LUCAS GENERALIZED NUMBERS IN NARAYANA’S COWS SEQUENCE

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ABSTRACT. Let \( \{N_n\}_{n \geq 0} \) be the Narayana’s cows sequence given by \( N_0 = 0 \), \( N_1 = N_2 = 1 \) and

\[ N_{n+3} = N_{n+2} + N_n, \quad \text{for integers } n \geq 0 \]

and let \( \{U_n\}_{n \geq 0} \) be the generalized Lucas sequence with parameters integers \( a \geq 1, b = \pm 1 \) given by \( U_0 = 0 \), \( U_1 = 1 \) and

\[ U_{n+2} = aU_{n+1} + bU_n, \quad \text{for integers } n \geq 0. \]

In this paper we give effective bounds for the Diophantine equation

\[ N_m = U_n, \]

in positive unknowns \( m \) and \( n \). We then solve explicitly that equation with Fibonacci, Pell and Balancing sequences cases.

1. INTRODUCTION AND PRELIMINARIES

1.1. Instruction. The Narayana’s cows sequence \( (N_m)_{m \geq 0} \) is a modern translation to a problem stated by the Indian mathematician Narayana Pandit (1340-1400) in his famous book titled the Ganita Kaumudi of Narayana Pandit publish by P. Singh [11]. It is the sequence A000930 in the OEIS satisfying the recurrence relation

\[ N_{m+3} = N_{m+2} + N_m \quad \text{for integers } n \geq 0 \] (1.1)

with initial terms \( N_0 = 0 \) and \( N_1 = N_2 = 1 \). The first few terms of \( (N_m)_{m \geq 0} \) are

\[ 0, 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, \ldots \]

The generalized Lucas sequence \( (U_n)_{n \geq 0} \) with integers parameters \( a \geq 1, b = \pm 1 \) is given by \( U_0 = 0, U_1 = 1 \) and

\[ U_{n+2} = aU_{n+1} + bU_n, \quad \text{for integers } n \geq 0. \]

This is Fibonacci sequence, Pell sequence and Balancing sequence when \((a, b) = (1, 1)\), \((a, b) = (2, 1)\) and \((a, b) = (6, -1)\) respectively. The cases \((a, b) = (1, -1)\) and \((a, b) = (2, -1)\) are not interested. So we only consider in this study the...
generalized Lucas sequence \((U_n)_{n \geq 0}\) with integers parameters \(a \geq 1, b = \pm 1\) except \((a, b) = (1, -1), (2, -1)\).

The Fibonacci and Narayana’s cows sequences are very similar definitions, but with a delay in the recursion which makes Narayana’s cows sequence a third-order linear recurrence sequence. This is related to the “delayed morphism” of Allouche and Johnson [9];

In this note, we study the Diophantine equation
\[ N_m = U_n \] (1.2)
in positive integers unknowns \(m\) and \(n\).

1.2. Preliminaries. In this subsection, we gather some tools which lead us to prove the main results of the paper.

1.2.1. Some properties of considered sequences \((N_m)_{m \geq 0}\) and \((U_m)_{m \geq 0}\). For a complex number \(z\) we write \(\overline{z}\) for its complex conjugate. The characteristic polynomial of Narayana’s cows sequence
\[ X^3 - X^2 - 1 = (X - \gamma)(X - \delta)(\overline{X} - \overline{\delta}) \]
with \(\gamma \approx 1.46557\) and \(|\delta| = |\overline{\delta}| = \gamma^{-1/2} < 1\). Narayana’s cows sequence has Binet’s formula
\[ N_m = c_1\gamma^m + c_2\delta^m + c_3\overline{\delta}^m \quad \text{for} \quad m \geq 0, \] (1.3)
where
\[ c_1 = \frac{\gamma}{(\gamma - \delta)(\gamma - \overline{\delta})}, \quad c_2 = \frac{\delta}{(\delta - \gamma)(\delta - \overline{\delta})}, \quad \text{and} \quad c_3 = \overline{c_2} = \frac{\overline{\delta}}{(\overline{\delta} - \gamma)(\overline{\delta} - \delta)}. \]

We will use the following version of formula (1.3)
\[ N_m = C_\gamma\gamma^{m+2} + C_\delta\delta^{m+2} + C_{\overline{\delta}}\overline{\delta}^{m+2} \quad \text{for} \quad m \geq 0, \] (1.4)
where
\[ C_\gamma = \frac{1}{\gamma^3 + 2}, \quad C_\delta = \frac{1}{\delta^3 + 2} \quad \text{and} \quad C_{\overline{\delta}} = \overline{C_\delta}. \]
The coefficient \(C_\gamma\) has the minimal polynomial \(31X^3 - 31X^2 + 10X - 1\) over \(\mathbb{Z}\) and all the zeros of this polynomial lie strictly inside the unit circle. Numerically, we can calculate
\[ C_\gamma^{-1} \approx 5.1479 \quad \text{and} \quad |C_\delta| \approx 0.407506. \]
One can prove using the induction (see [7]) that the \(n\)th Narayana number satisfies the following relation
\[ \gamma^{m-2} \leq N_m \leq \gamma^{m-1} \quad \text{for} \quad m \geq 1, \] (1.5)

The Binet formula of Lucas generalized sequence with integer parameters \(a \geq 1\) and \(b = \pm 1\) is given by
\[ U_n = \frac{\alpha^n - \beta^n}{\sqrt{\Delta}} \quad \text{for} \quad n \geq 0, \] (1.6)
where $\alpha = \frac{a + \sqrt{\Delta}}{2} \quad \text{and} \quad \beta = a - \alpha$ with $\Delta = a^2 + 4b$. From the assumption that $(a, b) \not\in \{(1, -1), (2, -1)\}$, we have $\Delta \geq 1$.

**Lemma 1.1.** For integer $n \geq 1$, the inequalities below hold

$$\alpha^{n-2} \leq U_n \leq \alpha^n. \quad (1.7)$$

**Proof.** The property is true if $n = 1$. Indeed, $\alpha > 1 = U_1$ then $\alpha^{-1} < U_1 < \alpha$.

Now assuming that the property is true in order $n$, we have $\alpha^{n-2} \leq U_n \leq \alpha^n$ and $\alpha^{n-3} \leq U_{n-1} \leq \alpha^{n-1}$. Then, for $b = 1$, $a\alpha^{n-2} + b\alpha^{n-3} \leq U_{n+1} \leq a\alpha^{n} + b\alpha^{n-1}$ and for $b = -1$, $a\alpha^{n-2} - \alpha^{n-1} \leq U_{n+1} \leq a\alpha^{n} - \alpha^{n-3}$.

**Case** $b = 1$.

We have $a\alpha^{n-1} - aa^{n-2} + b\alpha^{n-3} = -2\alpha^{n-3} < 0$ and $U_{n+1} - a\alpha^{n+1} = \frac{\alpha^{2(n+1)}(1 - \sqrt{\Delta}) + (-1)^n}{(\alpha - \beta)\alpha^{n+1}}$.

As $a \geq 1$ then $\Delta \geq 5$ and $1 - \sqrt{\Delta} < -1.23$. If $n$ is odd, $\alpha^{2(n+1)}(1 - \sqrt{\Delta}) + (-1)^n < 0$ otherwise $\alpha^{2(n+1)}(1 - \sqrt{\Delta}) + (-1)^n < -1.23\alpha^{2(n+1)} + 1 < 0$ as $\alpha > 1$.

**Case** $b = -1$.

We have $a\alpha^n + b\alpha^{n-3} - \alpha^{n+1} = \alpha n - 3 ((1 - 2\alpha^2)\alpha - 1) < 0$ as $\alpha > 1$. Also,

$$a^{n-1} - U_{n+1} = \frac{\alpha^{2n} ((2 - a)\alpha - a + 1)}{(\alpha - \beta)\alpha^{n+1}} < 0.$$

Indeed, $a \geq 2$ and $\alpha > 0$ then $(2 - a)\alpha < 0$ and $-a + 1 < 0$.

In both cases, we have $\alpha^{n-1} < U_{n+1} < \alpha^{n+1}$. Then, the property is true in order $n + 1$. That proves the Lemma 1.1. \hfill \Box

The splitting field of the polynomial $X^3 - X^2 - 1$ is $\mathbb{Q}(\gamma, \delta)$ and contains $C_\gamma, C_\delta$ and $C_\delta$. The splitting field of $X^2 - X - 1$ is $\mathbb{Q}(\alpha)$. Since $\gamma$ and $\alpha$ are reals then the field $K = \mathbb{Q}(\gamma, \alpha)$ is a real extension of $\mathbb{Q}$ with degree $d_{K/\mathbb{Q}} = 6$. More the field $\mathbb{Q}(\gamma, \delta, \alpha)$ is Galois extension and a $\mathbb{Q}$-automorphism is for example the permutation

$$\sigma : \gamma \mapsto \delta, \; \delta \mapsto \gamma, \; \bar{\delta} \mapsto \bar{\delta} \; \text{and} \; x \mapsto x, \; \text{for} \; x \in \mathbb{Q}(\alpha). \quad (1.8)$$

1.2.2. **Some useful results on linear forms in logarithms of algebraic numbers.** We use Baker’s theory of linear forms in logarithms of algebraic numbers for the proof of our result. Let $\eta$ be an algebraic number with minimal primitive polynomial

$$f(X) = a_0 \prod_{i=1}^{d} (X - \eta^{(i)}) \in \mathbb{Z}[X],$$

where $a_0 > 0$, and $\eta = \eta^{(1)}, \cdots, \eta^{(d)}$ are conjugates of $\eta$. Then,

$$h(\eta) = \frac{1}{d} \left( \log a_0 + \sum_{i=1}^{d} \log \max \{|\eta^{(i)}|, 1\} \right)$$

is called the logarithmic height of $\eta$. In particular, if $\eta = P/Q$ is a rational number with $\gcd(P, Q) = 1$ and $Q > 1$, then $h(\eta) = \max \{|P|, Q\}$. The following are some properties of logarithmic height function. For $\eta, \lambda$ algebraic and $s \in \mathbb{Z}$ we have
• $h(\eta + \lambda) \leq h(\eta) + h(\lambda) + \log 2$,
• $h(\eta^{\lambda^\pm 1}) \leq h(\eta) + h(\lambda)$,
• $h(\eta^n) = |s|h(\eta)$.

Now, let $K$ be a real number field of degree $d_K, \eta_1, \ldots, \eta_t$ positive elements of $K$ and $b_1, \ldots, b_t$ integers non-zero all. Let $B \leq \max \{|b_1|, \ldots, |b_t|\}$ and

$$\Lambda = \eta_1^{b_1} \cdots \eta_t^{b_t} - 1.$$ 

Let $A_1, \ldots, A_t$ be real numbers with

$$A_i \geq \max \{d_K h(\eta_i), |\log \eta_i|, 0.16\}, \quad i = 1, 2, \ldots, t.$$ 

With the above notations, Matveev in [5] (see also [12, Theorem 9.4]) proved the following result which is the second tool we need.

**Theorem 1.2.** Assume that $\Lambda \neq 0$. Then

$$\log |\Lambda| > -1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot d_K^2 \cdot (1 + \log d_K) \cdot (1 + \log B) A_1 \cdots A_t.$$ 

In this paper we always use $t = 3$. Since $K = \mathbb{Q}(\gamma, \alpha)$ has degree $d_K/\mathbb{Q} = 6$, where $\gamma$ and $\alpha$ are defined in Subsection 1.2.1. We thus, once and for all fix the constant

$$C := 1.43908 \times 10^{13} > 1.4 \cdot 30^{3+3} \cdot 3^{4.5} \cdot 6^2 \cdot (1 + \log 6).$$

**Lemma 1.3.** [10, Lemma 7] Let $\ell$ be integer and $H$ a real. If $\ell \geq 1$, $H > (4\ell^2)^\ell$ and $H > L/(\log L)^\ell$, then

$$L < 2^\ell H(\log H)^\ell.$$ 

Finally, we need a version of the reduction method of Baker-Davenport based on Lemma in [1]. We shall use the one given by Bravo, Gómez and Luca in [6] (See also Dujella and Pethö [2]). For a real number $x$, we write $\|x\|$ for the distance from $x$ to the nearest integer.

**Lemma 1.4.** Let $M$ be a positive integer. Let $\tau, \mu, A > 0$, $B > 1$ be given real numbers. Assume that $p/q$ is a convergent of $\tau$ such that $q > 6M$ and that $\varepsilon := \|q\mu\| - M \|q\tau\| > 0$. Then there is no solution to the inequality

$$0 < |n\tau - m + \mu| < \frac{A}{B^\omega}$$

in positive integers $n, m$ and $\omega$ satisfying

$$n \leq M \quad \text{and} \quad \omega \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$
2. Main results

We use the theory of lower bounds for linear forms in logarithms of algebraic numbers and a version of the reduction procedure from Baker and Davenport [1]. We then prove:

**Theorem 2.1.** Let $n$ and $m$ be positive integers solution of Diophantine equation (1.2). Then we have $n \leq m$. Moreover if $m \geq 7$,\[ m + 2 < 1.546 \cdot 10^{15} \left( \log \max \{ \sqrt[3]{3} \sqrt[15]{\Delta}, \alpha \} \right)^2 \log \left( 7.7281 \cdot 10^{14} \left( \log \max \{ \sqrt[3]{3} \sqrt[15]{\Delta}, \alpha \} \right)^2 \right), \]
where $\Delta := a^2 + 4$ and $\alpha = \frac{a + \sqrt{a^2 + 4}}{2}$.

**Proof.** Let $m, n \geq 2$ be integers that satisfy the equation (1.2). Then from the inequalities in (1.5) and (1.7), we have\[ \gamma^{m-2} < \alpha^n \quad \text{and} \quad \alpha^{n-2} < \gamma^{m-1}. \]
So taking logarithms in that both inequalities we get the following inequalities.
\[ (m + 2) \frac{\log \gamma}{\log \alpha} < n < 2 + (m - 1) \frac{\log \gamma}{\log \alpha}. \quad (2.1) \]
According to assumption that $a \in \mathbb{N} \setminus \{0\}$, $b = \pm 1$ and $(a, b) \notin \{(2, -1), (1, -1)\}$, we have\[ \Delta \geq \frac{a + \sqrt{5}}{2} \geq 1 + \frac{\sqrt{5}}{2}. \]
We thus obtain (the first step of the Theorem 2.1)\[ n < 2 + (m - 1) \frac{\log \gamma}{\log \frac{1 + \sqrt{5}}{2}} < m. \]

Now inserting Binet’s formula of $(N_m)_{m \geq 0}$ and $(U_n)_{n \geq 0}$ in the equation (1.2) we get\[ C_{\gamma} \gamma^{m+2} - \frac{\alpha^n}{\sqrt{\Delta}} = -C_{\delta} \delta^{m+2} - C_{\beta} \beta^{m+2} - \frac{\beta^n}{\sqrt{\Delta}}. \]
Taking absolute values on both sides and dividing by $C_{\gamma} \gamma^{m+2}$, leads to\[ \left| (\sqrt{\Delta} C_{\gamma})^{-1} \alpha^n \gamma^{-m-2} - 1 \right| < \left( 2 |C_{\delta}| \gamma^{-m-2} + \frac{\alpha^n}{\sqrt{\Delta}} \right) C_{\gamma}^{-1} \gamma^{-m-2} \]
\[ < \left( 2 |C_{\delta}| \gamma^{-m-2} + \gamma^{-m-2} \right) C_{\gamma}^{-1} \gamma^{-m-2}, \quad \text{since} \quad \Delta \geq 1 \quad \text{and} \quad \gamma^{m-2} < \alpha^n \]
\[ < (2 |C_{\delta}| + 1) C_{\gamma}^{-1} \gamma^{-m-2}, \quad \text{for} \quad m \geq 7. \]
Putting\[ \Lambda_1 := (\sqrt{\Delta} C_{\gamma})^{-1} \alpha^n \gamma^{-m-2} - 1, \]
we then get, for $m \geq 7$\[ |\Lambda_1| < 9.35 \gamma^{-m-2}. \quad (2.2) \]
We have $\Lambda_1 \neq 0$. Indeed if $\Lambda_1 = 0$ then $\alpha^n = \sqrt{\Delta C_\gamma \gamma^{m+2}}$ and applying the $\mathbb{Q}$-automorphism $\sigma$ of the Galois extension $\mathbb{Q}(\gamma, \delta, \alpha)$ defined in (1.8), we have

$$\alpha^n = \sqrt{\Delta C_\delta \delta^{m+2}}$$

This implies that $n = \frac{-(m+2) \log \gamma}{\log \alpha} + \frac{\log |C_\delta| \sqrt{\Delta}}{\log \alpha}$ which is impossible cause of the first inequality in (2.1).

We now can apply Theorem 1.2 for $\Lambda_1$ with

$$\eta_1 = \sqrt{\Delta}C_\gamma, \quad \eta_2 = \alpha, \quad \eta_3 = \gamma, \quad b_1 = -1, \quad b_2 = n, \quad b_3 = -m - 2.$$  

We have $B = \max \{ |b_1|, |b_2|, |b_3| \} = m + 2$. We must now estimate $h(\eta_i)$ for $i = 1, 2, 3$. Using properties of the height, we have

$$h(\eta_1) \leq \log \sqrt{\Delta} + h(C_\gamma) \leq \log \sqrt{\Delta} + \frac{\log 31}{3} = \log (\sqrt[3]{31\sqrt{\Delta}}),$$

$$h(\eta_2) \leq \frac{\log \alpha}{2} \quad \text{and} \quad h(\eta_3) \leq \frac{\log \gamma}{3}.$$

Thus, we can take

$$A_1 = 6 \log (\sqrt[3]{31\sqrt{\Delta}}), \quad A_2 = 3 \log \alpha \quad \text{and} \quad A_3 = 2 \log \gamma \approx 0.76.$$  

Applying Theorem 1.2, we have

$$\log |\Lambda_1| > -C \cdot 0.76 \cdot (3 \log \alpha) \left( 6 \log (\sqrt[3]{31\sqrt{\Delta}}) \right) (1 + \log (m + 2)),$$

$$> -1.969 \cdot 10^{14} (\log \alpha) \left( \log (\sqrt[3]{31\sqrt{\Delta}}) \right) (1 + \log (m + 2)),$$

$$> -2.954 \cdot 10^{14} (\log \alpha) \left( \log (\sqrt[3]{31\sqrt{\Delta}}) \right) \log (m + 2).$$

Let put $\theta := \max \{ \sqrt[3]{31\sqrt{\Delta}}, \alpha \}$. We have

$$\log |\Lambda_1| > -2.954 \cdot 10^{14} (\log \theta)^2 \log (m + 2), \quad (2.3)$$

We take logarithm in (2.2), we have

$$\log |\Lambda_1| < -(m + 2) \log \gamma + \log (9.35).$$

Combining this with (2.3) we obtain

$$m + 2 < 7.7281 \cdot 10^{14} (\log \theta)^2 \cdot \log (m + 2).$$

Hence, by applying Lemma 1.3 we get

$$m + 2 < 1.546 \cdot 10^{15} (\log \theta)^2 \log (7.7281 \cdot 10^{14} (\log \theta)^2). \quad (2.4)$$

This completes the proof of the Theorem 2.1. \(\square\)

Applying the effective bounds from the Theorem 2.1 for Fibonacci sequence, Pell sequence and Balancing sequence, we obtain Corollaries 2.2, 2.3 and 2.4 below.

**Corollary 2.2.** The only Fibonacci numbers in Narayana’s cows sequence are 1, 2, 3 and 13.
Corollary 2.3. The only Pell numbers in Narayana’s cows sequence are 1 and 2.

Corollary 2.4. The only Balancing numbers in Narayana’s cows sequence are 1 and 6.

Proof of the Corollaries 2.2, 2.3 and 2.4.

Here we apply the Theorem 2.1 with some special case of the sequence \((U_n)_{n \geq 0}\), namely Fibonacci \((F_n)_{n \geq 0}\), Pell \((P_n)_{n \geq 0}\), and Balancing \((B_n)_{n \geq 0}\) sequences. As we can see, this upper bounds of solutions provided by the Theorem 2.1 is too large that must be reduced before by Baker reduction method.

Let consider \(\Lambda_1\) defined in (2.2). For a positive real \(x\), if \(|x - 1| < \frac{1}{2}\) then \(|\log x| < 1.5 |x - 1|\) (see [8, Lemma 4]). Hence we have, from there

\[
|n \log \alpha - (m + 2) \log \gamma - \log \left(\sqrt{\Delta C_\gamma}\right)| < 1.5 \cdot 9.35 \gamma^{-m-2},
\]

and divide by \(\log \gamma\), we get

\[
\left|\frac{n \log \alpha}{\log \gamma} - (m + 2) - \frac{\log \left(\sqrt{\Delta C_\gamma}\right)}{\log \gamma}\right| < 37.351 \gamma^{-m-2}. \tag{2.5}
\]

So we then apply the Lemma 1.4 with

\[
\omega := m + 2, \quad \tau := \frac{\log \alpha}{\log \gamma}, \quad \mu := \frac{\log \left(\sqrt{\Delta C_\gamma}\right)}{\log \gamma}, \quad A := 37.351, \quad B := \gamma
\]

and

\[
M := 1.546 \cdot 10^{15} \left(\log \max\left\{\sqrt[3]{31} \sqrt{\Delta}, \alpha\right\}\right)^2 \log \left(7.7281 \cdot 10^{14} \left(\log \max\left\{\sqrt[3]{31} \sqrt{\Delta}, \alpha\right\}\right)^2\right).
\]

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Table 1.
**Fibonacci sequence case:** We find that 39-th convergent of $\frac{p_{39}}{q_{39}} = \frac{2789144197090847260}{3511279069780293827}$ satisfies $q_{39} > 6M$ and $\varepsilon = 0.458791087262574 > 0$. Hence the inequality (2.5) has no solution for
\[ m + 2 \geq \frac{\log(37.351 \cdot q_{39}/\varepsilon)}{\log \gamma} \geq \frac{\log(37.351 \cdot q_{39}/0.458791087262574)}{\log \gamma} \geq 122.622 \]
Thus we obtain $m \leq 120$ and consequently $n \leq 120$. We now compute
\[ \{ N_m : 1 \leq m \leq 120 \} \cap \{ F_n : 1 \leq n \leq 120 \} = \{ 1, 2, 3, 5, 13 \} \]
This is the proof of Corollary 2.2.

**Pell sequence case:** We find that 32-th convergent of $\frac{p_{32}}{q_{32}} = \frac{7211923117474819860}{16629117764559080077}$ satisfies $q_{32} > 6M$ and $\varepsilon = 0.221867342218888 > 0$. So the inequality (2.5) has no solution for
\[ m + 2 \geq \frac{\log(37.351 \cdot q_{32}/\varepsilon)}{\log \gamma} \geq \frac{\log(37.351 \cdot q_{32}/0.221867342218888)}{\log \gamma} \geq 127.008 \]
Thus we obtain $m \leq 125$ and consequently $n \leq 125$. We now compute
\[ \{ N_m : 1 \leq m \leq 125 \} \cap \{ P_n : 1 \leq n \leq 125 \} = \{ 1, 2 \} \]
This completes the proof of Corollary 2.3.

**Balancing sequence case:** We find that 36-th convergent of $\frac{p_{36}}{q_{36}} = \frac{3605961558737409930}{16629117764559080077}$ satisfies $q_{36} > 6M$ and $\varepsilon = 0.336040242950998 > 0$. Therefore the inequality (2.5) has no solution for
\[ m + 2 \geq \frac{\log(37.351 \cdot q_{36}/\varepsilon)}{\log \gamma} \geq \frac{\log(37.351 \cdot q_{36}/0.336040242950998)}{\log \gamma} \geq 124.108 . \]
Thus we obtain $m \leq 122$ and consequently $n \leq 122$. We now compute
\[ \{ N_m : 1 \leq m \leq 122 \} \cap \{ B_n : 1 \leq n \leq 122 \} = \{ 1, 6 \} \]
This finishes the proof of Corollary 2.4. □

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