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GENERAL ROTATIONAL SURFACES IN EUCLIDEAN 4-SPACE WITH GENERALIZED 1-TYPE GAUSS MAP

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Abstract. In this paper, we consider rotational surfaces in the Euclidean 4-space $\mathbb{E}^4$ with the profile curve contained in a 2-plane. We obtain some classification results of such surfaces in terms of having generalized 1-type Gauss map.

1. Introduction and preliminaries

The notion of finite type map from a Riemannian manifold into a Euclidean space was firstly introduced by B. Y. Chen *et. al.* in [4]. Some of the initial results on this subject were published in [3]. By the definition, a map $\phi : M \to \mathbb{E}^N$ into a Euclidean space is said to be finite type if it can be expressed as

$$\phi = \phi_0 + \phi_1 + \phi_2 + \cdots + \phi_k$$

for some eigenvectors $\phi_1, \phi_2, \ldots, \phi_k$ of the Laplace operator Δ of M, where $\phi_0 \in \mathbb{E}^N$ is a constant vector. More precisely, if these eigenvectors are corresponding from k distinct eigenvalues of Δ, then ϕ is said to be of k-type.

Submanifolds of Euclidean spaces with '*finite type*' Gauss map have caught a special interest after they firstly studied in [5]. Many works have been published in this topic so far, [2, 11, 17, 18, 19]. Note that from the definition above one may observe that a submanifold M of $\mathbb{E}^m$ has 1-type Gauss map G if the differential equation

$$\Delta G = \lambda(G + C) \tag{1.1}$$

is satisfied for a constant λ and a constant vector C. However, further studies have yielded that the Gauss map of some important surfaces such as helicoid and catenoid in $\mathbb{E}^3$ satisfies the weaker condition obtained by replacing λ appearing in (1.1) with a smooth function. By considering these results, in [12] Kim and Yoon defined submanifolds with pointwise 1-type Gauss map. Namely, a submanifold said to have '*pointwise 1-type*' Gauss map if the equation

$$\Delta G = f(G + C)$$

is satisfied for a smooth function f and a constant vector C. Note that a pointwise 1-type Gauss map G is said to be of '*the first kind*' (resp. '*the second kind*') if

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$C = 0$ (resp. $C \neq 0$). Submanifolds in Euclidean spaces with pointwise 1-type Gauss map were recently studied in [1, 6, 7, 10, 14].

On the other hand, rotational surfaces in the 4-dimensional Euclidean space $\mathbb{E}^4$ are defined by Moore in [13] by considering the studies of Cole in [8] about the rotations in $\mathbb{E}^4$. A general rotational surface in $\mathbb{E}^4$ with the profile curve $\alpha = (x, y, z, w)$ is parametrized as

$$F(s,t) = (x(s)\cos at - y(s)\sin at, x(s)\sin at + y(s)\cos at, z(s)\cos bt - \omega(s)\sin bt, z(s)\sin bt + \omega(s)\cos bt) \quad (1.2)$$

for some constants a and b called as '*the rates of rotation*' in fixed planes of the rotation.

Rotational surfaces in Euclidean 4-space have been studied in some papers in terms of the type of their Gauss map. For example, rotational surfaces with finite type Gauss map were investigated in [19]. In [20], Yoon studied flat Vranceanu rotational surfaces. In particular, it is proved that such a surface has pointwise 1-type Gauss map if and only if it is an open part of a Clifford tor surface, [20]. Also, in [1], Arslan *et al.* proved a classification theorem for flat rotational surfaces with pointwise 1-type Gauss map. Moreover, Dursun and the third named author considered simple rotational surfaces, [9]. Finally, in [10], this study extended to the general rotational surfaces with meridian curves in the two-dimensional plane.

Recent studies have shown that the Gauss map of some hypersurfaces of Euclidean spaces satisfies the equation

$$\Delta G = f_1 G + f_2 C \quad (1.3)$$

for some smooth functions f_1 and f_2. A submanifold is said to have '*generalized 1-type*' Gauss map if its Gauss map satisfies the condition (1.3), [21]. After this definition was given, hypersurfaces of pseudo-Euclidean spaces have been considered in terms of having generalized 1-type Gauss map, [15, 16, 22]. In this paper, we study submanifolds with codimension 2. In particular, we focus on general rotational surfaces in $\mathbb{E}^4$ with generalized 1-type Gauss map. We obtain general rotational surfaces with generalized 1-type Gauss map. Also, we show that if the Gauss map of a minimal surfaces is generalized 1-type Gauss map, it must necessarily be pointwise 1-type.

1.1. **Submanifolds in Euclidean spaces.** Let $\mathbb{E}^m$ denote the Euclidean m-space given with the metric tensor

$$\tilde{g} = \langle\, , \,\rangle = \sum \_i = 1^m x_i^2$$

and the Levi-Civita connection $\tilde{\nabla}$, where $(x_1, x_2, \ldots, x_m)$ is the Cartesian coordinate system in $\mathbb{R}^m$.

For a given n-dimensional submanifold M of $\mathbb{E}^m$, the Gauss and Weingarten Formulæ

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad (1.4)$$

$$\widetilde{\nabla}_X \xi = -A_\xi(X) + D_X \xi, \quad (1.5)$$

are satisfied for all X, Y tangent to M and ξ normal to M, where ∇ is the Levi-Civita connection of M, h, A and D denote the second fundamental form, shape operator and normal connection, respectively.

Consider the case $m = n+2$ and let $\{e_1, e_{2,} \ldots, e_{n+2}\}$ be an oriented local orthonormal frame on M such that $e_1, e_{2,} \ldots, e_n$ are tangent to M and e_{n+1}, e_{n+2} are normal to M. We use the following convention on the range of indices: $1 \leq i, j, k, \ldots \leq n$, $n+1 \leq r, s, t, \ldots \leq n+2$. We define the connection form ω_{ij} by

$$\omega_{ij}(X) = \langle \tilde{\nabla}_X e_i, e_j \rangle$$

and let h^{β}_{ij} denote the components of the second fundamental form. Then, (1.4) and (1.5) turn into

$$\tilde{\nabla}_{e_k} e_i = \sum_{j=1}^{n} \omega_{ij}(e_k) e_j + \sum_{r=n+1}^{n+2} h^r_{ik} e_r$$

and

$$\tilde{\nabla}_{e_k} e_s = -\sum_{j=1}^{n} h^s_{jk}(e_k) e_j + \sum_{r=n+1}^{n+2} \omega_{sr}(e_k) e_r,$$

respectively. On the other hand, the Laplacian of M is defined by

$$\Delta = \sum_{i=1}^{n} \left(\nabla_{e_i} e_i - e_i e_i \right).$$

The mean curvature vector H and the squared length $\|h\|^2$ of the second fundamental form h are defined, respectively, by

$$H = \frac{1}{n} \sum_{i,r} h^r_{ii} e_r \qquad \text{and} \qquad \|h\|^2 = \sum_{i,j,r} h^r_{ij} h^r_{ji}.$$

The Codazzi equation of M in $\mathbb{E}^{n+2}$ is given by

$$h^r_{ij,k} = h^r_{jk,i},$$

where we put

$$h^r_{jk,i} = e_i(h^r_{jk}) + \sum_{s=n+1}^{n+2} h^s_{jk} \omega_{sr}(e_i) - \sum_{\gamma=1}^{n} \left(\omega_{j\gamma}(e_i) h^r_{\gamma k} + \omega_{k\gamma}(e_i) h^r_{j\gamma} \right).$$

Also, from the Ricci equation of M in $\mathbb{E}^{n+2}$ we have

$$R^D(e_j, e_k; e_r, e_s) = \langle [A_{e_r}, A_{e_s}](e_j), e_k \rangle = \sum_{i=1}^{n} \left(h^r_{ik} h^s_{ij} - h^r_{ij} h^s_{ik} \right),$$

where R^D is the normal curvature tensor.

Now, consider the case $n = 2$. Then, the Codazzi equations become

$$\begin{aligned} e_1(h^3_{22}) &= \omega_{21}(e_2)(h^3_{22} - h^3_{11}) + h^4_{12} \omega_{43}(e_2), \\ e_1(h^4_{12}) &= 2\omega_{21}(e_2) h^4_{12} - h^3_{11} \omega_{43}(e_2). \end{aligned} \tag{1.6}$$

1.2. **Generalized Rotational Surfaces.** Moore defined a rotational surface in $\mathbb{E}^4$ as a surface parametrized by (1.2), [13]. When the profile curve of the rotational surface is planar, (1.2) turns into

$$F(s,t) = (x(s)\cos at, x(s)\sin at, z(s)\cos bt, z(s)\sin bt)\,. \tag{1.7}$$

Now, let M be a rotational surface parametrized by (1.7). We also consider the profile curve $\beta = (x,z)$ of M and denote its curvature by κ. Without loss of generality, we assume that β is parametrized by its arc-length; that is, the equation

$$(x')^2 + (z')^2 = 1$$

is satisfied and the curvature function κ of β is given by equation (1.8)

$$\kappa(s) = x'(s)z''(s) - x''(s)z'(s),\ \ s \in I. \tag{1.8}$$

In this case, an orthonormal moving frame $\{e_1, e_2, e_3, e_4\}$ on M is defined as

$$e_1 = \frac{\partial}{\partial s}\ , \qquad e_2 = \frac{1}{\sqrt{a^2x^2+b^2z^2}}\frac{\partial}{\partial t}, \tag{1.9a}$$

$$e_3 = (-z'\cos at, -z'\sin at, x'\cos bt, x'\sin bt), \tag{1.9b}$$

$$e_4 = \frac{1}{\sqrt{a^2x^2+b^2z^2}}\left(-bz\sin at, bz\cos at, ax\sin bt, -ax\cos bt\right), \tag{1.9c}$$

where e_1, e_2 are tangent and e_3, e_4 are normal to M. With a direct calculation, we obtain the connection forms of M and components of the second fundamental form of M as

$$h^3_{11} = \kappa, \qquad h^3_{22} = \frac{a^2xz' - b^2zx'}{a^2x^2+b^2z^2}, \qquad h^3_{12} = 0, \tag{1.10}$$

$$h^4_{12} = \frac{ab\left(zx' - xz'\right)}{a^2x^2+b^2z^2}, \qquad h^4_{11} = h^4_{22} = 0, \tag{1.11}$$

$$\omega_{12}(e_1) = 0, \qquad \omega_{12}(e_2) = \frac{a^2xx' + b^2zz'}{a^2x^2+b^2z^2},$$

$$\omega_{43}(e_1) = 0, \qquad \omega_{43}(e_2) = \frac{ab\left(xx' + zz'\right)}{a^2x^2+b^2z^2}. \tag{1.12}$$

Consequently, the shape operators of M has the matrix representation

$$A_3 = \begin{pmatrix} h^3_{11} & 0 \\ 0 & h^3_{22} \end{pmatrix} \qquad \text{and} \qquad A_4 = \begin{pmatrix} 0 & h^4_{12} \\ h^4_{12} & 0 \end{pmatrix},$$

from which we obtain the mean curvature vector H and normal curvature K^D of M as

$$H = \frac{(h^3_{11} + h^3_{22})}{2}e_3,$$

$$K^D(e_1, e_2; e_3, e_4) = h^4_{12}(h^3_{22} - h^3_{11}), \tag{1.13}$$

where $K^D = R^D(e_1, e_2; e_3, e_4)$, [10].

2. General rotational surfaces with generalized 1-type Gauss map

In this section, it will be investigated whether the Gauss map $G = e_1 \wedge e_2$ of general rotational surfaces satisfies the condition of the Laplacian (1.3).

Assume that M is a rotational surface in $\mathbb{E}^4$ parametrized by (1.7). We consider the orthonormal moving frame $\{e_1, e_2, e_3, e_4\}$ given in (1.9). For a constant vector $C \in \Lambda(4,2) \equiv \mathbb{E}^6$, we define c_{ij} by

$$c_{ij} = \langle C, e_i \wedge e_j \rangle. \tag{2.1}$$

Consequently, we have

$$C = \sum_{1 \le i < j \le 4} c_{ij} e_i \wedge e_j, \tag{2.2}$$

Before we proceed we would like to state the following result directly obtained from [5, Lemma 3.1].

Lemma 2.1. *The Gauss map $G = e_1 \wedge e_2$ of an oriented surface M in $\mathbb{E}^4$ satisfies the equation*

$$\Delta G = \|h\|^2 G + 2R^D(e_1, e_2; e_3, e_4) e_3 \wedge e_4 - 2(D_{e_1} H \wedge e_2 + e_1 \wedge D_{e_2} H), \tag{2.3}$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal moving frame.

Proposition 2.2. *Let M be a rotational surface in $\mathbb{E}^4$ given by (1.7) with generalized 1-type Gauss map. Then, the functions f_1, f_2 and the constant vector C in (1.3) satisfy*

$$\begin{aligned} f_1 + f_2 c_{12} &= \|h\|^2, && (2.4)\\ f_2 c_{14} &= -(h^3_{11}\omega_{34}(e_2) + h^3_{22}\omega_{34}(e_2)), &&\\ f_2 c_{23} &= (e_1(h^3_{11}) + e_1(h^3_{22})), &&\\ f_2 c_{34} &= 2h^4_{12}(h^3_{22} - h^3_{11}), && (2.5)\\ e_1(c_{12}) &= -\kappa c_{23} + h^4_{12} c_{14}, && (2.6)\\ e_1(c_{14}) &= \kappa c_{34} - h^4_{12} c_{12}, &&\\ e_1(c_{23}) &= -h^4_{12} c_{34} + \kappa c_{12}, &&\\ e_1(c_{34}) &= -\kappa c_{14} + h^4_{12} c_{23}, && (2.7) \end{aligned}$$

and

$$\omega_{12}(e_2)c_{23} - h^4_{12}c_{34} - h^3_{22}c_{12} + \omega_{34}(e_2)c_{14} = 0, \tag{2.8}$$

$$-\omega_{12}(e_2)c_{14} + h^3_{22}c_{34} + h^4_{12}c_{12} - \omega_{34}(e_2)c_{23} = 0, \tag{2.9}$$

where the functions $c_{ij}, 1 \le i < j \le 4$ are defined by (2.1).

Proof. Assume that the Gauss map $G = e_1 \wedge e_2$ of M is generalized 1-type, i.e., the equation (1.3) is satisfied for some smooth functions $f_1,\ f_2$ and a constant vector C. Then, by combining (1.3) and (2.3) and inner product both sides of this equation with $e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4$ and $e_3 \wedge e_4$ we get (2.4)-(2.5) and as well as

$$f_2 c_{13} = -(e_2(h^3_{11}) + e_2(h^3_{22})), \tag{2.10}$$

$$f_2 c_{24} = (h^3_{11}\omega_{34}(e_1) + h^3_{22}\omega_{34}(e_1)). \tag{2.11}$$

Since the h^3_{11} and h^3_{22} functions are only dependent on s, we have $e_2(h^3_{11}) = e_2(h^3_{22}) = 0$. By combining these equations with (2.10), we get $c_{13} = 0$. Moreover, (1.12) and (2.11) imply $c_{24} = 0$. Moreover, applying e_1 to the equation (2.1), we get (2.6)-(2.7) and applying e_2 to the equation (2.1), using $c_{13} = c_{24} = 0$, we get (2.8)-(2.9). □

2.1. **Rotational Surfaces with Flat Normal Bundle.** Let M be a rotational surfaces (1.7) and assume that its normal bundle is flat. In this case, the normal curvature K^D of M vanishes identically. Therefore, (1.13) implies either $h^4_{12} = 0$ or $h^3_{11} = h^3_{22}$.

Theorem 2.3. *Let M be a general rotational surface defined by (1.7) with flat normal bundle. Then M has generalized 1-type Gauss map if and only if it can be parametrized by*

$$F(s,t) = (s\cos at, s\sin at, cs\cos bt, cs\sin bt)\,, \tag{2.12}$$

for a non-zero constant $c \in \mathbb{R}$.

Proof. We assume that a general rotational surface M with flat normal bundle has generalized 1-type Gauss map. Then we have two discrete case, that is $h^4_{12} = 0$ and $h^3_{11} = h^3_{22}$.

Case I. $h^4_{12} = 0$. In this case (1.11) implies that the meridian curve β of M is found as a straight line passing through the origin. Therefore, we have $\kappa = 0$. Also, by combining $h^4_{12} = 0$ with the equation (2.5), we get $c_{34} = 0$. Therefore, for $h^4_{12} = 0$, the Codazzi equation (1.6) becomes

$$e_1\left(h^3_{22}\right) = -h^3_{22}\omega_{12}(e_2).$$

Thus, the necessary condition for having generalized 1-type Gauss map of M is we have

$$f_1 + f_2c_{12} = \left(h^3_{22}\right)^2, \tag{2.13}$$

$$f_2c_{14} = -h^3_{22}\omega_{34}(e_2), \tag{2.14}$$

$$f_2c_{23} = -h^3_{22}\omega_{12}(e_2). \tag{2.15}$$

Since $\kappa = h^3_{11} = 0$, $h^4_{12} = 0$ and $c_{34} = 0$, c_{12}, c_{14} and c_{23} are found as constants from (2.6)-(2.7) equations. Put $c_{12} = c_1$, $c_{14} = c_2$ and $c_{23} = c_3$. Then, we have

$$C = c_1e_1 \wedge e_2 + c_2e_1 \wedge e_4 + c_3e_2 \wedge e_3.$$

From equations (2.14) and (2.15), we have

$$\omega_{34}(e_2)c_{23} = \omega_{12}(e_2)c_{14} \tag{2.16}$$

and considering that $z = cx$ in equation (2.16), we get

$$(a^2 + b^2c^2)c_2 - ab(1 + c^2)c_3 = 0 \tag{2.17}$$

Also, from equation (2.8), we get

$$\omega_{12}(e_2)c_3 - h^3_{22}c_1 + \omega_{34}(e_2)c_2 = 0 \tag{2.18}$$

Similarly, if we use the equation $z = cx$ in equation (2.18), we get

$$(a^2 + b^2c^2)c_3 - c(a^2 - b^2)c_1 - ab(1 + c^2)c_2 = 0. \tag{2.19}$$

If we take $a^2+b^2c^2=\lambda_1$, $ab(1+c^2)=\lambda_2$ and $c(a^2-b^2)=\lambda_3$ in equations (2.17) and (2.19), solve it depending on the parameter $c_3=1$, we can write the constant vector C as in the expression (2.20)

$$C=\left(\begin{array}{cccccc}\frac{\lambda_1^2-\lambda_2^2}{\lambda_1\lambda_3}, & 0, & \frac{\lambda_2}{\lambda_1}, & 1, & 0, & 0\end{array}\right). \tag{2.20}$$

Also, since $c_3=1$, from the expression (2.15)

$$f_2=-h_{22}^3\omega_{12}(e_2) \tag{2.21}$$

and, the f_1 function from the expression (2.13), we obtain that

$$f_1=\left(h_{22}^3\right)^2+\frac{\lambda_1^2-\lambda_2^2}{\lambda_1\lambda_3}h_{22}^3\omega_{12}(e_2) \tag{2.22}$$

For $h_{12}^4=0$, generalized rotational surfaces with flat normal bundle, can be found the constant C vector given by (2.20) expression, respectively, f_1 and f_2 functions given by (2.22) and (2.21) expressions, so generalized rotational surfaces given by (2.12) are understood to have a generalized 1-type Gauss map.

Case II. $h_{11}^3=h_{22}^3$. In this case, from $e_1(h_{11}^3)=e_1(h_{22}^3)$ and the Codazzi equation (1.6) we get $e_1(h_{11}^3)=-h_{12}^4\omega_{34}(e_2)$.

Therefore equations (2.4)-(2.5) implies

$$\begin{aligned}
f_1+f_2c_{12} &= \|h\|^2,\\
f_2c_{14} &= -2h_{11}^3\omega_{34}(e_2),\\
f_2c_{23} &= -2h_{12}^4\omega_{34}(e_2),\\
e_1(c_{12}) &= -\kappa c_{23}+h_{12}^4c_{14},\\
e_1(c_{14}) &= -h_{12}^4c_{12},\\
e_1(c_{23}) &= \kappa c_{12},\\
e_1(c_{34}) &= -\kappa c_{14}+h_{12}^4c_{23}=0.
\end{aligned} \tag{2.23}$$

In equation (2.23), considering $\kappa=h_{11}^3$, we get

$$h_{11}c_{14}=h_{12}^4c_{23}. \tag{2.24}$$

By multiplying both sides of the (2.24) equation by f_2, we get

$$h_{11}^3=\varepsilon h_{12}^4\ ,\qquad \varepsilon=\mp1. \tag{2.25}$$

Since $h_{11}^3=h_{22}^3$, the equation (2.25) is again written as

$$h_{22}^3=\varepsilon h_{12}^4. \tag{2.26}$$

Thus, using (1.10) and (1.11) in (2.26), we have

$$\begin{gathered}
h_{22}^3=\varepsilon h_{12}^4,\\
\frac{a^2xz'-b^2zx'}{a^2x^2+b^2z^2}=\varepsilon\frac{ab\left(zx'-xz'\right)}{a^2x^2+b^2z^2},\\
a\left(a+\varepsilon b\right)xz'=b(\varepsilon a+b)zx',\\
\frac{z'}{z}=\varepsilon\frac{b}{a}\frac{x'}{x},
\end{gathered}$$

$$z = cx^{\varepsilon \frac{b}{a}}. \tag{2.27}$$

Since it was shown in article [10] that the parabola expressed with (2.27) as a meridian curve is a minimal surface, $H = 0$, that is, $h^3_{11} + h^3_{22} = 0$. Also, since we assume $h^3_{11} = h^3_{22}$, it is found as $h^3_{11} = h^3_{22} = \varepsilon h^4_{12} = 0$. Hence M is an open part of a plane. However, this yields a contradiction. □

2.2. **Minimal Rotational Surfaces.** In this subsection, we are going to study minimal generalized rotational surfaces. In the following theorem, we give the classification of such surfaces with generalized 1-type Gauss map.

Theorem 2.4. *Let M be a generalized rotational surface in $\mathbb{E}^4$ parametrized by (1.7) and assume that M is minimal. If M has generalized 1-type Gauss map, then it is pointwise 1-type.*

Proof. Assume that M has generalized 1-type Gauss map, i.e., (1.3) is satisfied for some smooth functions f_1 and f_2 a constant vector $C \neq 0$. Towards contradiction assume that the open subset

$$\mathcal{M} = \{p \in M | f_1(p) \neq f_2(p) \text{ and } f_2(p) \neq 0\}$$

is not empty. Note that because of the minimality of M we have $h^3_{11} = -h^3_{22}$ and $e_1(h^3_{11}) = -e_1(h^3_{22})$. Therefore, the equations (2.4)-(2.5) turns into

$$\begin{aligned} f_1 + f_2 c_{12} &= \|h\|^2, \\ f_2 c_{34} &= 4h^3_{22}h^4_{12}, \\ c_{14} = c_{23} &= 0. \end{aligned}$$

By considering these equations, we observe that the equations(2.6)-(2.7) become

$$e_1(c_{12}) = 0,$$

$$e_1(c_{14}) = \kappa c_{34} - h^4_{12} c_{12} = 0, \tag{2.28}$$

$$e_1(c_{23}) = -h^4_{12} c_{34} + \kappa c_{12} = 0, \tag{2.29}$$

$$e_1(c_{34}) = 0.$$

From (2.28), (2.29) and $h^3_{11} = -h^3_{22}$, we have $h^3_{22} = \varepsilon h^4_{12}$. Next, by considering (1.10) and (1.11), we solve $h^3_{22} = \varepsilon h^4_{12}$ and obtain that the profile curve $\beta = (x, z)$ of $\mathcal{M}$ is

$$z = cx^{\varepsilon \frac{b}{a}}.$$

However, it is shown in the article [10] that $\mathcal{M}$ has pointwise 1-type Gauss map. This contradicts with our initial assumption. □

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