Gulf Journal of Mathematics ISSN: 2309-4966
Vol 15, Issue 1 (2023) 131-145 https://doi.org/10.56947/gjom.v15i1.1384

ON LIE-RINEHART-POISSON ALGEBRAS STRUCTURES

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ABSTRACT. We define the Schouten-Nijenhuis bracket on the algebra of the module of Kähler differentials. We give the main features of Poisson manifolds by using the universal property of derivation. We prove the equivalence between a Lie-Rinehart algebra structure and a Poisson structure and we recover Lichnerowicz's notion of Poisson manifold. We show that a symplectic Lie-Rinehart algebra structure induce a nondegenerate Poisson structure and conversely.

1. INTRODUCTION

The concept of a Poisson structure is currently of much interest and is being studied by Lichnerowicz, Weinstein and others (see [9, 14, 14]). The Poisson bracket is a derivation of the underlying commutative algebra.

A Lie-Rinehart algebra is a generalization of Lie algebras. The term Lie-Rinehart algebra was coined by Huebschmann [4]. However, this algebraic structure, which was introduced by Herz [3] under the name Lie pseudo-algebra, also known as Lie algebroid [12] in a differential geometric context.

Let A be a commutative algebra with unit 1_A over a commutative field $\mathbb{K}$ with characteristic zero and let E be an A-module. We denote by $Der_{\mathbb{K}}(A, E)$ the A-module of derivations on A with coeffcients in E and $Der_{\mathbb{K}}(A)$ the A-module of derivations of A, that is $d \in Der_{\mathbb{K}}(A)$ if d is a linear mapping and $d(ab) = d(a)b + ad(a)$, for all $a, b \in A$, (see [6, Definition 2.5]) and for more details see [1, 5].

The bracket $[,] : Der_{\mathbb{K}}(A) \times Der_{\mathbb{K}}(A) \longrightarrow Der_{\mathbb{K}}(A)$, such that, for any $d_1, d_2 \in Der_{\mathbb{K}}(A)$, $[d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1$ is a Lie bracket.

We recall that a Lie-Rinehart algebra is a triple $(\mathcal{G}, \rho, [,])$ where $\mathcal{G}$ is simultaneously an A-module and a Lie algebra over $\mathbb{K}$, which Lie algebra bracket $[,]$, and $\rho : \mathcal{G} \longrightarrow Der_{\mathbb{K}}(A)$ is simultaneously a morphism of A-modules and $\mathbb{K}$-Lie algebras satisfying

$$[x, a \cdot y] = [\rho(x)(a)] \cdot y + a \cdot [x, y]$$

for any $a \in A$ and $x\ , y \in \mathcal{G}$ (see [4, 13]).

Date: Received: Sep 2, 2022; Accepted: Feb 19, 2023.

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2010 *Mathematics Subject Classification.* Primary 17B63, 16W25; Secondary 17B66.

Key words and phrases. Derivation, Poisson algebra, Schouten-Nijenhuis bracket, Lie-Rinehart algebra.

Let $(\mathcal{G}, \rho)$ be a Lie-Rinehart algebra and $\mathfrak{L}_{sks}(\mathcal{G}, A) = \bigoplus_{p\in\mathbb{N}} \mathfrak{L}^p_{sks}(\mathcal{G}, A)$, where $\mathfrak{L}^p_{sks}(\mathcal{G}, A)$ is the module of skew-symmetric A-multilinear maps of degree p from $\mathcal{G}$ into A and let

$$d_\rho : \mathfrak{L}_{sks}(\mathcal{G}, A) \longrightarrow \mathfrak{L}_{sks}(\mathcal{G}, A)$$

be the cohomology operator associated to the representation ρ, given by

$$\begin{aligned} d_\rho\eta(x_1, x_2, ..., x_{p+1}) &= \sum_{i=1}^{p+1}(-1)^{i-1}\rho(x_i)\left[\eta(x_1, x_2, ..., \widehat{x_i}, ..., x_{p+1})\right] \\ &+ \sum_{1\leq i<j\leq p+1}(-1)^{i+j}\eta(\left[x_i, x_j\right], x_1, x_2, ..., \widehat{x_i}, ..., \widehat{x_j}, ..., x_{p+1}). \end{aligned} \tag{1.1}$$

for any $\eta \in \mathfrak{L}^p_{alt}(\mathcal{G}, A)$ and $x_1, x_2, ..., x_{p+1} \in \mathcal{G}$. The pair $(\mathfrak{L}_{sks}(\mathcal{G}, A), d_\rho)$ is a differential algebra.

The goal of this paper is to characterize the notion of Poisson structures in terms of Lie-Rinehart algebras with use the Schouten-Nijenhuis bracket on the module of Kähler differentials.

The paper is organized as follows. In Section 2, we briefly recall the universal property of derivation and we focus on the Schouten bracket over module of Kähler differentials. In Section 3, We define the Koszul bracket, compute the Jacobiator of this bracket and with the use of the Schouten-Nijenhuis bracket, we give the condition of the existence of the Lie-Rinehart algebra structure. In Section 4, we recall the notion of Poisson structure and we establish, for a Poisson algebra, the existence of a Lie-Rinehart algebra structure on the module of Kähler differentials. We prove the equivalence between a Lie-Rinehart algebra structure and a Poisson structure. Finally, in Section 5, we study the relation between Poisson structures and symplectic Lie-Rinehart algebra structures. We describe its relation with the Schouten-Nijenhuis bracket and we show that it is equivalent to give a symplectic Lie-Rinehart algebra structure and to give a nondegenerate Poisson structure.

2. Preliminaries

Throughout this paper, $\mathbb{K}$ will be a field and A will be a unital, commutative $\mathbb{K}$-algebra; we assume $\mathbb{K}$ to have characteristic zero. We denote by $\Omega_{\mathbb{K}}(A)$ the module of Kähler differentials of commutative algebra A, that is, the quotient space $\Omega_{\mathbb{K}}(A) = I/I^2$, where I is the A-submodule of $A \bigotimes_{\mathbb{K}} A$ generated by the elements of the form $a \otimes 1_A - 1_A \otimes a$ with $a \in A$ (see [2, 11]). The linear map $d_{A/\mathbb{K}} : A \longrightarrow \Omega_{\mathbb{K}}(A)$ defined, for $a \in A$, by

$$d_{A/\mathbb{K}}(a) = \overline{a \otimes 1_A - 1_A \otimes a}$$

is the canonical derivation which the image of $d_{A/\mathbb{K}}$ generates the A-module $\Omega_{\mathbb{K}}(A)$, that is, for $x \in \Omega_{\mathbb{K}}(A)$, $x = \sum_{i\in I:finite} a_i \cdot d_{A/\mathbb{K}}(b_i)$, with $a_i, b_i \in A$.

Theorem 2.1. [2] *The pair* $(\Omega_{\mathbb{K}}(A), d_{A/\mathbb{K}})$ *satisfies the following universal property: for every* A*-module* E *and for every derivation* $D : A \longrightarrow E$*, there exists a unique* A*-linear map* $\widetilde{D} : \Omega_{\mathbb{K}}(A) \longrightarrow E$ *such that*

$$\widetilde{D} \circ d_{A/\mathbb{K}} = D. \tag{2.1}$$

Moreover, the linear mapping

$$Hom_A(\Omega_{\mathbb{K}}(A), E) \longrightarrow Der_{\mathbb{K}}(A, E), \psi \longmapsto \psi \circ d_{A/\mathbb{K}}$$

is an isomorphism of A*-modules. In particular,* $(\Omega_{\mathbb{K}}(A))^* \simeq Der_{\mathbb{K}}(A)$.

For each integer $p \geq 1$, let us denote by $\Lambda^p(\Omega_{\mathbb{K}}(A))$ the space of multilinear skew-symmetric mappings from $[\Omega_{\mathbb{K}}(A)]^p$ into A. Also, we set $\Lambda^0(\Omega_{\mathbb{K}}(A)) = A$ and $\Lambda(\Omega_{\mathbb{K}}(A)) = \bigoplus_{n\in\mathbb{N}} \Lambda^n(\Omega_{\mathbb{K}}(A))$ be the exterior algebra of the A-module $\Omega_{\mathbb{K}}(A)$. The elements of $\Lambda^p(\Omega_{\mathbb{K}}(A))$, with $p > 0$, are called Kähler p-forms, or simply Kähler forms [2]. The set $\Lambda^p(\Omega_{\mathbb{K}}(A))$ is generated by elements of the form $d_{A/\mathbb{K}}(a_1) \wedge d_{A/\mathbb{K}}(a_2) \wedge \cdots \wedge d_{A/\mathbb{K}}(a_p)$, for any $a_1, ..., a_p \in A$. The $\mathbb{K}$-linear map $d^1_{A/\mathbb{K}} : \Omega_{\mathbb{K}}(A) \longrightarrow \Lambda^2[\Omega_{\mathbb{K}}(A)]$ defined by

$$d^1_{A/\mathbb{K}}(x) = \sum_{i\in I:finite} \left[d_{A/\mathbb{K}}(a_i) \wedge d_{A/\mathbb{K}}(b_i)\right]$$

satisfies,

$$d^1_{A/\mathbb{K}}(a \cdot x) = d_{A/\mathbb{K}}(a) \wedge x + a \cdot d^1_{A/\mathbb{K}}(x); \quad d^1_{A/\mathbb{K}} \circ d_{A/\mathbb{K}} = 0,$$

for any $x = \sum_{i\in I:finite} a_i \cdot d_{A/\mathbb{K}}(b_i) \in \Omega_{\mathbb{K}}(A)$ and $a \in A$.

For $D \in Der_{\mathbb{K}}(A)$, the map $\sigma_D : \Omega_{\mathbb{K}}(A) \times \Omega_{\mathbb{K}}(A) \longrightarrow \Omega_{\mathbb{K}}(A)$ defined by

$$\sigma_D(x, y) = \widetilde{D}(x)y - \widetilde{D}(y)x$$

is skew-symmetric A-bilinear, for any $x, y \in \Omega_{\mathbb{K}}(A)$. Therefore, according the universal property of exterior algebra [1], there exists a unique A-linear map

$$i_D : \Lambda^2(\Omega_{\mathbb{K}}(A)) \longrightarrow \Omega_{\mathbb{K}}(A)$$

such that, for any $x, y \in \Omega_{\mathbb{K}}(A)$,

$$i_D(x \wedge y) = \sigma_D(x, y) = \widetilde{D}(x)y - \widetilde{D}(y)x.$$

We define the Lie derivative with respect to a derivation $D \in Der_{\mathbb{K}}(A)$ by the $\mathbb{K}$-linear map

$$\mathfrak{L}_D : i_D \circ d^1_{A/\mathbb{K}} + d_{A/\mathbb{K}} \circ \widetilde{D} : \Omega_{\mathbb{K}}(A) \longrightarrow \Omega_{\mathbb{K}}(A).$$

For any $D \in Der_{\mathbb{K}}(A), a \in A$ and for any $x = \sum_{i\in I:finite} a_i d_{A/\mathbb{K}}(b_i) \in \Omega_{\mathbb{K}}(A)$, we get

$$\mathfrak{L}_D(x) = \sum_{i\in I:finite} \left[D(a_i)d_{A/\mathbb{K}}(b_i) + a_i d_{A/\mathbb{K}}(D(b_i))\right]; \tag{2.2}$$

$$(\mathfrak{L}_{a\cdot D})(x) = a \cdot \mathfrak{L}_D(x) + \widetilde{D}(x)d_{A/\mathbb{K}}(a); \tag{2.3}$$

$$\mathfrak{L}_D(a \cdot x) = [D(a))]\, x + a \cdot \mathfrak{L}_D(x); \tag{2.4}$$

$$\mathfrak{L}_D\left[d_{A/\mathbb{K}}(a)\right] = d_{A/\mathbb{K}}[D(a)]. \tag{2.5}$$

Denote by $\mathcal{D}er^{p}_{sks}(A)$ the A-module of the skew-symmetric p-derivations of A, that is, $D \in \mathcal{D}er^{p}_{sks}(A)$ if

$$D^{i} = D(a_1, ..., \widehat{a}_i, ..., a_p) : A \longrightarrow A, a_i \longmapsto D(a_1, a_2, ..., a_{i-1}, a_i, a_{i+1}, ..., a_p)$$

is a derivation, for any $a_1, ..., a_p \in A$ (see [11]).

Theorem 2.2. [11] *For any $D \in \mathcal{D}er^{p}_{sks}(A)$, there exists a unique skew-symmetric A-multilinear map of degree p, $\tilde{D} : [\Omega_{\mathbb{K}}(A)]^{p} \longrightarrow A$ such that*

$$\tilde{D}\left(d_{A/\mathbb{K}}(a_1), ..., d_{A/\mathbb{K}}(a_p)\right) = D(a_1, ..., a_p) \tag{2.6}$$

and there exists a unique A-linear map $\overline{D} : \Lambda^{p}(\Omega_{\mathbb{K}}(A)) \longrightarrow E$ such that

$$\overline{D}\left(d_{A/\mathbb{K}}(a_1) \wedge d_{A/\mathbb{K}}(a_2) \wedge ... \wedge d_{A/\mathbb{K}}(a_p)\right) = D(a_1, ..., a_p), \tag{2.7}$$

for any $a_1, ..., a_p \in A$.

When we denote $\tilde{D} = P \in \Lambda^{p}(\Omega_{\mathbb{K}}(A))$ and $i_P = \bar{D}$, then for any $a_1, ..., a_p \in A$,

$$\begin{aligned} i_P\left(d_{A/\mathbb{K}}(a_1) \wedge d_{A/\mathbb{K}}(a_2) \wedge ... \wedge d_{A/\mathbb{K}}(a_p)\right) &= P\left(d_{A/\mathbb{K}}(a_1), d_{A/\mathbb{K}}(a_2), ..., d_{A/\mathbb{K}}(a_p)\right) \\ &= D(a_1, ..., a_p). \end{aligned}$$

For $\pi \in \Lambda^{2}(\Omega_{\mathbb{K}}(A))$ and $a, b, c \in A$, we have

$$\begin{aligned} i_\pi\left(d_{A/\mathbb{K}}(a) \wedge d_{A/\mathbb{K}}(b) \wedge d_{A/\mathbb{K}}(c)\right) &= i_\pi\left(d_{A/\mathbb{K}}(a) \wedge d_{A/\mathbb{K}}(b)\right) \cdot d_{A/\mathbb{K}}(c) \\ &\quad -i_\pi\left(d_{A/\mathbb{K}}(a) \wedge d_{A/\mathbb{K}}(c)\right) \cdot d_{A/\mathbb{K}}(b) \\ &\quad +i_\pi\left(d_{A/\mathbb{K}}(b) \wedge d_{A/\mathbb{K}}(c)\right) \cdot d_{A/\mathbb{K}}(a), \end{aligned}$$

that is,

$$i_\pi\left(d_{A/\mathbb{K}}(a) \wedge d_{A/\mathbb{K}}(b) \wedge d_{A/\mathbb{K}}(c)\right) = \oint \pi\left(d_{A/\mathbb{K}}\left(\pi\left(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b)\right)\right), d_{A/\mathbb{K}}(c)\right) \tag{2.8}$$

where the symbol $\oint$ means the cyclic sum in a, b, c.

Let $P \in \Lambda^{p}(\Omega_{\mathbb{K}}(A))$ and $Q \in \Lambda^{q}(\Omega_{\mathbb{K}}(A))$ be the Kähler forms. We define the Schouten-Nijenhuis bracket of P and Q by the mapping

$$[.,.]_S : \Lambda^{p}(\Omega_{\mathbb{K}}(A)) \times \Lambda^{q}(\Omega_{\mathbb{K}}(A)) \longrightarrow \Lambda^{p+q-1}(\Omega_{\mathbb{K}}(A))$$

such that

$$[P, Q]_S = P \circ Q - (-1)^{(p-1)(q-1)} Q \circ P \tag{2.9}$$

where

$$\begin{aligned} &(Q \circ P)\left(d_{A/\mathbb{K}}(a_1), d_{A/\mathbb{K}}(a_2), ..., d_{A/\mathbb{K}}(a_{p+q-1})\right) \\ &= \sum_{\sigma \in S_{p,q-1}} (-1)^{\sigma} \tilde{Q}\left[\tilde{P}\left(a_{\sigma(1)}, a_{\sigma(2)}, ... a_{\sigma(p)}\right), a_{\sigma(p+1)}, ..., a_{\sigma(p+q-1)}\right] \end{aligned}$$

and $\tilde{P} = D \in \mathcal{D}er^{p}_{sks}(A)$ is a unique p-derivation such that

$$D(a_1, ..., a_p) = P\left(d_{A/\mathbb{K}}(a_1), d_{A/\mathbb{K}}(a_2), ..., d_{A/\mathbb{K}}(a_p)\right).$$

Throughout this section, we denote $[.,.]_S$ by an unadorned bracket $[,]$. The description of interior pruduct $P \in \Lambda(\Omega_{\mathbb{K}}(A))$ with the Schouten-Nijenhuis bracket is similar to the interior pruduct defined in [8, 10]. Then if P and Q are two elements of the $\Lambda\left(\Omega_{\mathbb{K}}(A)\right)$, then

$$i_{[P,Q]} = \left[\left[i_P, d_{A/\mathbb{K}}\right], i_Q\right]. \tag{2.10}$$

If $P \in \Lambda^p\left(\Omega_{\mathbb{K}}(A)\right)$, then i_P is of degree $-p$. So

$$\left[i_P, d_{A/\mathbb{K}}\right] = i_P \circ d_{A/\mathbb{K}} - (-1)^{-p} d_{A/\mathbb{K}} \circ i_P. \tag{2.11}$$

Now, assuming $Q \in \Lambda^q\left(\Omega_{\mathbb{K}}(A)\right)$, we have

$$\begin{aligned} \left[\left[i_P, d_{A/\mathbb{K}}\right], i_Q\right] &= \left[i_P, d_{A/\mathbb{K}}\right] \circ i_Q - (-1)^{-q(1-p)} i_Q \circ \left[i_P, d_{A/\mathbb{K}}\right] \\ &= i_P \circ d_{A/\mathbb{K}} \circ i_Q - (-1)^{-p} d_{A/\mathbb{K}} \circ i_{P\wedge Q} \\ &\quad - (-1)^{-q(1-p)} i_{P\wedge Q} \circ d_{A/\mathbb{K}} + (-1)^{-q(1-p)-p} i_Q \circ d_{A/\mathbb{K}} \circ i_P \end{aligned} \tag{2.12}$$

For any $P = Q = \pi \in \Lambda^2\left(\Omega_{\mathbb{K}}(A)\right)$ and $\eta \in \left[\Omega_{\mathbb{K}}(A)\right]^p$, we get

$$i_{[\pi,\pi]}\eta = 2 i_\pi d_{A/\mathbb{K}} i_\pi \eta. \tag{2.13}$$

Proposition 2.3. *If $\pi \in \Lambda^2\left(\Omega_{\mathbb{K}}(A)\right)$ and $a, b, c \in A$, then*

$$\frac{1}{2}[\pi,\pi]\left(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b), d_{A/\mathbb{K}}(c)\right) = \oint \pi\left(d_{A/\mathbb{K}}\left(\pi\left(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b)\right)\right), d_{A/\mathbb{K}}(c)\right) \tag{2.14}$$

where the symbol $\oint$ means the cyclic sum in a, b, c.

Proof. For $\eta = d_{A/\mathbb{K}}(a) \wedge d_{A/\mathbb{K}}(b) \wedge d_{A/\mathbb{K}}(c)$, we have

$$\begin{aligned} i_\pi \eta &= \pi\left(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b)\right) \cdot d_{A/\mathbb{K}}(c) - \pi\left(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(c)\right) \cdot d_{A/\mathbb{K}}(b) \\ &\quad + \pi\left(d_{A/\mathbb{K}}(b), d_{A/\mathbb{K}}(c)\right) \cdot d_{A/\mathbb{K}}(a), \end{aligned}$$

$$\begin{aligned} d_{A/\mathbb{K}} i_\pi \eta &= d_{A/\mathbb{K}}\left(\pi\left(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b)\right)\right) \wedge d_{A/\mathbb{K}}(c) \\ &\quad - d_{A/\mathbb{K}}\left(\pi\left(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(c)\right)\right) \wedge d_{A/\mathbb{K}}(b) \\ &\quad + d_{A/\mathbb{K}}\left(\pi\left(d_{A/\mathbb{K}}(b), d_{A/\mathbb{K}}(c)\right)\right) \wedge d_{A/\mathbb{K}}(a) \end{aligned}$$

and

$$i_\pi \delta_M i_\pi \eta = \oint \pi\left(d_{A/\mathbb{K}}\left(\pi\left(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b)\right)\right), d_{A/\mathbb{K}}(c)\right).$$

According to (2.13), we obtain (2.14). □

3. Lie-Rinehart algebra structure on $\Omega_{\mathbb{K}}(A)$

If $x \in \Omega_{\mathbb{K}}(A)$ and $\omega_0 \in \Lambda^2(\Omega_{\mathbb{K}}(A))$, then the map $\rho_{\omega_0}(x) : A \longrightarrow A$ such that

$$\rho_{\omega_0}(x)(a) = \omega_0\left(x, d_{A/\mathbb{K}}(a)\right) \tag{3.1}$$

is a derivation, for any $a \in A$. Moreover, the map

$$\rho_{\omega_0} : \Omega_{\mathbb{K}}(A) \longrightarrow Der_{\mathbb{K}}(A), x \longmapsto \rho_{\omega_0}(x)$$

is a morphism of A-modudes. Let $\overline{\rho_{\omega_0}(x)}$ be the unique A-linear map such that

$$\overline{\rho_{\omega_0}(x)} \circ d_{A/\mathbb{K}} = \rho_{\omega_0}(x). \tag{3.2}$$

Proposition 3.1. *For any $x, y \in \Omega_{\mathbb{K}}(A)$, we have*

$$\omega_0(x, y) = \overline{\rho_{\omega_0}(x)}(y). \tag{3.3}$$

Proof. For $y = \sum_{i \in I} a_i \cdot d_{A/\mathbb{K}}(b_i) \in \Omega_{\mathbb{K}}(A)$,

$$\omega_0(x, y) = \sum_{i \in I} a_i \cdot \omega_0\left(x, d_{A/\mathbb{K}}(b_i)\right)$$

By (3.1) and (3.3)

$$\omega_0(x, y) = \overline{\rho_{\omega_0}(x)}\left(\sum_{i \in I} f_i \cdot d_{A/\mathbb{K}}(b_i)\right),$$

that is, $\omega_0(x, y) = \overline{\rho_{\omega_0}(x)}(y)$. □

The 2-form $\omega_0 \in \Lambda^2(\Omega_{\mathbb{K}}(A))$ induces on $\Omega_{\mathbb{K}}(A)$ a bracket $[.,.]_\pi$ called the Koszul bracket defined by

$$[x, y]_{\omega_0} = \mathfrak{L}_{\rho_{\omega_0}(x)} y - \mathfrak{L}_{\rho_{\omega_0}(y)} x - d_{A/\mathbb{K}}(\omega_0(x, y)). \tag{3.4}$$

Proposition 3.2. *For any $\omega_0 \in \Omega_{\mathbb{K}}(A)$, $x, y \in \Omega_{\mathbb{K}}(A)$ and $a \in A$. Then,*

$$[x, a \cdot y]_{\omega_0} = a \cdot [x, y]_{\omega_0} + [\rho_{\omega_0}(x)(a)] \cdot y. \tag{3.5}$$

Proof. From the relations (2.3) and (2.4), we have

$$\begin{aligned}
[x, a \cdot y]_{\omega_0} &= \mathfrak{L}_{\rho_{\omega_0}(x))}(a \cdot y) - \mathfrak{L}_{\rho_{\omega_0}(a \cdot y)} x - d_{A/\mathbb{K}}(\omega_0(x, a \cdot y)) \\
&= [\rho_{\omega_0}(x)(a)] \cdot y + a \cdot \mathfrak{L}_{\rho_{\omega_0}(x)}(y) \\
&\quad - a \cdot \mathfrak{L}_{\rho_{\omega_0}(y)} x - a \cdot d_{A/\mathbb{K}}(\omega_0(x, y)) \\
&= a \cdot [x, y]_{\omega_0} + [\rho_{\omega_0}(x)(a)] \cdot y.
\end{aligned}$$

□

For $k \geq 0$ and $x \in \Omega_{\mathbb{K}}(A)$, we define the Lie derivative with respect to $\rho_{\omega_0}(x) \in Der_{\mathbb{K}}(A)$ to be the unique graded endomorphism $\mathfrak{L}_{\rho_{\omega_0}(x)}$ of degree 0 of the graded algebra $\Lambda(\Omega_{\mathbb{K}}(A))$ such that

$$\begin{aligned}
\left(\mathfrak{L}_{\rho_{\omega_0}(x)} \eta\right)(x_1, x_2, ..., x_k) &= \rho_{\omega_0}(x) \cdot \eta(x_1, x_2, ..., x_k) \\
&\quad - \sum_{i=1}^{k} \eta(x_1, ..., x_{i-1}, [x, x_i]_{\omega_0}, x_{i+1}, ..., x_k)
\end{aligned}$$

for any $\eta \in \Lambda^k(\Omega_{\mathbb{K}}(A))$ and $x_1, x_2, ..., x_k \in \Omega_{\mathbb{K}}(A)$. For $a \in A$, we have

$$\mathfrak{L}_{\rho_{\omega_0}(x)}(a) = \rho_{\omega_0}(x)(a) = \omega_0\left(x, d_{A/\mathbb{K}}(a)\right).$$

We define the operator $d_{\rho_{\omega_0}}$ associated with the pair $(\rho_{\omega_0}, [,]_{\omega_0})$ as follows: for any $x_1, ..., x_{k+1} \in \Omega_{\mathbb{K}}(A)$ and $P \in \Lambda^k(\Omega_{\mathbb{K}}(A))$,

$$\begin{aligned}
d_{\rho_{\omega_0}} P(x_1, ..., x_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i-1} \rho_{\omega_0}(x_i) \cdot P(x_1, ..., \widehat{x_i}, ..., x_{k+1}) \\
&\quad + \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} P([x_i, x_j]_{\omega_0}, x_1, ..., \widehat{x_i}, ..., \widehat{x_j}, ..., x_{k+1}).
\end{aligned} \tag{3.6}$$

For $a \in A$,

$$d_{\rho_{\omega_0}}(a) = \rho_{\omega_0}(x)(a) = \omega_0\left(x, d_{A/\mathbb{K}}(a)\right) = \mathfrak{L}_{\rho_{\omega_0}(x)}(a).$$

For any $\omega_0 \in \Lambda^2(\Omega_{\mathbb{K}}(A))$ and $P \in \Lambda^k(\Omega_{\mathbb{K}}(A))$. Then,

$$d_{\rho_{\omega_0}} P = -[\omega_0, P]_S. \tag{3.7}$$

In particular, if $P = \omega_0$, then for any $x, y, z \in \Omega_{\mathbb{K}}(A)$,

$$[\omega_0, \omega_0]_S(x, y, z) = -\oint \rho_{\omega_0}(x) \cdot \omega_0(y, z) + \oint \omega_0([x, y]_{\omega_0}, z) \tag{3.8}$$

where the symbol $\oint$ means the cyclic sum in x, y, z.

For any $x, y, z \in \Omega_{\mathbb{K}}(A)$, we have

$$\omega_0\left(d_{A/\mathbb{K}}(\omega_0(x, y)), z\right) = -\rho_{\omega_0}(z) \cdot \omega_0(x, y) \tag{3.9}$$

$$\omega_0\left(\mathfrak{L}_{\rho_{\omega_0}(x)} y, z\right) = \rho_{\omega_0}(x) \cdot \omega_0(y, z) - \left(\overline{[\rho_{\omega_0}(z), \rho_{\omega_0}(x)]}\right)(y) \tag{3.10}$$

Proposition 3.3. *For any $x, y, z \in \Omega_{\mathbb{K}}(A)$,*

$$\omega_0([x, y]_{\omega_0}, z) - \left(\overline{[\rho_{\omega_0}(x), \rho_{\omega_0}(y)]}\right)(z) = \frac{1}{2}[\omega_0, \omega_0]_S(x, y, z). \tag{3.11}$$

Proof. For $x, y, z \in \Omega_{\mathbb{K}}(A)$, by (3.4), we have

$$\omega_0([x, y]_{\omega_0}, z) = \omega_0\left(\mathfrak{L}_{\rho_{\omega_0}(x)} y, z\right) - \omega_0\left(\mathfrak{L}_{\rho_{\omega_0}(y)} x, z\right) - \omega_0\left(d_{A/\mathbb{K}}(\omega_0(x, y)), z\right).$$

Put

$$\Phi(x, y, z) = \left(\overline{\rho_{\omega_0}([x, y]_{\omega_0})} - \left(\overline{[\rho_{\omega_0}(x), \rho_{\omega_0}(y)]}\right)\right)(z), \tag{3.12}$$

for $x, y, z \in \Omega_{\mathbb{K}}(A)$. From (3.9) and (3.10), the formula (3.12) becomes

$$\begin{aligned} \Phi(x, y, z) &= \omega_0([x, y]_{\omega_0}, z) - \left(\overline{[\rho_{\omega_0}(x), \rho_{\omega_0}(y)]}\right)(z) \\ &= \oint \rho_{\omega_0}(x) \cdot \omega_0(y, z) - \oint \left(\overline{[\rho_{\omega_0}(y), \rho_{\omega_0}(z)]}\right)(x) \end{aligned}$$

A straightforward computation, using (3.8), shows that

$$\Phi(x, y, z) = -[\omega_0, \omega_0]_S(x, y, z) + \oint \omega_0([x, y]_{\omega_0}, z) - \oint \left(\overline{[\rho_{\omega_0}(y), \rho_{\omega_0}(z)]}\right)(x)$$

that is

$$\Phi(x, y, z) = -[\omega_0, \omega_0]_S(x, y, z) + \Phi(x, y, z) + \Phi(y, z, x) + \Phi(z, x, y).$$

Hence, since Φ is an alternating map, then

$$[\omega_0, \omega_0]_S(x, y, z) = 2\Phi(x, y, z).$$

Thus,

$$\omega_0([x, y]_{\omega_0}, z) - \left(\overline{[\rho_{\omega_0}(x), \rho_{\omega_0}(y)]}\right)(z) = \frac{1}{2}[\omega_0, \omega_0]_S(x, y, z).$$

$\square$

Proposition 3.4. *For any $x, y, z \in \Omega_{\mathbb{K}}(A)$,*

$$[d_{A/\mathbb{K}}(\omega_0(x, y)), z]_{\omega_0} = \mathfrak{L}_{\rho_{\omega_0}(d_{A/\mathbb{K}}(\omega_0(x, y)))} z. \tag{3.13}$$

Proof. From (3.4), for $x, y, z \in \Omega_{\mathbb{K}}(A)$, we have

$$\begin{aligned}[d_{A/\mathbb{K}}(\omega_0(x,y)), z]_{\omega_0} &= \mathfrak{L}_{\rho_{\omega_0}(d_{A/\mathbb{K}}(\omega_0(x,y)))} z - \mathfrak{L}_{\rho_{\omega_0}(z)} d_{A/\mathbb{K}}(\omega_0(x,y)) \\ &\quad - d_{A/\mathbb{K}}(\omega_0(d_{A/\mathbb{K}}(\omega_0(x,y)), z).\end{aligned}$$

Since

$$\mathfrak{L}_{\rho_{\omega_0}(z)} d_{A/\mathbb{K}} (\omega_0(x,y)) = d_{A/\mathbb{K}} \mathfrak{L}_{\rho_{\omega_0}(z)} (\omega_0(x,y)) = d_{A/\mathbb{K}} (\rho_{\omega_0}(z)) (\omega_0(x,y)),$$

then

$$[d_{A/\mathbb{K}}(\omega_0(x,y)), z]_{\omega_0} = \mathfrak{L}_{\rho_{\omega_0}(d_{A/\mathbb{K}}(\omega_0(x,y)))} z.$$

□

Proposition 3.5. *For any $x, y, z \in \Omega_{\mathbb{K}}(A)$,*

$$\begin{aligned}[[x,y]_{\omega_0}, z]_{\omega_0} &= \mathfrak{L}_{\rho_{\omega_0}([x,y]_{\omega_0})} z - \mathfrak{L}_{\rho_{\omega_0}(z)} (\mathfrak{L}_{\rho_{\omega_0}(x)} y - \mathfrak{L}_{\rho_{\omega_0}(y)} x) \\ &\quad - d_{A/\mathbb{K}}(\omega_0(\mathfrak{L}_{\rho_{\omega_0}(x)} y, z) + \omega_0(z, \mathfrak{L}_{\rho_{\omega_0}(y)} x)).\end{aligned} \tag{3.14}$$

Proof. From (3.4), for $x, y, z \in \Omega_{\mathbb{K}}(A)$,

$$\mathfrak{L}_{\rho_{\omega_0}(z)} d_{A/\mathbb{K}}(\omega_0(x,y)) - d_{A/\mathbb{K}}(\omega_0([x,y]_{\omega_0}, z) = -d_{A/\mathbb{K}} \left[(\omega_0(\mathfrak{L}_{\rho_{\omega_0}(x)} y - \mathfrak{L}_{\rho_{\omega_0}(y)} x, z)\right],$$

and

$$\begin{aligned}[[x,y]_{\omega_0}, z]_{\omega_0} &= \mathfrak{L}_{\rho_{\omega_0}([x,y]_{\omega_0})} z - \mathfrak{L}_{\rho_{\omega_0}(z)} \left(\mathfrak{L}_{\rho_{\omega_0}(x)} y - \mathfrak{L}_{\rho_{\omega_0}(y)} x - d_{A/\mathbb{K}} (\omega_0(x,y))\right) \\ &\quad - \mathfrak{L}_{\rho_{\omega_0}(z)} d_{A/\mathbb{K}}(\omega_0(x,y)) - d_{A/\mathbb{K}} \left[(\omega_0(\mathfrak{L}_{\rho_{\omega_0}(x)} y - \mathfrak{L}_{\rho_{\omega_0}(y)} x, z)\right],\end{aligned}$$

that is

$$\begin{aligned}[[x,y]_{\omega_0}, z]_{\omega_0} &= \mathfrak{L}_{\rho_{\omega_0}([x,y]_{\omega_0})} z - \mathfrak{L}_{\rho_{\omega_0}(z)} \left(\mathfrak{L}_{\rho_{\omega_0}(x)} y - \mathfrak{L}_{\rho_{\omega_0}(y)} x\right) \\ &\quad - d_{A/\mathbb{K}} \left[(\omega_0(\mathfrak{L}_{\rho_{\omega_0}(x)} y - \mathfrak{L}_{\rho_{\omega_0}(y)} x, z)\right].\end{aligned}$$

□

Denote by J_{ω_0} the Jacobiator of the triple $(\Omega_{\mathbb{K}}(A), \rho_{\omega_0}, [,]_{\omega_0})$, that is,

$$J_{\omega_0}(x,y,z) = \oint([[x,y]_{\omega_0}, z]_{\omega_0}) = [[x,y]_{\omega_0}, z]_{\omega_0} + [[y,z]_{\omega_0}, x]_{\omega_0} + [[z,x]_{\omega_0}, y]_{\omega_0}$$

for $x, y, z \in \Omega_{\mathbb{K}}(A)$ and $\oint$ the cyclic sum in x, y, z.

Proposition 3.6. *For any $x, y, z \in \Omega_{\mathbb{K}}(A)$, we have*

$$\begin{aligned}J_{\omega_0}(x,y,z) &= \oint \left[\mathfrak{L}_{\rho_{\omega_0}(z)}(\mathfrak{L}_{\rho_{\omega_0}(x)} y) - \mathfrak{L}_{\rho_{\omega_0}(x)}(\mathfrak{L}_{\rho_{\omega_0}(z)} y) - \mathfrak{L}_{[\rho_{\omega_0}(z), \rho_{\omega_0}(x)]} y\right] \\ &\quad - \oint d_{A/\mathbb{K}} \left(\omega_0(\mathfrak{L}_{\rho_{\omega_0}(x)} y, z) + \omega_0(y, \mathfrak{L}_{\rho_{\omega_0}(x)} z)\right).\end{aligned} \tag{3.15}$$

Proof. By straightforward calculations taking the cyclic sum in x, y, z on the two sides of the equality (3.14), we have

$$\begin{aligned}
J_{\omega_0}(x,y,z) \;=\;& \mathfrak{L}_{\rho_{\omega_0}([x,y]_{\omega_0})}z - \mathfrak{L}_{\rho_{\omega_0}(z)}(\mathfrak{L}_{\rho_{\omega_0}(x)}y) + \mathfrak{L}_{\rho_{\omega_0}(z)}(\mathfrak{L}_{\rho_{\omega_0}(y)}x) \\
& -d_{A/\mathbb{K}}(\omega_0(\mathfrak{L}_{\rho_{\omega_0}(x)}y, z) + \omega_0(z, \mathfrak{L}_{\rho_{\omega_0}(y)}x)). \\
& +\mathfrak{L}_{\rho_{\omega_0}([y,z]_{\omega_0})}x - \mathfrak{L}_{\rho_{\omega_0}(x)}(\mathfrak{L}_{\rho_{\omega_0}(y)}z) + \mathfrak{L}_{\rho_{\omega_0}(x)}(\mathfrak{L}_{\rho_{\omega_0}(z)}y) \\
& -d_{A/\mathbb{K}}(\omega_0(\mathfrak{L}_{\rho_{\omega_0}(y)}z, x) + \omega_0(x, \mathfrak{L}_{\rho_{\omega_0}(z)}y)) \\
& +\mathfrak{L}_{\rho_{\omega_0}([z,x]_{\omega_0})}y - \mathfrak{L}_{\rho_{\omega_0}(y)}(\mathfrak{L}_{\rho_{\omega_0}(z)}x) + \mathfrak{L}_{\rho_{\omega_0}(y)}(\mathfrak{L}_{\rho_{\omega_0}(x)}z) \\
& -d_{A/\mathbb{K}}(\omega_0(\mathfrak{L}_{\rho_{\omega_0}(z)}x, y) + \omega_0(y, \mathfrak{L}_{\rho_{\omega_0}(x)}z)).
\end{aligned}$$

It is follows that,

$$\begin{aligned}
J_{\omega_0}(x,y,z) \;=\;& \oint \left[\mathfrak{L}_{\rho_{\omega_0}(z)}(\mathfrak{L}_{\rho_{\omega_0}(x)}y) - \mathfrak{L}_{\rho_{\omega_0}(x)}(\mathfrak{L}_{\rho_{\omega_0}(z)}y) - \mathfrak{L}_{[\rho_{\omega_0}(z),\rho_{\omega_0}(x)]}y\right] \\
& - \oint d_{A/\mathbb{K}}\left(\omega_0(\mathfrak{L}_{\rho_{\omega_0}(x)}y, z) + \omega_0(y, \mathfrak{L}_{\rho_{\omega_0}(x)}z)\right).
\end{aligned}$$

□

Lemma 3.7. *If* $[\omega_0, \omega_0]_S = 0$, *then*

$$\rho_{\omega_0} : \left(\Omega_{\mathbb{K}}(A), [,]_{\omega_0}\right) \longrightarrow \left(Der_{\mathbb{K}}(A), [,]\right)$$

is a Lie algebras morphism.

Proof. Since the image of $d_{A/\mathbb{K}}$ generates $\Omega_{\mathbb{K}}(A)$, then for any $z \in \Omega_{\mathbb{K}}(A)$, $z = d_{A/\mathbb{K}}(a)$ with $a \in A$. When $[\omega_0, \omega_0]_S = 0$, then the equation (3.11) becomes

$$\left(\overline{\rho_{\omega_0}\left([x,y]_{\omega_0}\right)}\right) \circ d_{A/\mathbb{K}}(a) - \left(\overline{[\rho_{\omega_0}(x), \rho_{\omega_0}(y)]}\right) \circ d_{A/\mathbb{K}}(a) = 0$$

for all $a \in A$, that is, by the Theorem 2.1,

$$\rho_{\omega_0}\left([x,y]_{\omega_0}\right) = [\rho_{\omega_0}(x), \rho_{\omega_0}(y)]$$

for all $x, y \in \Omega_{\mathbb{K}}(A)$. □

Theorem 3.8. *The triple* $\left(\Omega_{\mathbb{K}}(A), \rho_{\omega_0}, [,]_{\omega_0}\right)$ *has a Lie-Rinehart algebra structure if and only if* $[\omega_0, \omega_0]_S = 0$.

Proof. If the triple $\left(\Omega_{\mathbb{K}}(A), \rho_{\omega_0}, [,]_{\omega_0}\right)$ is a Lie-Rinehart algebra structure, then

$$\rho_{\omega_0}([x,y]_{\omega_0}) = [\rho_{\omega_0}(x), \rho_{\omega_0}(y)]$$

and by the equation (3.11), $[\omega_0, \omega_0]_S(x,y,z) = 0$, for any $x, y, z \in \Omega_{\mathbb{K}}(A)$. Therefore, $[\omega_0, \omega_0]_S = 0$.

Conversely, if $[\omega_0, \omega_0]_S = 0$, the equations (3.8) and (3.11) becomes

$$\oint \rho_{\omega_0}(x) \cdot \omega_0(y,z) = \oint \omega_0([x,y]_{\omega_0}, z),$$

$$\rho_{\omega_0}([x,y]_{\omega_0}) = [\rho_{\omega_0}(x), \rho_{\omega_0}(y)]$$

and

$$\mathfrak{L}_{\rho_{\omega_0}(z)}(\mathfrak{L}_{\rho_{\omega_0}(x)}y) - \mathfrak{L}_{\rho_{\omega_0}(x)}(\mathfrak{L}_{\rho_{\omega_0}(z)}y) = \mathfrak{L}_{[\rho_{\omega_0}(z),\rho_{\omega_0}(x)]}y.$$

Then, the equation

$$J_{\omega_0}(x,y,z) = \oint \mathfrak{L}_{\rho_{\omega_0}([x,y]_{\omega_0})-[\rho_{\omega_0}(x),\rho_{\omega_0}(y)]}z - \oint d_{A/\mathbb{K}}(\omega_0(\mathfrak{L}_{\rho_{\omega_0}(x)}y,z)+\omega_0(y,\mathfrak{L}_{\rho_{\omega_0}(x)}z).$$

becomes

$$\begin{aligned} J_{\omega_0}(x,y,z) &= -2d_{A/\mathbb{K}}[\oint([[x,y]_{\omega_0},z]_{\omega_0}) - \oint([[x,y]_{\omega_0},z]_{\omega_0})] \\ &= 0, \end{aligned}$$

that is,

$$[[x,y]_{\omega_0},z]_{\omega_0} + [[y,z]_{\omega_0},x]_{\omega_0} + [[z,x]_{\omega_0},y]_{\omega_0} = 0$$

Since

$$[x,a\cdot y]_{\omega_0} = a\cdot[x,y]_{\omega_0} + [\rho_{\omega_0}(x)(a)]\cdot y.$$

then, the triple $(\Omega_{\mathbb{K}}(A), \rho_{\omega_0}, [,]_{\omega_0})$ has a Lie-Rinehart algebra structure. □

4. Poisson structures on Lie-Rinehart algebra

We recall that a Poisson bracket is a Lie bracket $\{,\}$ on a commutative algebra A satisfying the Leibniz identity

$$\{a,b\cdot c\} = \{a,b\}\cdot c + b\cdot\{a,c\} \tag{4.1}$$

for any $a,b,c\in A$ (see[7, 9]). The Leibniz identity means that, for a given function $a\in A$, the inner derivation $ad(a): A\longrightarrow A, b\longmapsto\{a,b\}$ is a derivation of a commutative algebra A. For any element $a\in A$, the derivation $ad(a)$ is called the Hamiltonian derivation corresponding to the element a.

If A is a Poisson algebra, the bracket $\{,\}$ is a skew-symmetric 2-derivation. By the Theorem 2.2, there exists $\pi\in\Lambda^2(\Omega_{\mathbb{K}}(A))$ such that

$$\{a,b\} = \pi(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b)), \tag{4.2}$$

for any $a,b\in A$.

Consider the Jacobiator $J(.,.,.)$ defined as

$$J(a,b,c) = \{\{a,b\},c\} + \{\{b,c\},a\} + \{\{c,a\},b\}$$

for $a,b,c\in A$.

Lemma 4.1. *For any $a,b\in A$, we have*

$$J(a,b,c) = \left(\frac{1}{2}[\pi,\pi]\right)\left(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b), d_{A/\mathbb{K}}(c)\right).$$

Proof. Using the equation (4.2), for any $a,b,c\in A$, we have

$$\{\{a,b\},c\} = \pi(d_{A/\mathbb{K}}(\{a,b\}), d_{A/\mathbb{K}}(c)).$$

Using the skew-symmetry of π and grouping relevant terms together, we get

$$\begin{aligned} J(a,b,c) &= \oint \pi\left(d_{A/\mathbb{K}}\left(\pi\left(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b)\right)\right), d_{A/\mathbb{K}}(c)\right) \\ &= \left(\frac{1}{2}[\pi,\pi]\right)\left(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b), d_{A/\mathbb{K}}(c)\right) \end{aligned}$$

□

Theorem 4.2. *For any $a, b \in A$, the bracket*

$$\{a,b\} = \pi(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b))$$

satisfies the Jacobi identity if and only if

$$[\pi, \pi] = 0. \tag{4.3}$$

Proof. Assume that the bracket $\{,\}$ satisfies the Jacobi identity, then from the Lemma 4.1, $[\pi, \pi] = 0$.

Conversely, assume the equation (4.3), then from the Lemma 4.1, we have,

$$\{\{a,b\},c\} + \{\{b,c\},a\} + \{\{c,a\},b\}.$$

□

The skew-symmetric 2-form π on $\Omega_{\mathbb{K}}(A)$ is called Poisson 2-form of the Poisson algebra A. If $A = C^{\infty}(M)$ then pair (M, π) is called Poisson manifold.

If $(A, \{,\})$ Poisson algebra, then the map $ad : A \longrightarrow Der_{\mathbb{K}}(A), a \longmapsto ad(a)$ is a derivation. Thus, by the Theorem 2.1, there exists a unique A-linear map $\overline{ad} : \Omega_{\mathbb{K}}(A)] \longrightarrow Der_{\mathbb{K}}(A)$ such that

$$\overline{ad} \circ d_{A/\mathbb{K}} = ad. \tag{4.4}$$

Then, from (4.2), we have, for any $a \in A$ and $x, y \in \Omega_{\mathbb{K}}(A)$

$$\left[\overline{ad}(x)\right](a) = \pi\left(x, d_{A/\mathbb{K}}(a)\right), \tag{4.5}$$

$$\overline{\left[\overline{ad}(x)\right]}(y) = \pi(x, y). \tag{4.6}$$

Theorem 4.3. [11]*If π is the Poisson 2-form of a Poisson algebra A, then the bracket $[,]_{\pi}$ on $\Omega_{\mathbb{K}}(A)$ defined by*

$$[x,y]_{\pi} = \mathfrak{L}_{\overline{ad}(x)} y - \mathfrak{L}_{\overline{ad}(y)} x - d_{A/\mathbb{K}}(\pi(x,y))$$

is a Lie algebra bracket, for any $x, y \in \Omega_{\mathbb{K}}(A)$.

Moreover the triple $\left(\Omega_{\mathbb{K}}(A), \overline{ad}, [,]_{\pi}\right)$ is a Lie-Rinehart algebra.

Theorem 4.4. *The following statements are equivalent:*

(i) A is a Poisson algebra.

(ii) There exists a skew-symmetric 2-form

$$\omega_0 : \Omega_{\mathbb{K}}(A) \times \Omega_{\mathbb{K}}(A)) \longrightarrow A$$

such that the triple $(\Omega_{\mathbb{K}}(A), \rho_{\omega_0}, [,]_{\omega_0})$ is a Lie-Rinehart algebra.

(iii) The skew-symmetric 2-form $\omega_0 \in \Lambda^2(\Omega_{\mathbb{K}}(A))$ satisfies

$$[\omega_0, \omega_0]_S = 0,$$

where $[,]_S$ denotes the Schouten-Nijenhuis bracket.

Proof. (i)⇒(ii) If A is a Poisson algebra, from the Theorem 4.3 and by putting $\rho_{\omega_0} = \overline{ad}$, then the triple $(\Omega_{\mathbb{K}}(A), \rho_{\omega_0}, [,]_{\omega_0})$ is a Lie-Rinehart algebra.

(ii)⇒(iii) From the Theorem 3.8, the proof is obvious (iii)⇒(i) If $\omega_0 \in \Lambda^2\left(\Omega_{\mathbb{K}}(A)\right)$, then from the universal Theorem 2.2, there exists a skew-symmetric 2-derivation $\{,\} : A \times A \longrightarrow A$ such that

$$\{a,b\} = \omega_0(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b))$$

for any $a,b,c \in A$. Then

$$\begin{aligned}\{\{a,b\},c\} &= \omega_0(d_{A/\mathbb{K}}(\{a,b\}), d_{A/\mathbb{K}}(c)) \\ &= \omega_0\left(d_{A/\mathbb{K}}\left(\omega_0\left(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b)\right)\right), d_{A/\mathbb{K}}(c)\right)\end{aligned}$$

and

$$\begin{aligned}J(a,b,c) &= \oint \omega_0\left(d_{A/\mathbb{K}}\left(\omega_0\left(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b)\right)\right), d_{A/\mathbb{K}}(c)\right) \\ &= \frac{1}{2}[\omega_0,\omega_0]_S\left(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b), d_{A/\mathbb{K}}(c)\right).\end{aligned}$$

Since ρ_{ω_0} is a morphim of Lie algebras, then from the Theorem 3.8, we have $[\omega_0,\omega_0]_S = 0$ and we deduce that $\{\{a,b\},c\} + \{\{b,c\},a\} + \{\{c,a\},b\}$. Therefore A is a Poisson algebra. □

If A is a Poisson algebra, then $d_{\rho_{\omega_0}}\omega_0 = -[\omega_0,\omega_0]_S = 0$. In this case, We say that $(\Omega_{\mathbb{K}}(A), \rho_{\omega_0}, [,]_{\omega_0}, \omega_0)$ is called Lie-Rinehart-Poisson algebra

5. Poisson structures on a symplectic Lie-Rinehart algebra

When $(\mathcal{G},\rho,[,])$ is a Lie-Rinehart algebra, for any $x \in \mathcal{G}$, the map $i_x : \mathfrak{L}_{sks}(\mathcal{G},A) \longrightarrow \mathfrak{L}_{sks}(\mathcal{G},A)$ defined by

$$(i_x\eta)(x_1,x_2,...,x_{p-1}) = \eta(x,x_1,x_2,...,x_{p-1}),$$

for $x_1,x_2,...,x_{p-1}$ elements of $\mathcal{G}$ and for any $\eta \in \mathfrak{L}^p_{sks}(\mathcal{G},A)$, is a derivation of degree -1. The map

$$\mathfrak{L}_x = [i_x, d_\rho] : \mathfrak{L}_{sks}(\mathcal{G},A) \longrightarrow \mathfrak{L}_{sks}(\mathcal{G},A)$$

is a derivation and of degree *zero* satisfying, for any $y \in \mathcal{G}$, $a \in A$,

$$[\mathfrak{L}_x, i_y] = i_{[x,y]};\ \ \mathfrak{L}_x \circ d_\rho = d_\rho \circ \mathfrak{L}_x \ \text{ and } \ \mathfrak{L}_x(a) = [\rho(x)](a).$$

Let $(\mathcal{G},\rho,[,])$ be a Lie-Rinehart algebra. A symplectic form on $(\mathcal{G},\rho,[,])$ is a nondegenerate 2-form $\eta \in \mathfrak{L}^2_{sks}(\mathcal{G},A)$ such that $d_\rho\eta = 0$. In this case, $(\mathcal{G},\rho,[,],\eta)$ is called symplectic Lie-Rinehart algebra.

Example 5.1. When (M,ω) is symplectic manifold, then $(\mathfrak{X}(M), id_{\mathfrak{X}(M)}, [,], \omega)$ is a symplectic Lie-Rinehart algebra. A nondegenerate and closed 2-form ω is called a symplectic form and the pair $(C^\infty(M),\omega)$ is called the corresponding symplectic structure, or symplectic algebra.

When $(\mathcal{G},\rho,[,],\eta)$ is a symplectic Lie-Rinehart algebra, for $a \in A$, we denote x_a the unique element of $\mathcal{G}$ such that $i_{x_a}\eta = d_\rho(a)$.

The map $\varphi : A \longrightarrow \mathcal{G}, a \longmapsto x_a$ is derivation. According to the Theorem 2.1, there exists a unique A-linear map $\tilde{\varphi} : \Omega_{\mathbb{K}}(A) \longrightarrow \mathcal{G}$ such that $\tilde{\varphi} \circ d_{A/\mathbb{K}} = \varphi$ that is,

$$\tilde{\varphi}\left[d_{A/\mathbb{K}}(a)\right] = \varphi(a) = x_a. \tag{5.1}$$

If $(\mathcal{G}, \rho, [,], \eta)$ is a symplectic Lie-Rinehart algebra, then there exists a morphism of A-modules

$$\rho_\pi = \rho \circ \tilde{\varphi} : \Omega_{\mathbb{K}}(A) \longrightarrow Der_{\mathbb{K}}(A)$$

and a Kähler 2-form

$$\pi = \eta \circ (\tilde{\varphi} \times \tilde{\varphi}) : \Omega_{\mathbb{K}}(A) \times \Omega_{\mathbb{K}}(A) \longrightarrow A$$

such that

$$\pi(x, y) = \eta(\tilde{\varphi}(x), \tilde{\varphi}(y)), \tag{5.2}$$

for any $x, y \in \Omega_{\mathbb{K}}(A)$. It is easy to verify that if the form π is nondegenerate and for any $x, y, z \in \Omega_{\mathbb{K}}(A)$, we have

$$\eta([\tilde{\varphi}(x), \tilde{\varphi}(y)], \tilde{\varphi}(z)) = -\frac{1}{2}[\pi, \pi](x, y, z) + \pi([x, y]_\pi, z). \tag{5.3}$$

Lemma 5.2. *If $(\mathcal{G}, \rho, [,], \eta)$ is a symplectic Lie-Rinehart algebra, for any $x, y, z \in \Omega_{\mathbb{K}}(A)$, then*

$$[\pi, \pi](x, y, z) = 2d_\rho\eta\left(\tilde{\varphi}(x), \tilde{\varphi}(y), \tilde{\varphi}(z)\right). \tag{5.4}$$

Proof. From (1.1), we have

$$\begin{aligned} d_\rho\left(\tilde{\varphi}(x), \tilde{\varphi}(y), \tilde{\varphi}(z)\right) &= \rho(\tilde{\varphi}(x)) \cdot \eta(\tilde{\varphi}(y), \tilde{\varphi}(z)) - \rho(\tilde{\varphi}(y)) \cdot \eta(\tilde{\varphi}(x), \tilde{\varphi}(z)) \\ &\quad + \rho(\tilde{\varphi}(z)) \cdot \eta(\tilde{\varphi}(x), \tilde{\varphi}(y)) - \eta([\tilde{\varphi}(x), \tilde{\varphi}(y)], \tilde{\varphi}(z)) \\ &\quad + \eta([\tilde{\varphi}(x), \tilde{\varphi}(z)], \tilde{\varphi}(y)) - \eta([\tilde{\varphi}(y), \tilde{\varphi}(z)], \tilde{\varphi}(x)) \end{aligned}$$

From (5.2) and (5.3),

$$\begin{aligned} d_\rho\left(\tilde{\varphi}(x), \tilde{\varphi}(y), \tilde{\varphi}(z)\right) &= \oint \rho_\pi(x) \cdot \pi(y, z) - \oint \pi([x, y]_\pi, z) \\ &\quad + \frac{1}{2}[\pi, \pi](x, y, z) - \frac{1}{2}[\pi, \pi](x, y, z) + \frac{1}{2}[\pi, \pi](y, z, x) \end{aligned}$$

Using (3.8), we get

$$d_\rho\eta\left(\tilde{\varphi}(x), \tilde{\varphi}(y), \tilde{\varphi}(z)\right) = \frac{1}{2}[\pi, \pi](x, y, z).$$

□

Theorem 5.3. *For any $a, b \in A$, the bracketThe Kähler 2-form π is a nondegenerate Poisson 2-form if and only if η is a symplectic form on a Lie-Rinehart algebra $(\mathcal{G}, \rho, [,])$.*

Proof. If π is Poisson 2-form, $[\pi, \pi] = 0$ i.e., $[\pi, \pi](x, y, z) = 0$, for any $x, y, z \in \Omega_{\mathbb{K}}(A)$. From the Lemma 5.2, we deduce that the identity $d\eta = 0$. It is follows that η is a symplectic form on a Lie-Rinehart algebra $(\mathcal{G}, \rho, [,])$.

Conversely, assume that η is a symplectic form on a Lie-Rinehart algebra $(\mathcal{G}, \rho, [,])$, then $d\eta = 0$. From (5.4), $[\pi, \pi](x, y, z) = 0$, for any $x, y, z \in \Omega_{\mathbb{K}}(A)$. Thus π is Poisson 2-form. □

If $(\mathcal{G}, \rho, [,], \eta)$ is a symplectic Lie-Rinehart algebra, then for any $a, b \in A$, the bracket

$$\{a, b\} = \eta(x_a, x_b) = [\rho(x_a)](b) = ad(a)(b)$$

satisfies $[x_a, x_b] = x_{\{a,b\}}$ and $2 \oint \{a, \{b, c\}\} = (d_\rho \eta)(x_a, x_b, x_c) = 0$. Therefore $\{,\}$ is a Poisson bracket and we get

$$\pi\left(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b)\right) = \eta(d_\rho(a), d_\rho(b)) \text{ and } d_\rho \circ \tilde{\varphi} = \overline{ad}. \tag{5.5}$$

If $(\mathcal{G}, \rho, [,], \eta)$ is a symplectic Lie-Rinehart algebra, then map $\tilde{\varphi}$ is a morphism of Lie algebras and satisfies

$$(\tilde{\varphi}^* \eta)(x, y) = \eta[\tilde{\varphi}(x), \tilde{\varphi}(y)] = \pi(x, y)$$

for any $x, y \in \Omega_{\mathbb{K}}(A)$. Thus $\tilde{\varphi}^* \eta = \pi$, that is $\tilde{\varphi}$ is a morphism of Lie-Rinehart algebras.

Acknowledgement. The auteurs thank the referee for the relevant suggestions which have improved the document.

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