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LUCAS GENERALIZED NUMBERS IN NARAYANA'S COWS SEQUENCE

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Abstract. Let $\{N_n\}_{n\geq 0}$ be the Narayana's cows sequence given by $N_0 = 0$, $N_1 = N_2 = 1$ and

$$N_{n+3} = N_{n+2} + N_n, \quad \text{for integers } n \geq 0$$

and let $\{U_n\}_{n\geq 0}$ be the generalized Lucas sequence with parameters integers $a \geq 1, b = \pm 1$ given by $U_0 = 0,\ U_1 = 1$ and

$$U_{n+2} = aU_{n+1} + bU_n, \quad \text{for integers } n \geq 0.$$

In this paper we give effective bounds for the Diophantine equation

$$N_m = U_n,$$

in positive unknowns m and n. We then solve explicitly that equation with Fibonacci, Pell and Balancing sequences cases.

1. Introduction and preliminaries

1.1. **Instruction.** The Narayana's cows sequence $(N_m)_{m\geq 0}$ is a modern translation to a problem state by the Indian mathematician Narayana Pandit (1340-1400) in his famous book titled the Ganita Kaumudi of Narayana Pandit publish by P. Singh [11]. It is the sequence A000930 in the OEIS satisfying the recurrence relation

$$N_{m+3} = N_{m+2} + N_m \;\text{ for integers }\; n \geq 0 \tag{1.1}$$

with initial terms $N_0 = 0$ and $N_1 = N_2 = 1$. The first few terms of $(N_m)_{m\geq 0}$ are

$$0, 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, \ldots$$

The generalized Lucas sequence $(U_n)_{n\geq 0}$ with integers parameters $a \geq 1$, $b = \pm 1$ is given by $U_0 = 0,\ U_1 = 1$ and

$$U_{n+2} = aU_{n+1} + bU_n, \;\text{ for integers }\; n \geq 0.$$

This is Fibonacci sequence, Pell sequence and Balancing sequence when $(a, b) = (1, 1)$, $(a, b) = (2, 1)$ and $(a, b) = (6, -1)$ respectively. The cases $(a, b) = (1, -1)$ and $(a, b) = (2, -1)$ are not interested. So we only consider in this study the

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generalized Lucas sequence $(U_n)_{n\geq 0}$ with integers parameters $a \geq 1$, $b = \pm 1$ except $(a, b) = (1, -1)$, $(2, -1)$.

The Fibonacci and Narayana's cows sequences are very similar definitions, but with a delay in the recursion which makes Narayana's cows sequence a third-order linear recurrence sequence. This is related to the "delayed morphism" of Allouche and Johnson [9];

In this note, we study the Diophantine equation

$$N_m = U_n \tag{1.2}$$

in positive integers unknowns m and n.

1.2. **Preliminaries.** In this subsection, we gather some tools which lead us to prove the main results of the paper.

1.2.1. *Some properties of considered sequences* $(N_m)_{m\geq 0}$ *and* $(U_m)m \geq 0$. For a complex number z we write $\overline{z}$ for its complex conjugate. The characteristic polynomial of Narayana's cows sequence

$$X^3 - X^2 - 1 = (X - \gamma)(X - \delta)(X - \overline{\delta})$$

with γ (≈ 1.46557) and $|\delta| = \left|\overline{\delta}\right| = \gamma^{-1/2} < 1$. Narayana's cows sequence has Binet's formula

$$N_m = c_1\gamma^m + c_2\delta^m + c_3\overline{\delta}^m \quad \text{for} \quad m \geq 0, \tag{1.3}$$

where

$$c_1 = \frac{\gamma}{(\gamma - \delta)(\gamma - \overline{\delta})}, \; c_2 = \frac{\delta}{(\delta - \gamma)(\delta - \overline{\delta})}, \text{ and } c_3 = \overline{c_2} = \frac{\overline{\delta}}{(\overline{\delta} - \gamma)(\overline{\delta} - \delta)}.$$

We will use the following version of formula (1.3)

$$N_m = C_\gamma\gamma^{m+2} + C_\delta\delta^{m+2} + C_{\overline{\delta}}\overline{\delta}^{m+2} \quad \text{for} \quad m \geq 0, \tag{1.4}$$

where

$$C_\gamma = \frac{1}{\gamma^3 + 2}, \quad C_\delta = \frac{1}{\delta^3 + 2} \quad \text{and} \quad C_{\overline{\delta}} = \overline{C_\delta}.$$

The coefficient C_γ has the minimal polynomial $31X^3 - 31X^2 + 10X - 1$ over $\mathbb{Z}$ and all the zeros of this polynomial lie strictly inside the unit circle. Numerically, we can calculate

$$C_\gamma^{-1} \approx 5.1479 \quad \text{and} \quad |C_\delta| \approx 0.407506.$$

One can prove using the induction (see [7]) that the *nth* Narayana number satisfies the following relation

$$\gamma^{m-2} \leq N_m \leq \gamma^{m-1} \quad \text{for} \quad m \geq 1, \tag{1.5}$$

The Binet formula of Lucas generalized sequence with integer parameters $a \geq 1$ and $b = \pm 1$ is given by

$$U_n = \frac{\alpha^n - \beta^n}{\sqrt{\Delta}} \quad \text{for} \quad n \geq 0, \tag{1.6}$$

where $\alpha = \dfrac{a+\sqrt{\Delta}}{2}$ and $\beta = a - \alpha$ with $\Delta = a^2 + 4b$. From the assumption that $(a, b) \notin \{(1, -1), (2, -1)\}$, we have $\Delta \geq 1$.

Lemma 1.1. *For integer $n \geq 1$, the inequalities below hold*

$$\alpha^{n-2} \leq U_n \leq \alpha^n. \tag{1.7}$$

Proof. The property is true if $n = 1$. Indeed, $\alpha > 1 = U_1$ then $\alpha^{-1} < U_1 < \alpha$. Now assuming that the property is true in order n, we have $\alpha^{n-2} \leq U_n \leq \alpha^n$ and $\alpha^{n-3} \leq U_{n-1} \leq \alpha^{n-1}$. Then, for $b = 1$, $a\alpha^{n-2} + b\alpha^{n-3} \leq U_{n+1} \leq a\alpha^n + b\alpha^{n-1}$ and for $b = -1$, $a\alpha^{n-2} - \alpha^{n-1} \leq U_{n+1} \leq a\alpha^n - \alpha^{n-3}$.

Case $b = 1$.

We have $\alpha^{n-1} - a\alpha^{n-2} + b\alpha^{n-3} = -2\alpha^{n-3} < 0$ and $U_{n+1} - \alpha^{n+1} = \dfrac{\alpha^{2(n+1)}(1-\sqrt{\Delta}) + (-1)^n}{(\alpha-\beta)\alpha^{n+1}}$.

As $a \geq 1$ then $\Delta \geq 5$ and $1 - \sqrt{\Delta} < -1.23$. If n is odd, $\alpha^{2(n+1)}(1-\sqrt{\Delta}) + (-1)^n < 0$ otherwise $\alpha^{2(n+1)}(1-\sqrt{\Delta}) + (-1)^n) < -1.23\alpha^{2(n+1)} + 1 < 0$ as $\alpha > 1$.

Case $b = -1$.

We have $a\alpha^n + b\alpha^{n-3} - \alpha^{n+1} = \alpha n - 3\left((1-2\alpha^2)\alpha - 1\right) < 0$ as $\alpha > 1$. Also,

$$\alpha^{n-1} - U_{n+1} = \frac{\alpha^{2n}\left((2-a)\alpha - a + 1\right)}{(\alpha-\beta)\alpha^{n+1}} < 0.$$

Indeed, $a \geq 2$ and $\alpha > 0$ then $(2-a)\alpha < 0$ and $-a + 1 < 0$.
In both cases, we have $\alpha^{n-1} < U_{n+1} < \alpha^{n+1}$. Then, the property is true in order $n + 1$. That proves the Lemma 1.1. □

The splitting field of the polynomial $X^3 - X^2 - 1$ is $\mathbb{Q}(\gamma, \delta)$ and contains C_γ, C_δ and $C_{\bar{\delta}}$. The splitting field of $X^2 - X - 1$ is $\mathbb{Q}(\alpha)$. Since γ and α are reals then the field $\mathbb{K} = \mathbb{Q}(\gamma, \alpha)$ is a real extension of $\mathbb{Q}$ with degree $d_{\mathbb{K}/\mathbb{Q}} = 6$. More the field $\mathbb{Q}(\gamma, \delta, \alpha)$ is Galois extension and a $\mathbb{Q}$-automorphism is for example the permutation

$$\sigma: \ \gamma \mapsto \delta, \ \delta \mapsto \gamma, \ \bar{\delta} \mapsto \bar{\delta} \ \text{ and } \ x \mapsto x, \ \text{ for } \ x \in \mathbb{Q}(\alpha). \tag{1.8}$$

1.2.2. *Some useful results on linear forms in logarithms of algebraic numbers.* We use Baker's theory of linear forms in logarithms of algebraic numbers for the proof of our result. Let η be an algebraic number with minimal primitive polynomial

$$f(X) = a_0 \prod_{i=1}^{d} \left(X - \eta^{(i)}\right) \in \mathbb{Z}[X],$$

where $a_0 > 0$, and $\eta = \eta^{(1)}, \cdots, \eta^{(d)}$ are conjugates of η. Then,

$$\mathrm{h}(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log \max\left\{\left|\eta^{(i)}\right|, 1\right\}\right)$$

is called the logarithmic height of η. In particular, if $\eta = P/Q$ is a rational number with $\gcd(P, Q) = 1$ and $Q > 1$, then $\mathrm{h}(\eta) = \max\{|P|, Q\}$. The following are some properties of logarithmic height function. For η, λ algebraic and $s \in \mathbb{Z}$ we have

- $\mathrm{h}(\eta+\lambda) \leq \mathrm{h}(\eta)+\mathrm{h}(\lambda)+\log 2$,
- $\mathrm{h}(\eta\lambda^{\pm 1}) \leq \mathrm{h}(\eta)+\mathrm{h}(\lambda)$,
- $\mathrm{h}(\eta^s) = |s|\,\mathrm{h}(\eta)$.

Now, let $\mathbb{K}$ be a real number field of degree $d_{\mathbb{K}}$, $\eta_1,\ldots,\eta_t$ positive elements of $\mathbb{K}$ and $b_1,\ldots,b_t$ integers non-zero all. Let $B \leq \max\{|b_1|,\ldots,|b_t|\}$ and

$$\Lambda = \eta_1^{b_1}\cdots\eta_t^{b_t} - 1.$$

Let $A_1,\ldots,A_l$ be real numbers with

$$A_i \geq \max\{d_{\mathbb{K}}\mathrm{h}(\eta_i), |\log\eta_i|, 0.16\}, \quad i=1,2,\ldots,t.$$

With the above notations, Matveev in [5] (see also [12, Theorem 9.4]) proved the following result which is the second tool we need.

Theorem 1.2. *Assume that $\Lambda \neq 0$. Then*

$$\log|\Lambda| > -1.4\cdot 30^{t+3}\cdot t^{4.5}\cdot d_{\mathbb{K}}^2\cdot(1+\log d_{\mathbb{K}})\cdot(1+\log B)A_1\cdots A_t.$$

In this paper we always use $t=3$. Since $\mathbb{K}=\mathbb{Q}(\gamma,\alpha)$ has degree $d_{\mathbb{K}/\mathbb{Q}}=6$, where γ and α are defined in Subsection 1.2.1. We thus, once and for all fix the constant

$$C := 1.43908\times 10^{13} > 1.4\cdot 30^{3+3}\cdot 3^{4.5}\cdot 6^2\cdot(1+\log 6).$$

Lemma 1.3. [10, Lemma 7] *Let ℓ be integer and H a real.*
If $\ell \geq 1$, $H > (4\ell^2)^\ell$ and $H > L/(\log L)^\ell$, then

$$L < 2^\ell H(\log H)^\ell.$$

Finally, we need a version of the reduction method of Baker-Davenport based on Lemma in [1]. We shall use the one given by Bravo, Gómez and Luca in [6] (See also Dujella and Pethö [2]). For a real number x, we write $\|x\|$ for the distance from x to the nearest integer.

Lemma 1.4. *Let M be a positive integer. Let τ, μ, $A>0$, $B>1$ be given real numbers. Assume that p/q is a convergent of τ such that $q>6M$ and that $\varepsilon := \|q\mu\| - M\|q\tau\| > 0$. Then there is no solution to the inequality*

$$0 < |n\tau - m + \mu| < \frac{A}{B^\omega}$$

in positive integers n, m and ω satisfying

$$n \leq M \qquad \text{and} \qquad \omega \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

2. Main results

We use the theory of lower bounds for linear forms in logarithms of algebraic numbers and a version of the reduction procedure from Baker and Davenport [1]. We then prove :

Theorem 2.1. *Let n and m be positive integers solution of Diophantine equation (1.2). Then we have $n \le m$. Moreover if $m \ge 7$,*

$$m+2 < 1.546 \cdot 10^{15} \left(\log \max\{\sqrt[3]{31}\sqrt{\Delta}, \alpha\}\right)^2 \log\left(7.7281 \cdot 10^{14}\left(\log \max\{\sqrt[3]{31}\sqrt{\Delta}, \alpha\}\right)^2\right),$$

where $\Delta := a^2 \pm 4$ and $\alpha = \frac{a+\sqrt{a^2\pm 4}}{2}$.

Proof. Let $m, n \ge 2$ be integers that satisfy the equation (1.2). Then from the inequalities in (1.5) and (1.7), we have

$$\gamma^{m-2} < \alpha^n \quad \text{and} \quad \alpha^{n-2} < \gamma^{m-1}.$$

So taking logarithms in that both inequalities we get the following inequalities.

$$(m+2)\frac{\log\gamma}{\log\alpha} < n < 2 + (m-1)\frac{\log\gamma}{\log\alpha}. \tag{2.1}$$

According to assumption that $a \in \mathbb{N}\backslash\{0\}$, $b = \pm 1$ and $(a, b) \notin \{(2, -1), (1, -1)\}$, we have

$$\Delta \ge \frac{a+\sqrt{5}}{2} \ge \frac{1+\sqrt{5}}{2}.$$

We thus obtain (the first step of the Theorem 2.1)

$$n < 2 + (m-1)\frac{\log\gamma}{\log\frac{1+\sqrt{5}}{2}} < m.$$

Now inserting Binet's formula of $(N_m)_{m\ge 0}$ and $(U_n)_{n\ge 0}$ in the equation (1.2) we get

$$C_\gamma\gamma^{m+2} - \frac{\alpha^n}{\sqrt{\Delta}} = -C_\delta\delta^{m+2} - C_{\overline{\delta}}\overline{\delta}^{m+2} - \frac{\beta^n}{\sqrt{\Delta}}.$$

Taking absolute values on both sides and dividing by $C_\gamma\gamma^{m+2}$, leads to

$$\begin{aligned}\left|(\sqrt{\Delta}C_\gamma)^{-1}\alpha^n\gamma^{-m-2} - 1\right| &< \left(2|C_\delta|\gamma^{\frac{-m-2}{2}} + \frac{\alpha^{-n}}{\sqrt{\Delta}}\right)C_\gamma^{-1}\gamma^{-m-2}\\ &< \left(2|C_\delta|\gamma^{\frac{-m-2}{2}} + \gamma^{-m-2}\right)C_\gamma^{-1}\gamma^{-m-2}, \quad \text{since } \Delta \ge 1 \text{ and } \gamma^{m-2} < \alpha^n\\ &< \left(2|C_\delta| + 1\right)C_\gamma^{-1}\gamma^{-m-2}, \quad \text{for } m \ge 7.\end{aligned}$$

Putting

$$\Lambda_1 := (\sqrt{\Delta}C_\gamma)^{-1}\alpha^n\gamma^{-m-2} - 1,$$

we then get, for $m \ge 7$

$$|\Lambda_1| < 9.35\gamma^{-m-2}. \tag{2.2}$$

We have $\Lambda_1 \neq 0$. Indeed if $\Lambda_1 = 0$ then $\alpha^n = \sqrt{\Delta} C_\gamma \gamma^{m+2}$ and applying the $\mathbb{Q}$-automorphism σ of the Galois extension $\mathbb{Q}(\gamma, \delta, \alpha)$ defined in (1.8), we have

$$\alpha^n = \sqrt{\Delta} C_\delta \delta^{m+2}$$
$$\alpha^n = \sqrt{\Delta}|C_\delta||\delta|^{m+2} = \sqrt{\Delta}|C_\delta|\gamma^{-(m+2)/2}.$$

This implies that $n = \frac{-(m+2)}{2}\frac{\log\gamma}{\log\alpha} + \frac{\log(|C_\delta|\sqrt{\Delta})}{\log\alpha}$ which is impossible cause of the first inequality in (2.1).
We now can apply Theorem 1.2 for Λ_1 with

$$\eta_1 = \sqrt{\Delta} C_\gamma, \quad \eta_2 = \alpha, \quad \eta_3 = \gamma, \quad b_1 = -1, \quad b_2 = n, \quad b_3 = -m-2.$$

We have $B = \max\{|b_1|, |b_2|, |b_3|\} = m+2$. We must now estimate $\mathrm{h}(\eta_i)$ for $i = 1, 2, 3$. Using properties of the height, we have

$$\mathrm{h}(\eta_1) \leq \mathrm{h}(\sqrt{\Delta}) + \mathrm{h}(C_\gamma) \leq \log\sqrt{\Delta} + \frac{\log 31}{3} = \log\left(\sqrt[3]{31}\sqrt{\Delta}\right),$$
$$\mathrm{h}(\eta_2) \leq \frac{\log\alpha}{2} \quad \text{and} \quad \mathrm{h}(\eta_3) \leq \frac{\log\gamma}{3}.$$

Thus, we can take

$$A_1 = 6\log\left(\sqrt[3]{31}\sqrt{\Delta}\right), \quad A_2 = 3\log\alpha \quad \text{and} \quad A_3 = 2\log\gamma \approx 0.76.$$

Applying Theorem 1.2, we have

$$\begin{aligned}\log|\Lambda_1| &> -C\cdot 0.76\cdot(3\log\alpha)\left(6\log\left(\sqrt[3]{31}\sqrt{\Delta}\right)\right)(1+\log(m+2)),\\ &> -1.969\cdot 10^{14}(\log\alpha)\left(\log\left(\sqrt[3]{31}\sqrt{\Delta}\right)\right)(1+\log(m+2)),\\ &> -2.954\cdot 10^{14}(\log\alpha)\left(\log\left(\sqrt[3]{31}\sqrt{\Delta}\right)\right)\log(m+2).\end{aligned}$$

Let put $\theta := \max\{\sqrt[3]{31}\sqrt{\Delta}, \alpha\}$. We have

$$\log|\Lambda_1| > -2.954\cdot 10^{14}(\log\theta)^2\log(m+2), \tag{2.3}$$

We take logarithm in (2.2), we have

$$\log|\Lambda_1| < -(m+2)\log\gamma + \log(9.35).$$

Combining this with (2.3) we obtain

$$m+2 < 7.7281\cdot 10^{14}(\log\theta)^2\cdot\log(m+2).$$

Hence, by applying Lemma 1.3 we get

$$m+2 < 1.546\cdot 10^{15}(\log\theta)^2\log\left(7.7281\cdot 10^{14}(\log\theta)^2\right). \tag{2.4}$$

This completes the proof of the Theorem 2.1. □

Applying the effective bounds from the Theorem 2.1 for Fibonacci sequence, Pell sequence and Balancing sequence, we obtain Corollaries 2.2, 2.3 and 2.4 below.

Corollary 2.2. *The only Fibonacci numbers in Narayana's cows sequence are* 1, 2, 3 *and* 13.

Corollary 2.3. *The only Pell numbers in Narayana's cows sequence are* 1 *and* 2.

Corollary 2.4. *The only Balancing numbers in Narayana's cows sequence are* 1 *and* 6.

Proof of the Corollaries 2.2, 2.3 and 2.4.
Here we apply the Theorem 2.1 with some special case of the sequence $(U_n)_{n\geq 0}$, namely Fibonacci $(F_n)_{n\geq 0}$, Pell $(P_n)_{n\geq 0}$, and Balancing $(B_n)_{n\geq 0}$ sequences.
As we can see, this upper bounds of solutions provided by the Theorem 2.1 is too large that must be reduced before by Baker reduction method.

Let consider Λ_1 defined in (2.2). For a positive real x, if $|x-1| < \frac{1}{2}$ then $|\log x| < 1.5\,|x-1|$ (see [8, Lemma 4]).
Hence we have, from there

$$\left|n\log\alpha - (m+2)\log\gamma - \log\left(\sqrt{\Delta}C_\gamma\right)\right| < 1.5\cdot 9.35\gamma^{-m-2},$$

and divide by $\log\gamma$, we get

$$\left|n\frac{\log\alpha}{\log\gamma} - (m+2) - \frac{\log\left(\sqrt{\Delta}C_\gamma\right)}{\log\gamma}\right| < 37.351\gamma^{-m-2}. \tag{2.5}$$

So we then apply the Lemma 1.4 with

$$\omega := m+2, \quad \tau := \frac{\log\alpha}{\log\gamma}, \quad \mu := \frac{\log\left(\sqrt{\Delta}C_\gamma\right)}{\log\gamma}, \quad A := 37.351, \quad B := \gamma$$

and

$$M := 1.546\cdot 10^{15}\left(\log\max\{\sqrt[3]{31}\sqrt{\Delta},\,\alpha\}\right)^2\log\left(7.7281\cdot 10^{14}\left(\log\max\{\sqrt[3]{31}\sqrt{\Delta},\,\alpha\}\right)^2\right).$$

Case	Δ	α	τ	μ	M
Fibonacci ($a=1$, $b=1$)	5	$\frac{1+\sqrt{5}}{2}$	$\frac{\log\alpha}{\log\gamma}$	$\frac{\log\left(\sqrt{5}C_\gamma\right)}{\log\gamma}$	$2.093\cdot 10^{17}$
Pell ($a=2$, $b=1$)	8	$1+\sqrt{2}$	$\frac{\log\alpha}{\log\gamma}$	$\frac{\log\left(2\sqrt{2}C_\gamma\right)}{\log\gamma}$	$2.645\cdot 10^{17}$
Balancing ($a=6$, $b=-1$)	32	$3+2\sqrt{2}$	$\frac{\log\alpha}{\log\gamma}$	$\frac{\log\left(4\sqrt{2}C_\gamma\right)}{\log\gamma}$	$4.659\cdot 10^{17}$

TABLE 1.

Fibonacci sequence case : We find that 39-th convergent of τ

$$\frac{p_{39}}{q_{39}} = \frac{3511279069780293827}{2789144197090847260},$$

satisfies $q_{39} > 6M$ and $\varepsilon = 0.458791087262574 > 0$. Hence the inequality (2.5) has no solution for

$$m + 2 \geq \frac{\log(37.351 \cdot q_{39}/\varepsilon)}{\log \gamma} \geq \frac{\log(37.351 \cdot q_{39}/0.458791087262574)}{\log \gamma} \geq 122.622$$

Thus we obtain $m \leq 120$ and consequently $n \leq 120$. We now compute

$$\{N_m : \ 1 \leq m \leq 120\} \cap \{F_n : \ 1 \leq n \leq 120\} = \{1, 2, 3, 5, 13\}$$

This is the proof of Corollary 2.2.

Pell sequence case : We find that 32-th convergent of τ

$$\frac{p_{32}}{q_{32}} = \frac{16629117764559080077}{7211923117474819860},$$

satisfies $q_{32} > 6M$ and $\varepsilon = 0.221867342218888 > 0$. So the inequality (2.5) has no solution for

$$m + 2 \geq \frac{\log(37.351 \cdot q_{32}/\varepsilon)}{\log \gamma} \geq \frac{\log(37.351 \cdot q_{32}/0.221867342218888)}{\log \gamma} \geq 127.008$$

Thus we obtain $m \leq 125$ and consequently $n \leq 125$. We now compute

$$\{N_m : \ 1 \leq m \leq 125\} \cap \{P_n : \ 1 \leq n \leq 125\} = \{1, 2\}$$

This completes the proof of Corollary 2.3.

Balancing sequence case : We find that 36-th convergent of τ

$$\frac{p_{36}}{q_{36}} = \frac{16629117764559080077}{3605961558737409930},$$

satisfies $q_{36} > 6M$ and $\varepsilon = 0.336040242950998 > 0$. Therefore the inequality (2.5) has no solution for

$$m + 2 \geq \frac{\log(37.351 \cdot q_{36}/\varepsilon)}{\log \gamma} \geq \frac{\log(37.351 \cdot q_{36}/0.336040242950998)}{\log \gamma} \geq 124.108.$$

Thus we obtain $m \leq 122$ and consequently $n \leq 122$. We now compute

$$\{N_m : \ 1 \leq m \leq 122\} \cap \{B_n : \ 1 \leq n \leq 122\} = \{1, 6\}$$

This finishes the proof of Corollary 2.4. □

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References

1. A. Baker and H. Davenport (1969). The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$, *Quart. J.Math. Oxford Ser.* 20, 129-137. https://doi.org/10.1093/qmath/20.1.129
2. A. Dujella, A. Pethö (1998). A generalization of a theorem of Baker and Davenport, *Quart. J. Math. Oxford* 49.3, 291-306. https://doi.org/10.1093/qmathj/49.3.291
3. Abdelalim, S., El Adlouni, H., and Diany, H. (2016). The Diophantine equation $x^3 + y^3 = 2z^2$. *Gulf Journal of Mathematics*, 4(4). https://doi.org/10.56947/gjom.v4i4.266
4. Diany, H., and Abdelalim, S. (2015). Characterization of the solution of the Diophantine equation $x^2 + y^2 = 2z^2$. *Gulf Journal of Mathematics*, 3(2). https://doi.org/10.56947/gjom.v3i2.154
5. E. M. Matveev (2000). An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers II, *Izv. Ross. Akad. Nauk Ser. Mat.* 64, 125-180. English Translation in *Izv. Math.* 64(6), 1217-1269. https://doi.org/10.4213/im314
6. J. J. Bravo, C. A. Gómez, F. Luca (2016). Powers of two as sums of two k-Fibonacci numbers, *Miskolc Math. Notes* 17.1, 85-100, https://doi.org/10.18514/MMN.2016.1505
7. J. J Bravo, P. Das and S. Guzmán (2020). Repdigits in Narayana's cows sequence and their consequences, *J. Integer Seq.* 23, Article 20.8.7. https://cs.uwaterloo.ca/journals/JIS/VOL23/Das/Bravo17.pdf
8. J. Odjoumani and V. Ziegler (2021). On prime powers in linear recurrence sequences, *Ann. Math. Québec*. https://doi.org/10.1007/s40316-021-00163-9
9. J. P. Allouche and T. Johnson (1996). Narayana's cows and delayed morphisms, *J. d'Inform. Music.*. https://hal.archives-ouvertes.fr/hal-02986050
10. S. G. Sánchez, F. Luca (2014). Linear combinations of factorials and S-units in a binary recurrence sequence, *Ann. Math. Québec* 38, 169-188. https://doi.org/10.1007/s40316-014-0025-z
11. P. Singh (1998). The Ganita kaumudi of Narayana Pandita, *Ganita-Bharati : Bull. Ind. soc. Hist. Math.*, Vol. 20. Nos.1-4, 25-82. https://archive.org/details/ganitakaumudinarayanapanditach113ed.parmanandsingh_680_X/mode/1up
12. Y. Bugeaud, M. Mignotte and S.Siksek (2006). Classical and modular approaches to exponential Diophantine equations I. Fibonacci and Lucas perfect powers, *Ann. of Math.* (2) 163, 969-1018. https://doi.org/10.4007/annals.2006.163.969

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