Gulf Journal of Mathematics
Vol 15, Issue 1 (2023) 93-108

ISSN: 2309-4966
https://doi.org/10.56947/gjom.v15i1.1382

SITT-RING PROPERTIES IN BI-AMALGAMATED RINGS ALONG IDEALS

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ABSTRACT. Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J and J' be two ideals of B and C, respectively, such that $f^{-1}(J) = g^{-1}(J')$. In this paper, we give a characterization for the amalgamation of A with B along J with respect to f (denoted by $A \bowtie^f J$) to be a SITT-ring and also we give a characterization for the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) (denoted by $A \bowtie^{f,g} (J, J')$) to be a SITT-ring. We also give some characterizations for strong weakly SIT-rings.

1. INTRODUCTION AND PRELIMINARIES

Throughout the paper, R denotes an associative ring with identity. An element a in a ring is an idempotent if $a^2 = a$. Idempotents are key elements in understanding the structure of a ring. When a ring has a rich supply of idempotents, one would usually expect something nice about the structure of the ring. An extreme case is the notion of Boolean rings, in which every element is an idempotent. Boolean rings are impotant in algebra and have applications in other areas. A ring R is weakly Boolean if for any $a \in R$, either a or $-a$ is an idempotent.

In 2016, Zhiling Ying, Tamer Kosan and Yiqiang Zhou [19] investigated that rings for which every element is a sum of an idempotent and a tripotent that commute and proved that every element of a ring R is a sum of an idempotent and a tripotent that commute and $2 \in J(R)$ if and only if R has the idetity $x^6 = x^4$ and $2 \in J(R)$ if and only if $R/J(R)$ is Boolean and $j^2 = 2j$ for all $j \in J(R)$ if and only if $R/J(R)$ is Boolean and $U(R)$ is a group of exponent 2. It can be shown that every element of a ring is a sum of an idempotent and a tripotent that commute iff $R \cong A \times B$, where $A/J(A)$ is Boolean with $U(A)$ a group of exponent 2, and B is a subdirect product of $\mathbb{Z}_3's$. For any idempotent e, both e and $-e$ are tripotents, i.e., the elements equal to their cubes.

Let A and B be two commutative rings with unity, let J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

Date: Received: Jun 18, 2022; Accepted: Dec 12, 2022.

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2010 *Mathematics Subject Classification.* Primary 16N20, 13A15.

Key words and phrases. idempotent, tripotent, strongly nil clean, subdirect product, Boolean ring.

called *the amalgamation of A with B along J with respect to f* (introduced and studied by D'Anna, Finocchiaro, and Fontana in [5, 6]). This construction is a generalization of *the amalgamated duplication of a ring along an ideal* (introduced and studied by D'Anna and Fontana in [2, 3, 4]). Moreover, other classical constructions (such as the $A+XB[X]$, $A+XB[[X]]$, and the $D+M$ constructions) can be studied as particular cases of the amalgamation ([5, Examples 2.5 and 2.6]) and other classical constructions, such as the Nagata's idealization (cf. [16, page 2]), and the CPI extensions (in the sense of Boisen and Sheldon [8]) are strictly related to it ([5, Example 2.7 and Remark 2.8]). On the other hand, the amalgamation $A \bowtie^f J$ is related to a construction proposed by Anderson in [1] and motivated by a classical construction due to Dorroh [10], concerning the embedding of a ring without identity in a ring with identity. An ample introduction on the genesis of the notion of amalgamation is given in [5, Section 2]. Also, the authors consider the iteration of the amalgamation process, giving some geometrical applications of it.

One of the key tools for studying $A \bowtie^f J$ is based on the fact that the amalgamation can be studied in the frame of pullback constructions [5, Section 4]. This point of view allows the authors in [5, 6] to provide an ample description of various properties of $A \bowtie^f J$, in connection with the properties of A, J and f. Namely, in [5], the authors studied the basic properties of this construction (e.g., characterizations for $A \bowtie^f J$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as an amalgamation. Moreover, in [6], they pursue the investigation on the structure of the rings of the form $A \bowtie^f J$, with particular attention to the prime spectrum, to the chain properties and to the Krull dimension.

Let $\alpha : A \longrightarrow C$, $\beta : A \longrightarrow C$ and $f : A \longrightarrow B$ be ring homomorphisms. In the aforementioned papers [5, 6], the authors studied amalgamated algebras within the frame of pullback $\alpha \times \beta$ such that $\alpha = \beta \circ f$ [5, Proposition 4.2 and 4.4]. In this motivation, the authors created the new constructions, called bi-amalgamated algebras which arise as pullbacks $\alpha \times \beta$ such that the following diagram of ring homomorphims

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ \downarrow g & & \downarrow \alpha \\ C & \xrightarrow{\quad \beta \quad} & D \end{array}$$

is commutative with $\alpha \circ \pi_B(\alpha \times \beta) = \alpha \circ f(A)$, where π_B denotes the canonical projection of $B \times C$ over B. Namely, let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J and J' be two ideals of B and C, respectively, such that $f^{-1}(J) = g^{-1}(J')$. The bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) is the subring of $B \times C$ given by

$$A \bowtie^{f,g} (J, J') := \{(f(a) + j, g(a) + j') | a \in A, (j, j') \in J \times J'\}$$

In this paper, the second section we introduce a class of rings which generalizes the so-called SIT-rings namely, weakly SIT-rings and SITT-rings. A ring R is called (strongly) weakly SIT-ring if every element of R is a sum or difference of an idempotent and a tripotent (that commute) and a ring R is called a SITT-ring if every element is a sum of an idempotent and two tripotents. Clearly, SIT-rings are weakly SIT-rings and SITT-rings, but the converse is not true in general. We give some characterizations of strongly weakly SIT-rings. Finally, we give some properties of a SITT-rings are presented. The third section, we give a characterization for $A \bowtie^f J$ to be (uniquely) SIT-ring. The fourth section investigates the characterization for $A \bowtie^{f,g} (J, J')$ to be (uniquely) SITT-ring.

In what follows, $\mathbb{Z}$ denote the ring of integers and for a positive integer n, $\mathbb{Z}_n$ is the ring of integers modulo n. We write $M_n(R)$ and $T_n(R)$ stands for the $n \times n$ matrix ring and $n \times n$ upper triangular matrix ring, respectively, over R. In addition, we write $Id(R)$, $Tr(R)$, $Nil(R)$ and $J(R)$ for the set of all idempotent elements, set of all tripotent elements of R, set of all nilpotent elements and the Jacobson radical of R, respectively.

2. SIT-ring and Weakly SIT-ring

We start with a definition.

Definition 2.1. A ring is said to be a (strong) SIT-ring if every element is a sum of an idempotent and a tripotent (that commute).

Theorem 2.2. *Let R be a ring with $2 \in J(R)$ and $U(R)$ forms a group of exponent 2. Then R is strong SIT-ring if and only if R is strongly nil clean.*

Proof. $\Rightarrow$ In view of [19, Theorem 3.6], $R/J(R)$ is Boolean. For $j \in J(R)$, we have $(1-j)^2 = 1$ by hypothesis, and $j^2 = 2j$. Replacing j by $j(j+1)$, we have $(j(1+j))^2 = 2j(1+j)$. We infer that $j(1+j)j = 2j$; that is, $j^2 + j^3 = 2j$. It follows that $j^3 = 0$. Hence, $J(R)$ is nil. In view of [12, Theorem 2.7], R is strongly nil clean.

$\Leftarrow$ For any $a \in R$, there exist $b \in Nil(R)$ and $e^2 = e$ such that $eb = be$ and $a - 1 = e + b$. By hypothesis, we have $(1+b)^2 = 1$, so $1+b$ is a tripotent. Hence, $a = e + (1+b)$ is a sum of an idempotent and a tripotent that commute. So R is a strong SIT-ring. □

If $U(R)$ does not form a group of exponent 2, the converse need not be true. We consider the following example.

Example 2.3. Let $\mathbb{Z}_4[i] = \mathbb{Z}_4[x]/(x^2-1) = \{a + bi | a, b \in \mathbb{Z}_4, i^2 = -1\}$ and the Jacobson radical $J(\mathbb{Z}_4[i]) = \{a + bi | a, b \in \mathbb{Z}_4, a^2 - b^2 \in J(\mathbb{Z}_4), i^2 = -1\}$. Thus $J(\mathbb{Z}_4[i]) = \{\bar{0}, \bar{2}i, \bar{1}+i, \bar{1}+\bar{3}i, \bar{2}, \bar{2}+\bar{2}i, \bar{3}+i, \bar{3}+\bar{3}i\}$. It is easy to check that $J(\mathbb{Z}_4[i])$ is nil. Hence, $N(\mathbb{Z}_4[i]) = J(\mathbb{Z}_4[i])$ and $\mathbb{Z}_4[i]/J(\mathbb{Z}_4[i])$ is Boolean. In view of [12, Theorem 2.7], $\mathbb{Z}_4[i]$ is strongly nil clean. In light of Theorem 2.2, $\mathbb{Z}_4[i]$ is not a SIT-ring because $U(\mathbb{Z}_4[i])$ is not a group of exponent 2.

Example 2.4. Let $R = \mathbb{Z}_4[x]/(x^2) = \{a + bx | a, b \in \mathbb{Z}_4\}$. Then set of all idempotents of R are $\{\bar{0}, \bar{1}\}$ and set of all tripotents of R are $\{\bar{0}, \bar{1}, \bar{3}, \bar{1}+\bar{2}x, \bar{3}+\bar{2}x\}$. According to the definition 2.1, R is not a SIT-ring because the element $\bar{1}+x \in R$ is not a sum of an idempotent and a tripotent.

Example 2.5. Let $\mathbb{Z}_{(3)} = \{\frac{m}{n} | m, n \in \mathbb{Z}_3, 3 \nmid n\}$. Then $\mathbb{Z}_{(3)}$ is a local ring with the Jacobson radical $J(\mathbb{Z}_{(3)}) = 3\mathbb{Z}_{(3)}$. Clearly, $\frac{2}{1} \in \mathbb{Z}_{(3)}$ satisfies $(\frac{2}{1})^2 - \frac{2}{1} = \frac{2}{1} \notin J(\mathbb{Z}_{(3)})$; hence $\mathbb{Z}_{(3)}/J(\mathbb{Z}_{(3)})$ is not Boolean. In view of [19, Theorem 3.6], $\mathbb{Z}_{(3)}$ is not a SIT-ring.

We now introduce the following definition.

Definition 2.6. A ring is said to be (strong) weakly SIT-ring if every element is a sum or difference of an idempotent and a tripotent (that commute).

Proposition 2.7. *The class of (strong) weakly SIT-rings are closed under homomorphic images and finite direct products . Any factor ring of a (strong) weakly SIT-ring is a (strongly) weakly SIT-ring.*

Lemma 2.8. *If R is a ring for which $5 = e + f$ or $5 = e - f$ where $e^2 = e$ and $f^3 = f$, then $120 = 0$.*

Proof. From $5 = e + f$, we see $ef = fe$, so $25 = (e + f)^2 = e + 2ef + f^2$. Thus, $4(e + f) = 20 = 25 - 5 = (e + 2ef + f^2) - (e + f) = 2ef + f^2 - f$. It follows that $4e + 5f - 2ef - f^2 = 0$. Thus, $0 = (4e + 5f - 2ef - f^2)ef^2 = 4ef^2 + 5ef - 2ef - ef^2 = 3ef^2 + 3ef$. So, $120 = 5^3 - 5 = (e + f)^3 - 5 = (e + 3ef + 3ef^2 + f) - 5 = [(e + f) - 5] + [3ef + 3ef^2] = 0$.

From $5 = e - f$, so $25 = (e - f)^2 = e - 2ef + f^2$. Thus, $4(e - f) = 20 = 25 - 5 = (e - 2ef + f^2) - (e - f) = -2ef + f^2 + f$. It follows that $4e - 5f + 2ef - f^2 = 0$. Thus, $0 = (4e - 5f + 2ef - f^2)ef^2 = 3ef^2 - 3ef$. So, $120 = 5^3 - 5 = (e - f)^3 - 5 = (e - 3ef + 3ef^2 - f) - 5 = [(e - f) - 5] + [3ef^2 - 3ef] = 0$. □

We now decompose (strong) weakly SIT-ring into product of (strong) weakly SIT-rings.

Lemma 2.9. *A ring R is a (strong) weakly SIT-ring if and only if $R \cong R_1 \times R_2 \times R_3$ where R_1, R_2 and R_3 are (strong) weakly SIT-rings, $2^3 = 0$ in R_1, $3 = 0$ in R_2 and $5 = 0$ in R_3.*

Proof. The sufficiency is clear by Proposition 2.7. For the necessity, assume that R is a (strong) weakly SIT-ring. Then, by Lemma 2.8 , $2^3.3.5 = 0$. Thus, $2^3R \bigcap 3R \bigcap 3R = 0$ and $R = 2^3R + 3R + 5R$. By the Chinese Remainder Theorem, $R \cong R/2^3R \times R/3R \times R/5R$. Let $R_1 = R/2^3R$, $R_2 = R/3R$ and $R_3 = R/5R$. Then R_1, R_2 and R_3 are (strong) weakly SIT-rings by Proposition 2.7 with $2^3 = 0$ in R_1, $3 = 0$ in R_2 and $5 = 0$ in R_3, and $R \cong R_1 \times R_2 \times R_3$. □

Theorem 2.10. *Let R be a ring. Then the following statements are equivalent.*

(1) *R is strongly weakly nil-clean;*
(2) *For any $a \in R$, $a \pm a^2$ is nilpotent;*
(3) *For any $a \in R$, there exists an idempotent $e \in \mathbb{Z}[x]$ such that $a \pm e$ is nilpotent;*

Proof. This was proved in [9, Theorem 2.1] . □

Theorem 2.11. *Let R be a ring. Then R is strongly weakly nil-clean iff $R/J(R)$ is weakly Boolean and $J(R)$ is nil.*

Proof. This was proved in [9, Theorem 3.2]. □

Following [9], an element of a ring is called (strongly) weakly nil clean if it is the sum or difference of a nilpotent and an idempotent (that commute), and the ring is called (strongly) weakly nil clean if each of its elements is (strongly) weakly nil clean.

Theorem 2.12. *Let R be a ring with $2 \in J(R)$. Then the following statements are equivalent.*

(1) *R is a strong weakly SIT-ring;*
(2) *R is a strong weakly SIT-ring with $2^3 = 0$;*
(3) *R has the identity $x^6 = x^4$;*
(4) *$R/J(R)$ is weakly Boolean and $j^2 = \pm 2j$ for all $j \in J(R)$;*
(5) *$R/J(R)$ is weakly Boolean and $U(R)$ is a group of exponent 2;*

Proof. (1) ⇒ (2). This is clear by Lemma 2.9.

(2) ⇒ (3). For $a \in R$, write $a = e + f$ or $a = e - f$ where $e^2 = e$, $f^3 = f$, and $ef = fe$. Then

$$\begin{aligned}
a^4 &= (e+f)^4 = e^4 + 4e^3f + 6e^2f^2 + 4ef^3 + f^4 \\
&= e + 4ef + 6ef^2 + 4ef + f^2 \\
&= e + 8ef + 6ef^2 + f^2 \\
&= e + 6ef + f^2 \\
a^6 &= a^4a^2 = (e + 6ef^2 + f^2)(e + 2ef + f^2) \\
&= (e + 6ef^2 + ef^2) + (2ef + 12ef + 2ef) + (ef^2 + 6ef^2 + f^2) \\
&= e + 6ef^2 + f^2
\end{aligned}$$

$$\begin{aligned}
a^4 &= (e-f)^4 = e^4 - 4e^3f + 6e^2f^2 - 4ef^3 + f^4 \\
&= e - 4ef + 6ef^2 - 4ef + f^2 \\
&= e - 8ef + 6ef^2 + f^2 \\
&= e + 6ef + f^2 \\
a^6 &= a^4a^2 = (e + 6ef^2 + f^2)(e - 2ef + f^2) \\
&= (e + 6ef^2 + ef^2) + (-2ef - 12ef - 2ef) + (ef^2 + 6ef^2 + f^2) \\
&= e + 6ef^2 + f^2
\end{aligned}$$

So $a^6 = a^4$. Hence, R has the identity $x^6 = x^4$.

(3) ⇒ (4). For $j \in J(R)$, we have $(1-j)^6 = (1-j)^4$, so $(1-j)^2 = 1$ as $(1-j)^4 \in U(R)$. It follows that $j^2 = 2j$. Hence, we have proved that $j^2 = 2j$ for all $j \in J(R)$ and we have $(1+j)^6 = (1+j)^4$, so $(1+j)^2 = 1$ as $(1+j)^4 \in U(R)$. It follows that $j^2 = -2j$. Hence, we have proved that $j^2 = -2j$ for all $j \in J(R)$. From $2^6 = 2^4$, we obtain $2^4 3 = 0$. As $2 \in J(R)$, $3 \in U(R)$, so we infer $2^4 = 0$. For $a \in R$, we have $a^6 = a^4$, so $(a - a^2)^4 = a^4(1-a)^4 = a^4(1 - 4a + 6a^2 - 4a^3 + a^4) = a^4 - 4a^5 + 6a^6 - 4a^7 + a^8 = a^4 - 4a^5 + 6a^4 - 4a^5 + a^4 = 8(a^4 - a^5)$, which is nilpotent

as 2 is nilpotent. Thus, $a-a^2$ is nilpotent and $(a+a^2)^4 = a^4(1+a)^4 = a^4(1+4a+6a^2+4a^3+a^4) = a^4+4a^5+6a^6+4a^7+a^8 = a^4+4a^5+6a^4+4a^5+a^4 = 8(a^4+a^5)$, which is nilpotent as 2 is nilpotent. Thus, $a+a^2$ is nilpotent. By Theorem 2.10, there exists $e^2 = e$ such that $ea = ae$ and $a \pm e$ is nilpotent. This shows that $a = e + (a - e)$ or $-a = e - (a + e)$ is strongly weakly nil clean. Therefore, R is strongly weakly nil clean. In light of Theorem 2.11, $R/J(R)$ is weakly Boolean.

(4) $\Rightarrow$ (5). For $u \in U(R)$, we have $u^2 \pm u \in J(R)$ since $R/J(R)$ is weakly Boolean. Then $u \in 1 \pm J(R)$; hence, $U(R) = 1 \pm J(R)$. Write $u = 1 \pm j$ for $j \in J(R)$. Then $u^2 = (1 - j)^2 = 1 - 2j + j^2 = 1$, as $j^2 = 2j$ and $u^2 = (1 + j)^2 = 1 + 2j + j^2 = 1$, as $j^2 = -2j$. Hence, $U(R)$ is a group of exponent 2.

(5) $\Rightarrow$ (1). For $j \in J(R)$, we have $(1 - j)^2 = 1$, so $j^2 = 2j$. Replacing j by $j(1 + j)$, we have $(j(1 + j))^2 = 2j(1 + j)$. We infer that $j(1 + j)j = 2j$; that is $j^2 + j^3 = 2j$. It follows that $j^3 = 0$ and we have $(1 + j)^2 = 1$, so $j^2 = -2j$. Replacing j by $j(1-j)$, we have $(j(1-j))^2 = 2j(1-j)$. We infer that $j(1 - j)j = 2j$; that is $j^2 + j^3 = 2j$. It follows that $j^3 = 0$. Hence, $J(R)$ is nil. Since $R/J(R)$ is weakly Boolean, R is strongly weakly nil clean by Theorem 2.11. Therefore, for any $a \in R$, there exist $b \in Nil(R)$ and $e^2 = e$ such that $eb = be$ and $a - 1 = e + b$ and $a + 1 = e - b$. By (5), $(1 + b)^2 = 1$, so $1 + b$ is a tripotent. So, R is a strong weakly SIT-ring. Moreover, since $R/J(R)$ is weakly Boolean, $2 \in J(R)$ as required. □

Proposition 2.13. *If R is a strong weakly SIT-ring with $3 = 0$ then R is a subdirect product of $\mathbb{Z}_3'$s.*

Proof. Let $a \in R$, and write $a = e + f$ or $a = e - f$ where $e^2 = e$, $f^3 = f$, and $ef = fe$. Then $a^3 = (e + f)^3 = e^3 + 3e^2f + 3ef^2 + f^3 = e + f = a$ or $a^3 = (e - f)^3 = e^3 - 3e^2f + 3ef^2 - f^3 = e - f = a$. Hence, R has the identity $x^3 = x$. Since $3 = 0$ in R, R is s subdirect product of $\mathbb{Z}_3'$s. □

Proposition 2.14. *If R is a strong weakly SIT-ring with $5 = 0$ then R is a subdirect product of $\mathbb{Z}_5'$s.*

Proof. Let $a \in R$, and write $a = e + f$ or $a = e - f$ where $e^2 = e$, $f^3 = f$, and $ef = fe$. Then $a^5 = (e+f)^5 = e^5+5e^4f+10e^3f^2+10e^2f^3+5ef^4+f^5 = e+f = a$ or $a^5 = (e - f)^5 = e^5 - 5e^4f + 10e^3f^2 - 10e^2f^3 + 5ef^4 - f^5 = e - f = a$. Hence, R has the identity $x^5 = x$. Since $5 = 0$ in R, R is s subdirect product of $\mathbb{Z}_5'$s. □

Proposition 2.15. *Every SIT-ring is a weakly SIT-ring but the converse is not true.*

Example 2.16. Consider the group ring $\mathbb{Z}_6C_2 = \{\Sigma_{g\in C_2} a_g g | a_g \in \mathbb{Z}_6\}$ $= \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, x, \bar{2}x, \bar{3}x, \bar{4}x, \bar{5}x, \bar{1}+x, \bar{2}+x, \bar{3}+x, \bar{4}+x, \bar{5}+x, \bar{1}+\bar{2}x, \bar{2}+\bar{2}x, \bar{3}+\bar{2}x, \bar{4}+\bar{2}x, \bar{5}+\bar{2}x, \bar{1}+\bar{3}x, \bar{2}+\bar{3}x, \bar{3}+\bar{3}x, \bar{4}+\bar{3}x, \bar{5}+\bar{3}x, \bar{1}+\bar{4}x, \bar{2}+\bar{4}x, \bar{3}+\bar{4}x, \bar{4}+\bar{4}x, \bar{5}+\bar{4}x, \bar{1}+\bar{5}x, \bar{2}+\bar{5}x, \bar{3}+\bar{5}x, \bar{4}+\bar{5}x, \bar{5}+\bar{5}x\}$. Here,

$Id(\mathbb{Z}_6C_2) = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{2}+\bar{2}x, \bar{2}+\bar{4}x, \bar{5}+\bar{4}x, \bar{5}+\bar{2}x\}$ and $Tr(\mathbb{Z}_6C_2) = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, x, \bar{2}x, \bar{3}x, \bar{4}x, \bar{5}x, \bar{2}+x, \bar{4}+x, \bar{1}+\bar{2}x, \bar{2}+\bar{2}x, \bar{4}+\bar{2}x, \bar{5}+\bar{2}x, \bar{2}+\bar{3}x, \bar{4}+\bar{3}x, \bar{2}+$

$\bar{4}x, \bar{3}+\bar{4}x, \bar{4}+\bar{4}x, \bar{5}+\bar{4}x, \bar{2}+\bar{5}x, \bar{4}+\bar{5}x\}$. Every element of $\mathbb{Z}_6C_2$ is a sum of an idempotent and a tripotent except $\bar{5}+\bar{5}x$. But the element $\bar{5}+\bar{5}x$ is the difference of an idempotent and a tripotent i.e $\bar{5}+\bar{5}x = \bar{1}-(\bar{2}+x)$. Therefore, the group ring $\mathbb{Z}_6C_2$ is a weakly SIT-ring.

3. SIT-ring properties in Amalgamated algebras along an ideal

We begin with the following examples:

Example 3.1. Let $A = \mathbb{Z}_2$ and $B = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ be the rings and $J = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ the ideal of B and $f : A \longrightarrow B$ defined by $f(a) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ where $a \in \mathbb{Z}_2$. Then $f(A)+J = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$. Hence, A and $f(A)+J$ are SIT-rings. Also, $A \bowtie^f J = \left\{ (0, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}), (0, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}), (0, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}), (0, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}), (1, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}), (1, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}), (1, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}), (1, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}) \right\}$, where $Id(A \bowtie^f J) = \left\{ (0, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}), (0, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}), (0, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}), (1, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}), (1, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}), (1, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}) \right\}$ and $Tr(A \bowtie^f J) = \left\{ (0, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}), (0, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}), (0, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}), (1, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}), (1, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}), (1, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}), (1, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}) \right\}$ is a SIT-ring.

Example 3.2. Let $A = \mathbb{Z}_2$ and $B = \begin{bmatrix} \mathbb{Z}_2 & 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 & 0 \\ 0 & 0 & \mathbb{Z}_2 \end{bmatrix}$ be the rings and $J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathbb{Z}_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ the ideal of B and $f : A \longrightarrow B$ defined by $f(a) = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$ where $a \in \mathbb{Z}_2$. Then $f(A)+J = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$. Hence, A and $f(A)+J$ are SIT-rings.

Also, $A \bowtie^f J = \left\{ (0, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}), (0, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}), (1, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}), \right.$
$\left. (1, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}) \right\}$, where $Id(A \bowtie^f J) = \left\{ (0, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}), (0, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}), \right.$
$\left. (1, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}), (1, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}) \right\} = Tr(A \bowtie^f J)$. Hence, $A \bowtie^f J$ is a SIT-ring.

Definition 3.3. A ring is called uniquely SIT-ring if each element in R can be written uniquely as the sum of an idempotent and a tripotent.

Proposition 3.4. *Let $f : A \to B$ be a ring homomorphism and J an ideal of B. If $A \bowtie^f J$ is a uniquely SIT-ring then A is a uniquely SIT-ring.*

Proof. The case $J = (0)$. In view of [5, Proposition 5.1 (3)], A is a SIT-ring. Otherwise, by the same reference, the ring A is a proper homomorphic image of $A \bowtie^f J$. Since the class of SIT-rings is closed under homomorphic images, the ring A is SIT-ring. Since $A \bowtie^f J$ is uniquely SIT-ring and consider $e+t = e'+t'$ where $e, e' \in Id(A)$ and $t, t' \in Tr(A)$. Then, $(e, f(e)) + (t, f(t)) = (e', f(e')) + (t', f(t'))$ and clearly $(e, f(e)), (e', f(e')) \in Id(A \bowtie^f J)$ and $(t, f(t)), (t', f(t')) \in Tr(A \bowtie^f J)$. Then, $(e, f(e)) = (e', f(e'))$ and $(t, f(t)) = (t', f(t'))$. Hence, $e = e'$ and $t = t'$. Consequently, A is uniquely SIT-ring. □

Proposition 3.5. *Let $f : A \to B$ be a ring homomorphism and J an ideal of B. If $A \bowtie^f J$ is a SIT-ring then A and $f(A) + J$ are SIT-rings.*

Proof. The cases $J = (0)$ and $f^{-1}(J) = (0)$ follow easily from [5, Proposition 5.1 (3)]. Otherwise, by the same reference, the rings A and $f(A) + J$ are proper homomorphic images of $A \bowtie^f J$ and so they are SIT-rings.

Proposition 3.6. *Assume that $\frac{f(A)+J}{J}$ is uniquely SIT-ring. Then $A \bowtie^f J$ is a SIT-ring if and only if A and $f(A) + J$ are SIT-rings.*

Proof. If $A \bowtie^f J$ is a SIT-ring, then so A and $f(A) + J$. Conversely, assume that A and $f(A) + J$ are SIT-rings and consider $(a, j) \in A \times J$. Since A is SIT-ring, we can write $a = e + t$, where $e \in Id(A)$ and $t \in Tr(A)$. On the other hand, since $f(A) + J$ is SIT-ring, $f(a) + j = f(x) + j_1 + f(y) + j_2$ with $f(x) + j_1$ and $f(y) + j_2$ are respectively an idempotent and a tripotent element of $f(A) + J$. It is clear that $\overline{f(x)} = \overline{f(x) + j_1}$ (resp. $\overline{f(e)}$) and $\overline{f(y)} = \overline{f(y) + j_2}$ (resp. $\overline{f(t)}$) are respectively an idempotent and a tripotent element of $\frac{f(A)+J}{J}$, and we have $\overline{f(a)} = \overline{f(e)} + \overline{f(t)} = \overline{f(x)} + \overline{f(y)}$. Thus, $\overline{f(e)} = \overline{f(x)}$ and $\overline{f(t)} = \overline{f(y)}$ since $\frac{f(A)+J}{J}$ is uniquely SIT-ring. Consider $j_1', j_2' \in J$ such that $f(x) = f(e) + j_1'$ and $f(y) = f(t) + j_2'$. We have, $(a, f(a)+j) = (e, f(e)+j_1'+j_1)+(t, f(t)+j_2'+j_2)$, and it is clear that $(e, f(e) + j_1' + j_1)$ is an idempotent and $(t, f(t) + j_2' + j_2)$ a tripotent element of $A \bowtie^f J$. □

Remark 3.7. Let $f : A \to B$ be a ring homomorphism and J an ideal of B.

(1) If $B = J$ then, $A \bowtie^f B$ is SIT-ring if and only if A and B are SIT-ring since $A \bowtie^f B = A \times B$.
(2) If $f^{-1}(J) = \{0\}$ then, $A \bowtie^f J$ is SIT-ring if and only if $f(A) + J$ is SIT-ring (by [5, Proposition 5.1(3)]).

Definition 3.8. Let A and B be two rings with unity. Let J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

called *the amalgamation of A with B along J with respect to f*. When $A = B$ and $f = id_A$, the amalgamated $A \bowtie^{id_A} J$ is called amalgamated duplication of a ring A along the ideal J, and denoted by $A \bowtie J$.

Corollary 3.9. *Let A be a ring and I an ideal such that A/I is a uniquely SIT-ring. Then, $A \bowtie I$ is SIT-ring if and only if A is a SIT-ring.*

Proposition 3.10. *Let $f : A \to B$ be a ring homomorphism and let (e) be an ideal of B generated by the idempotent element e. Then $A \bowtie^f (e)$ is SIT-ring if and only if A and $f(A) + (e)$ are SIT-ring.*
In particular, if e is an idempotent element of A then, $A \bowtie (e)$ is SIT-ring if and only if A is a SIT-ring.

Proof. In light of Proposition 3.5, we have only to show that $A \bowtie^f (e)$ is SIT-ring provided A and $f(A) + (e)$ are SIT-ring. Let $(a, f(a) + re)$ be an element of $A \bowtie^f (e)$ (with $a \in A$ and $r \in B$). Since A and $f(A) + (e)$ are SIT-ring, there exists s and t (resp. s' and t') in A (resp. $f(A) + (e)$) which are respectively idempotent and tripotent such that $a = s + t$ and $f(a) + re = s' + t'$. We have

$$(a, f(a) + re) = (s, f(s) + (s' - f(s))e) + (t, f(t) + (t' - f(t))e)$$

On the other hand,

$$\begin{aligned} [f(s) + (s' - f(s))e]^2 &= [f(s)(1 - e) + s'e]^2 \\ &= f(s)(1 - e) + s'e \\ &= f(s) + (s' - f(s))e \end{aligned}$$

and

$$\begin{aligned} [f(t) + (t' - f(t))e]^3 &= [f(t)(1 - e) + t'e]^3 \\ &= f(t)(1 - e) + t'e \\ &= f(t) + (t' - f(t))e \end{aligned}$$

Then, $(s, f(s) + (s' - f(s))e)$ and $(t, f(t) + (t' - f(t))e)$ are respectively idempotent and tripotent in $A \bowtie^f (e)$. Consequently, $A \bowtie^f (e)$ is SIT-ring, as desired. Finally, if $A = B$ and $f = id_A$ then $A \bowtie^f (e) = A \bowtie (e)$ and $f(A) + (e) = A$. Thus, the particular case is a direct consequence of what is above. □

Proposition 3.11. *Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B.*

(1) *If $J \subseteq Id(B)$ and $3 = 0$ then, $A \bowtie^f J$ is SIT-ring if and only if A is SIT-ring.*
(2) *The ring $A \bowtie^f J$ is tripotent if and only if A is tripotent and $J \subseteq Tr(B)$ and $3 = 0$ in B.*

Proof. Note first that if $J \subseteq Id(B)$ then $2J = (0)$. Indeed, let $j \in J$. Clearly, $2j \in J \subseteq Id(B)$. Then, $j + j = (j + j)^2 = j^2 + 2j^2 + j^2 = j + 2j + j$. Hence, $2j = 0$.

(1) Let $(a, f(a)+j)$ be an element of $A \bowtie^f J$ (with $a \in A$ and $j \in J$). We have $a = e+t$ where e and t are respectively an idempotent and a tripotent in A. We have $(f(e)+j)^2 = f(e)^2+j^2+2f(e)j = f(e)+j$ since $2j = 0$ and also $J \subseteq Tr(B)$ and $3 = 0$ in B then, we have $(f(t)+j)^3 = f(t)^3+3f(t)^2j+3f(t)j^2+j^3 = f(t)+j$. Hence, $(e, f(e)+j)$ and $(t, f(t)+j)$ are respectively an idempotent and tripotent in $A \bowtie^f J$, and we have $(a, f(a)+j) = ((e, f(e)+j)+(t, f(t)+j)$. Consequently, $A \bowtie^f J$ is SIT-ring.
The converse implication is clear.

(2) If $A \bowtie^f J$ is tripotent, for each $a \in A$, $(a, f(a)) = (a, f(a))^3 = (a^3, f(a)^3)$. Then, $a = a^3$. Hence, A is tripotent. Moreover, for each $j \in J$, $(0, j) = (0, j)^3 = (0, j^3)$. Thus, $j = j^3$. Hence, $J \subseteq Tr(B)$.
Now, assume that A is tripotent and $J \subseteq Tr(B)$ and $3 = 0$ in B. Hence, for each $a \in A$ and $j \in J$, $(a, f(a)+j)^3 = (a^3, f(a)^3+3f(j)^2j+3f(a)j^2+j^3) = (a, f(a)+j)$. Thus, $A \bowtie^f J$ is tripotent. □

4. SITT-ring properties in bi-amalgamated rings along ideals

The matrix ring $M_2(\mathbb{Z}_2)$ is not a SIT-ring. Because every element of $M_2(\mathbb{Z}_2)$ is sum of an idempotent and a tripotent except the element $\begin{pmatrix} \bar{0} & \bar{1} \\ \bar{1} & \bar{0} \end{pmatrix}$ in $M_2(\mathbb{Z}_2)$ is not a sum of an idempotent and a tripotent. But, the element $\begin{pmatrix} \bar{0} & \bar{1} \\ \bar{1} & \bar{0} \end{pmatrix}$ can be written as sum of an idempotent and two tripotents. In this situation, we introduce the following definition and some properties of SITT-ring.

Definition 4.1. A ring is called a (strong) SITT-ring if every element is a sum of an idempotent and two tripotents (that commute with one another).

Example 4.2. The matrix ring $M_2(\mathbb{Z}_2)$ is a SITT-ring but not a SIT-ring.

Proposition 4.3. *The class of SITT-rings is closed under direct product and homomorphic images.*

Proposition 4.4. *If R is a strong SITT-ring and $2 \in Nil(R)$, then $a^2 - a$ is nilpotent for all $a \in R$*

Proof. Let $a \in R$ and write $a = e + f + g$, where $e \in Id(R), f, g \in Tr(R)$. Then $a^2 = e + f^2 + g^2 (mod\ 2R)$. Thus $a^2 - a \equiv (f^2 - f) + (g^2 - g)(mod\ 2R)$, with $(f^2 - f)^2 = 2(f^2 - f)$, and $(g^2 - g)^2 = 2(g^2 - g)$. So $(f^2 - f)$ and $(g^2 - g)$ are nilpotent. Hence, $a^2 - a$ is nilpotent. □

Proposition 4.5. *If R is a strong SITT-ring and $3 \in Nil(R)$, then $a^3 - a$ is nilpotent for all $a \in R$.*

Proof. Let $a \in R$ and write $a = e + f + g$, where $e \in Id(R)$, $f, g \in Tr(R)$. Then

$$a^3 = (e + f + g) + 3ef + 3eg + 3ef^2 + 3f^2g + 3eg^2 + 3fg^2 + 6efg.$$

So $a^3 - a = 3ef + 3eg + 3ef^2 + 3f^2g + 3eg^2 + 3fg^2 + 6efg$. Since $3 \in Nil(R)$, $a^3 - a$ is nilpotent. □

Proposition 4.6. *([13, Theorem 2.1]) Let R be a ring with $J(R) = 0$ such that every nonzero right ideal contains a nonzero idempotent. If $a^n = 0$ but $a^{n-1} \neq 0$, then there exists $e^2 = e \in RaR$ such that $eRe \cong \mathbb{M}_n(T)$ for some nontrivial ring T.*

Theorem 4.7. *If R is a strong SITT-ring and $3 \in Nil(R)$, then $J(R)$ is nil and $R/J(R)$ is a subdirect product of $\mathbb{Z}_3$'s.*

Proof. By Proposition 4.5, $a^3 - a$ is nilpotent for all $a \in R$. Then for $j \in J(R)$, $j(j^2 - 1) = j^3 - j$ is nilpotent. So j is nilpotent as $j^2 - 1 \in U(R)$. Thus, $J(R)$ is nil. For any $a \in R$, $(a - a^3)^n = 0$ for some $n \geq 1$. Then $a^n(1 - a^2)^n = 0$, and it follows that $a^n \in a^{n+1}R \cap Ra^{n+1}$. So R, and hence $\bar{R} := R/J(R)$ are strongly π-regular. We next show that $\bar{R}$ is reduced. Assume that $\bar{a}^2 = 0$ for some $0 \neq \bar{a} \in R$. Then, by Proposition 4.6, there exists $\bar{0} \neq \bar{y}^2 = \bar{y} \in \bar{R}\bar{y}\bar{R}$ such that $\bar{y}\bar{R}\bar{y} \cong \mathbb{M}_2(T)$, where T is a nontrivial ring. Let $s = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in \mathbb{M}_2(T)$. We have $s^3 = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}$. So $s - s^3 = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$. We calculate that $(s - s^3)^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, which is not nilpotent. Therfore, $s - s^3$ is not nilpotent, a contradiction. This shows that $\bar{R}$ is reduced. So $s^3 = s$ for all $s \in \bar{R}$. Hence, $\bar{R}$ is a subdirect product of $\mathbb{Z}_3$'s (see [14, Ex. 12.11, p200]). □

Theorem 4.8. *Let R be a strongly clean ring and $2 \in Nil(R)$, If R is a strong SITT-ring, then $R/J(R)$ is Boolean with $J(R)$ is Nil.*

Proof. By Proposition 4.4, $a^2 - a$ is nilpotent. Then, by [12, Theorem 2.1], R is srongly nil-clean, so $R/J(R)$ is Boolean with $J(R)$ is Nil, by [12, Theorem 2.7]. □

Definition 4.9. If $\alpha : A \longrightarrow C$, $\beta : B \longrightarrow C$ are two ring homomorphisms. Then the subring $D := \alpha \times_C \beta := \{(a, b) \in A \times B \mid \alpha(a) = \beta(b)\}$ is called the pullback of α and β.(see[6])

Definition 4.10. Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be two ring homomorphisms and let J and J' be two ideals of B and C, respectively, such that $I_0 := f^{-1}J = g^{-1}J'$. The bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) is the subring of $(B \times C)$ given by $A \bowtie^{f,g} (J, J') := \{(f(a) + j, g(a) + j') | a \in A, (j, j') \in J \times J'\}$.

Following [11], the above definition was introduced by and studied by Kabbaj, Louartiti and Tamekkante in 2013.

Theorem 4.11. *Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be ring homomorphisms and J, J' be two ideals of B and C, respectively, such that $I := f^{-1}(J) = g^{-1}(J')$. Let $u : f(A) + J \longrightarrow A/I$ defined by $u(f(a) + j) = a + I$ and $u : g(A) + J' \longrightarrow A/I$ defind by $v(g(a)+j') = a+I$, where $a \in A$, $j \in J$ and $j' \in J'$. Then $A \bowtie^{f,g} (J, J')$ is the pullback of the maps u and v.*

Proof. Consider the following diagram with $\alpha(f(a) + j, g(a) + j') = f(a) + j$ and $v(f(a) + j, g(a) + j') = g(a) + j'$ where $(f(a) + j, g(a) + j') \in A \bowtie^{f,g} (J, J')$.

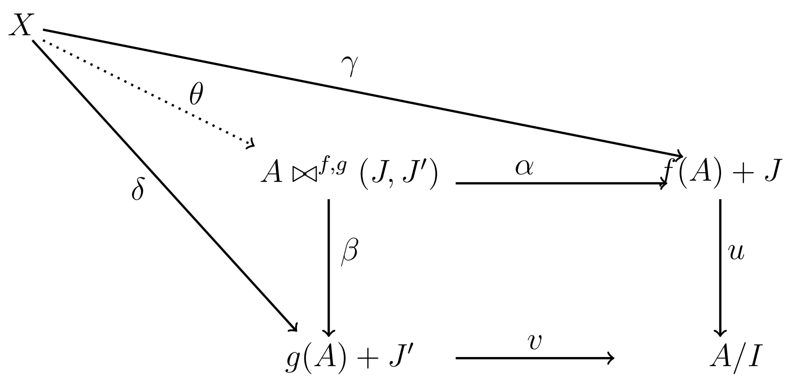

Fig. 1

Then $u\alpha = v\beta$. For if $(f(a) + j, g(a) + j') \in A \bowtie^{f,g} (J, J')$, then $u\alpha(f(a) + j, g(a)+j') = u(f(a)+j) = a+I$ and $v\beta(f(a)+j, g(a)+j') = v(g(a)+j') = a+I$. Let X be any ring and γ and δ ring homomorphisms such that $v\delta = u\gamma$. For any $x \in X$, set $\gamma(x) = f(a)+j \in A$ and $\delta(x) = g(a)+j'$. Then $v\delta(x) = v(f(a)+j) = a + I$ and $u\delta(x) = u(g(a) + j') = a + I$. Define $\theta : X \longrightarrow A \bowtie^{f,g} (J, J')$ by $\theta(x) = (f(a)+j, g(a)+j')$. Then $\alpha\theta(x) = \alpha(f(a)+j, g(a)+j') = f(a)+j = \gamma(x)$ and $\beta\theta(x) = \beta(f(a) + j, g(a) + j') = g(a) + j' = \delta(x)$. On the other hand, θ is unique, for if ζ satisfies $\alpha\zeta = \gamma$ and $\beta\zeta = \delta$, then $\zeta = \theta$ since β is a monomorphism. This completes the proof. $\square$

Example 4.12. Let $A = \mathbb{Z}_2$ and $B = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ and $C = \begin{bmatrix} \mathbb{Z}_2 & 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 & 0 \\ 0 & 0 & \mathbb{Z}_2 \end{bmatrix}$ be the rings and $J = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$ the ideal of B and $J' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathbb{Z}_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ the ideal of C. Let $f : A \longrightarrow B$ defined by $f(a) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ where $a \in \mathbb{Z}_2$ and $g : A \longrightarrow C$ defined by $g(a) = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$ where $a \in \mathbb{Z}_2$. Then $f(A) + J = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ and $g(A) + J' = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \right.$

$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix} \Bigg\}$. Hence, $f(A)+J$ and $g(A)+J'$ are SITT-rings. Also,

$$A \bowtie^{f,g} (J,J') = \Bigg\{ \Big(\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\Big), \Big(\begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}\Big), \Big(\begin{bmatrix}0&0\\0&0\end{bmatrix},$$
$$\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}\Big), \Big(\begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\Big), \Big(\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\Big), \Big(\begin{bmatrix}1&1\\0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\Big),$$
$$\Big(\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\Big), \Big(\begin{bmatrix}1&1\\0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\Big) \Bigg\},$$

where $Id(A \bowtie^{f,g} (J,J')) =$

$$\Bigg\{ \Big(\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\Big), \Big(\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}\Big), \Big(\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\Big),$$
$$\Big(\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\Big) \Bigg\}$$

and $Tr(A \bowtie^{f,g} (J,J')) =$

$$\Bigg\{ \Big(\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\Big), \Big(\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}\Big), \Big(\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\Big),$$
$$\Big(\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\Big), \Big(\begin{bmatrix}1&1\\0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\Big), \Big(\begin{bmatrix}1&1\\0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\Big) \Bigg\}$$

is a SITT-ring.

Proposition 4.13. *If $A \bowtie^{f,g} (J,J')$ is a SITT-ring then $f(A)+J$ and $g(A)+J'$ are a SITT-rings.*

Proof. Clearly, homomorphic image of a SITT-ring is a SITT-ring. Thus, in view of [11, Proposition 4.1], we have the following isomorphism of rings $\frac{A\bowtie^{f,g}(J,J')}{0\times J'} \cong f(A)+J$ and $\frac{A\bowtie^{f,g}(J,J')}{J\times 0} \cong g(A)+J'$. Hence, $f(A)+J$ and $g(A)+J'$ are SITT-rings. ☐

Definition 4.14. A ring is called uniquely SITT-ring if each element in R can be written uniquely as the sum of an idempotent and two tripotents.

Proposition 4.15. *Assume that A is SITT-ring and $\dfrac{f(A)+J}{J}$ and $\dfrac{g(A)+J'}{J'}$ are uniquely SITT-rings. Then $A \bowtie^{f,g} (J,J')$ is a SITT-ring if and only if $f(A)+J$ and $g(A)+J'$ are SITT-rings.*

Proof. If $A \bowtie^{f,g} (J,J')$ is a SITT-ring, then so $f(A)+J$ and $g(A)+J'$ by Proposition 4.13. Conversely, assume that $f(A)+J$ and $g(A)+J'$ are SITT-rings. Since A is SITT-ring, we can write $a = e + t_1 + t_2$, where $e \in Id(A)$ and $t_1, t_2 \in Tr(A)$. On the other hand, since $f(A)+J$ is SITT-ring, $f(a)+j = f(x)+j_1+f(y)+j_2+f(z)+j_3$ with $f(x)+j_1$ and $f(y)+j_2$, $f(z)+j_3$ are respectively an idempotent and a tripotent elements of $f(A)+J$. It is clear that $\overline{f(x)} = \overline{f(x)+j_1}$ (resp. $\overline{f(e)}$) and $\overline{f(y)} = \overline{f(y)+j_2}$ (resp. $\overline{f(t_1)}$), $\overline{f(z)} = \overline{f(z)+j_3}$ (resp. $\overline{f(t_2)}$) are respectively an idempotent and a tripotent elements of $\dfrac{f(A)+J}{J}$, and we have $\overline{f(a)} = \overline{f(e)} + \overline{f(t_1)} + \overline{f(t_2)} = \overline{f(x)} + \overline{f(y)} + \overline{f(z)}$. Thus, $\overline{f(e)} = \overline{f(x)}$, $\overline{f(t_1)} = \overline{f(y)}$ and $\overline{f(t_2)} = \overline{f(z)}$ since $\dfrac{f(A)+J}{J}$ is uniquely SITT-ring. Consider $j_1^*, j_2^*, j_3^* \in J$ such that $f(x) = f(e) + j_1^*$, $f(y) = f(t_1) + j_2^*$ and $f(z) = f(t_2) + j_3^*$ and also since $g(A) + J'$ is SITT-ring, $g(a) + j' = g(x) + j_1' + g(y) + j_2' + g(z) + j_3'$ with $g(x) + j_1'$ and $g(y) + j_2'$, $g(z) + j_3'$ are respectively an idempotent and a tripotent elements of $g(A)+J'$. It is clear that $\overline{g(x)} = \overline{g(x)+j_1'}$ (resp. $\overline{g(e)}$) and $\overline{g(y)} = \overline{g(y)+j_2'}$ (resp. $\overline{g(t_1)}$), $\overline{g(z)} = \overline{g(z)+j_3'}$ (resp. $\overline{g(t_2)}$) are respectively an idempotent and a tripotent elements of $\dfrac{g(A)+J'}{J'}$, and we have $\overline{g(a)} = \overline{g(e)} + \overline{g(t_1)} + \overline{g(t_2)} = \overline{g(x)} + \overline{g(y)} + \overline{g(z)}$. Thus, $\overline{g(e)} = \overline{g(x)}$, $\overline{g(t_1)} = \overline{g(y)}$ and $\overline{g(t_2)} = \overline{g(z)}$ since $\dfrac{g(A)+J'}{J'}$ is uniquely SITT-ring. Consider $j_1^{'*}, j_2^{'*}, j_3^{'*} \in J'$ such that $g(x) = g(e) + j_1^{'*}$, $g(y) = g(t_1) + j_2^{'*}$ and $g(z) = g(t_2) + j_3^{'*}$. We have, $(f(a)+j, g(a)+j') = (f(e)+j_1^*+j_1, g(e)+j_1^{'*}+j_1') + (f(t_1)+j_2^*+j_2, g(t_1)+j_2^{'*}+j_2') + (f(t_2)+j_3^*+j_3, g(t_2)+j_3^{'*}+j_3')$, and it is clear that $(f(e)+j_1^*+j_1, g(e)+j_1^{'*}+j_1')$ is an idempotent and $(f(t_1)+j_2^*+j_2, g(t_1)+j_2^{'*}+j_2')$ and $(f(t_2)+j_3^*+j_3, g(t_2)+j_3^{'*}+j_3')$ are tripotent elements of $A \bowtie^{f,g} (J, J')$. □

Proposition 4.16. *Let $f : A \to B$ and $f : A \to C$ be a ring homomorphisms and let (e_1) be an ideal of B generated by the idempotent element e_1 and (e_2) be an ideal of C generated by the idempotent element e_2. Assume that A is a SITT-ring. Then $A \bowtie^{f,g} ((e_1), (e_2))$ is SITT-ring if and only if $f(A) + (e_1)$ and $g(A) + (e_2)$ are SITT-ring.*

Proof. In light of Proposition 4.13, we have only to show that $A \bowtie^{f,g} ((e_1), (e_2))$ is SITT-ring provided $f(A) + (e_1)$ and $g(A) + (e_2)$ are SITT-rings. Let $(f(a) + r_1e_1, g(a) + r_2e_2)$ be an element of $A \bowtie^{f,g} ((e_1), (e_2))$ (with $a \in A$, $r_1 \in B$ and $r_2 \in C$). Since A is SITT-ring, we can write $a = s + t_1 + t_2$, where $s \in Id(A)$ and $t_1, t_2 \in Tr(A)$ and also since $f(A) + (e_1)$ and $g(A) + (e_2)$ are SITT-rings, we can write $f(a) + r_1e_1 = s' + t_1' + t_2'$, where $s' \in Id(f(A) + (e_1))$ and $t_1', t_2' \in Tr(f(A) + (e_1))$ and $g(a) + r_2e_2 = s'' + t_1'' + t_2''$, where $s'' \in Id(g(A) + (e_2))$ and $t_1'', t_2'' \in Tr(g(A) + (e_2))$. We have $(f(a) + r_1e_1, g(a) + r_2e_2) = (f(s) + (s' - f(s))e_1, g(s) + (s'' - g(s))e_2) + (f(t_1) + (t_1' - f(t_1))e_1, g(t_1) + (t_1'' - g(t_1))e_2) + (f(t_2) + (t_2' - f(t_2))e_1, g(t_2) + (t_2'' - g(t_2))e_2)$. On the other hand,

$$\begin{aligned} [f(s) + (s' - f(s))e_1]^2 &= [f(s)(1 - e_1) + s'e_1]^2 \\ &= f(s)(1 - e_1) + s'e_1 \\ &= f(s) + (s' - f(s))e_1 \end{aligned}$$

$$\begin{aligned} [f(t_1) + (t_1' - f(t_1))e_1]^3 &= [f(t_1)(1 - e_1) + t_1'e_1]^3 \\ &= f(t_1)(1 - e_1) + t_1'e_1 \\ &= f(t_1) + (t_1' - f(t_1))e_1 \end{aligned}$$

and

$$\begin{aligned}[f(t_2)+(t_2'-f(t_2))e_1]^3 &= [f(t_2)(1-e_1)+t_2'e_1]^3\\ &= f(t_2)(1-e_1)+t_2'e_1\\ &= f(t_2)+(t_2'-f(t_2))e_1\end{aligned}$$

Similarly, $g(s)+(s''-g(s))e_2$ is an idempotent of $g(A)+(e_2)$ and $g(t_1)+(t_1''-g(t_1))e_2, g(t_2)+(t_2''-g(t_2))e_2$ are tripotents of $g(A)+(e_2)$.Then, $(f(s)+(s'-f(s))e_1, g(s)+(s''-g(s))e_2)$ and $(f(t_1)+(t_1'-f(t_1))e_1, g(t_1)+(t_1''-g(t_1))e_2)$, $(f(t_2)+(t_2'-f(t_2))e_1, g(t_2)+(t_2''-g(t_2))e_2)$ respectively an idempotent and tripotents in $A\bowtie^{f,g}((e_1),(e_2))$. Consequently, $A\bowtie^{f,g}((e_1),(e_2))$ is SITT-ring, as desired. □

Acknowledgement. The first author is supported by Periyar University Research Fellowship (Letter No: PU/AD-3/URF/015723/2020 dated 16th November 2020). The second author is supported by DST FIST (Letter No: SR/FST/MSI-115/2016 dated 10th November 2017).

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