac{x^{\alpha+1}}{\alpha+1} \frac{1}{f(x)}, \tag{3.1}$$

$$\int_a^x t^\alpha f(t)\, dt \sim \frac{x^{\alpha+1}}{\alpha+1} f(x), \tag{3.2}$$

$$\sum_{i=1}^{n} i^\alpha \sim \frac{n^{\alpha+1}}{\alpha+1}. \tag{3.3}$$

Proof. Equations (3.1) and (3.2) can be proved with L'Hospital's rule and equation (2.64). Equation (3.3) is well-known [1]. The lemma is proved. □

Clearly, the definitions are consistent since $\sum_{i=1}^{n} d_i \sim A_n$, as it can be proved by use of Lemma 2.2 and Lemma 3.1.

In the following theorem we establish some formulas that regular integer sequences satisfy.

Theorem 3.2. *Regular integer sequences (either case 1 or case 2) satisfy the following formulas,*

$$A(x) = o(x),$$

$$A_{n+1} \sim A_n, \qquad \frac{d_n}{A_n} \to 0.$$

The series $\sum_{i=1}^{\infty} \frac{d_i^2}{A_i^2}$ *converges.*

$$d_n \to \infty.$$

$$d_{n+1} \sim d_n.$$

In case 1)

$$\sum_{i=1}^{n} \frac{d_i}{csi^{s-1}} \sim n, \qquad \sum_{i=1}^{n} \frac{csi^{s-1}}{d_i} \sim n.$$

In case 2)

$$\sum_{i=1}^{n} \frac{d_i}{si^{s-1}f(i)} \sim n, \qquad \sum_{i=1}^{n} \frac{si^{s-1}f(i)}{d_i} \sim n.$$

In both cases

$$A_n \sim \frac{1}{s} n d_n.$$

$$\sum_{i=1}^{n} \frac{i d_i}{A_i} \sim sn, \qquad \sum_{i=1}^{n} \frac{A_i}{i d_i} \sim \frac{1}{s} n.$$

$$\sum_{i=1}^{n} d_i^k \sim \frac{n}{(s-1)k+1} d_n^k.$$

In particular, if $s = 1$ *then*

$$\sum_{i=1}^{n} d_i^k \sim n d_n^k \sim n f(n)^k.$$

If $s = 1$ *then*

$$\sum_{i=1}^{n} \frac{1}{d_i} \sim \frac{n}{f(n)}.$$

In case 1)

$$d_n < c_1 A_n^{\frac{s-1}{s}},$$

where c_1 *is a positive constant.*

In case 2) for all $\epsilon > 0$

$$d_n < c_\epsilon A_n^{\frac{s-1}{s}+\epsilon},$$

where c_ϵ *is a positive constant depending of* ϵ*.*

Proof. The proofs are immediate from the definition, Lemma 2.2 and Lemma 3.1. The theorem is proved. □

Example 3.3. Some examples of regular integer sequences are: The integer sequences $A_n = n^k$ $(n \geq 1)$, where $k \geq 2$ is an arbitrary fixed positive integer and the sequence,

$$A_n = \lfloor \log 3 \rfloor + \lfloor \log 4 \rfloor + \cdots + \lfloor \log(n+2) \rfloor \qquad (n \geq 1),$$

where $A_n \sim n \log n$ and $d_n = A_{n+1} - A_n \sim \log n$.

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