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ALGEBRAIC STRUCTURE OF PLENARY TRAIN ALGEBRAS OF RANK 4

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Abstract. In this paper, we study plenary train algebras of rank 4 admitting an idempotent. We obtain the Peirce decomposition of these algebras relative to an idempotent and the product of the associated Peirce components. The effect of changing an idempotent is discussed under the e-stability hypothesis. We show that a back-crossing algebra is a plenary train algebra of rank 4 if and only if it is a principal train algebra of rank 4. Finally, we study the particular case of monogenic back-crossing train algebras.

1. Introduction

In the theory of non-associative algebras, the study of genetic phenomena has led researchers to introduce the notions of baric algebras, train and special train algebras ([3, 4]). We distinguish the principal train algebras and the plenary train algebras. In this paper, K denotes an infinite field of characteristic different from 2 and A is a commutative K-algebra.

If there exists a non-zero morphism of K-algebras $\omega : A \longrightarrow K$, we say that (A, ω) is a baric algebra. The principal powers of an element x in A are defined inductively by $x^1 = x$ and $x^n = x^{n-1}x \ \forall n \geq 2$. We also define the plenary powers of an element x of the algebra A by $x^{[1]} = x$ and $x^{[n]} = x^{[n-1]}x^{[n-1]} \ \forall n \geq 2$.

If x is an element of A, then the scalar $\omega(x)$ is called the weight of x. An element e of A is said to be an idempotent if $e^2 = e \neq 0$. We denote by H the set of elements of weight 1 and $Ip(A)$ the set of the idempotents elements of A.

In [6], H. Guzzo examines commutative principal train algebras of rank n. In [7], A. Suazo and A. Labra study the conditions of existence of an idempotent in plenary train algebras of rank 4. An algebra (A, ω) is called a plenary train algebra of rank n if there exists scalars $\alpha_1, \cdots, \alpha_{n-1}$ in K such that for all $x \in A$

$$x^{[n]} = \alpha_{n-1}\omega(x)^{2^{n-2}}x^{[n-1]} + \cdots + \alpha_2\omega(x)^{2^{n-1}-2}x^2 + \alpha_1\omega(x)^{2^{n-1}-1}x \tag{1.1}$$

where $\sum_{i=1}^{n-1} \alpha_i = 1$ [2].

In this paper, we study the algebraic structure of plenary train algebras of rank 4. We get a Peirce decomposition and a multiplication table for the Peirce

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components of this decomposition. Moreover, if the algebra A satisfied the identity $\left(x^2-\omega(x)x\right)^2=0$, we show that it is a principal train algebra of rank 4 with principal train roots $1,\lambda_1,\lambda_2$ if and only if A is a plenary train algebra of rank 4 with plenary train roots $1,2\lambda_1,2\lambda_2$. We examine the effect of changing an idempotent. We conclude by studying the particular case of back-crossing train algebras generated by one element.

2. Preliminaries

Lemma 2.1 ([5], Lemma 2)**.** *Let (A,ω) be a baric algebra and $p(x)$ a commutative polynomial of degree r. Then the following assertions are equivalent:*

(i) $p(x)=0$ *for any element x in H;*
(ii) $\hat{p}(x):=\omega(x)^r p(\omega(x)^{-1}x)=0$ *for any element x in A.*

Definition 2.2. We say that (A,ω) is a plenary train algebra of rank 4 if it satisfies the identity

$$x^{[4]}=\alpha\omega(x)^4x^{[3]}+\beta\omega(x)^6x^2+\gamma\omega(x)^7x, \tag{2.1}$$

with $\alpha,\beta,\gamma\in K$ such that $\alpha+\beta+\gamma=1$.

Theorem 2.3 ([5], Main Theorem)**.** *Let (A,ω) be a plenary train algebra satisfying (2.1). If $2-\alpha+\gamma\neq 0$ and $(\beta-1)(\gamma-1)\neq 0$ then A contains an idempotent.*

Example 2.4. Let A be a commutative $\mathbb{R}$-algebra of dimension 3 and $\{e,t,u\}$ be a basis of A in which the non-zero products are as follows:

$$e^2=e+t,\ et=\frac{1}{2}t,\ eu=-\frac{1}{2}u.$$

We define a non-zero homomorphism $\omega: A\longrightarrow\mathbb{R}$, by $\omega(e)=1$ and $\omega(t)=\omega(u)=0$. Let $x=e+at+bu$, $a,b\in\mathbb{R}$ be an element of weight 1 in A. Then we have the plenary powers of x: $x^{[2]}=e+(1+a)t-bu$, $x^{[3]}=e+(2+a)t+bu$ and $x^{[4]}=e+(3+a)t-bu$. The following identity, $x^{[4]}=x^{[3]}+x^{[2]}-x^{[1]}$, is satisfied. Thus (A,ω) is a plenary train algebra satisfying (2.1) for $(\alpha,\beta,\gamma)=(1,1,-1)$. Besides, we have $2-\alpha+\gamma=0$ and $(\beta-1)(\gamma-1)=0$. Taking $y=e+t+u$, then $\{y^{[1]},y^{[2]},y^{[3]}\}$ is an independent vector family. Thus (A,ω) is a plenary train algebra of rank 4 [8]. It is easily verified that A does not contain any idempotent.

Remark 2.5. In [7], the example given is a plenary train algebra of rank 3. We can observe that the identities $(x^2-\omega(x)x)^2=0$ and $x^4-\omega(x)x^3-\frac{1}{4}\omega(x)^2x^2+\frac{1}{4}\omega(x)^3x=0$ are valid for our example.

By partial linearizations of (2.1), we obtain the following identities:

$$\begin{aligned}8\left(x^2x^2\right)\left(x^2(xy)\right)=&\,4\alpha\omega(x^3y)x^2x^2+4\alpha\omega(x)^4x^2(xy)+6\beta\omega(x^5y)x^2\\&+2\beta\omega(x)^6xy+7\gamma\omega(x^6y)x+\gamma\omega(x)^7y\end{aligned} \tag{2.2}$$

$$\begin{aligned}32\big(x^2(xz)\big)\big(x^2(xy)\big) + 16\big(x^2x^2\big)\big((xz)(xy)\big) + 8\big(x^2x^2\big)\big(x^2(zy)\big) \\ = 12\alpha\omega(x^2zy)x^2x^2 + 16\alpha\omega(x^3y)x^2(xz) + 16\alpha\omega(x^3z)x^2(xy) \\ + 8\alpha\omega(x)^4(xz)(xy) + 4\alpha\omega(x)^4x^2(zy) + 30\beta\omega(x^4zy)x^2 \\ + 12\beta\omega(x^5y)xz + 12\beta\omega(x^5z)xy + 2\beta\omega(x)^6yz \\ + 42\gamma\omega(x^5yz)x + 7\gamma\omega(x^6y)z + 7\gamma\omega(x^6z)y\end{aligned} \tag{2.3}$$

$$\begin{aligned}&64\big((xt)(xz)\big)\big(x^2(xy)\big) + 64\big((xt)(xy)\big)\big(x^2(xz)\big) + 32\big(x^2(tz)\big)\big(x^2(xy)\big) \\ &+ 32\big(x^2(xz)\big)\big(x^2(ty)\big) + 64\big(x^2(xt)\big)\big((xz)(xy)\big) + 16\big(x^2x^2\big)\big((tz)(xy)\big) \\ &+ 16\big(x^2x^2\big)\big((xz)(ty)\big) + 32\big(x^2(xt)\big)\big(x^2(zy)\big) + 16\big(x^2x^2\big)\big((xt)(zy)\big) \\ &= 24\alpha\omega(xtzy)x^2x^2 + 48\alpha\omega(x^2zy)x^2(xt) + 48\alpha\omega(x^2ty)x^2(xz) \\ &+ 32\alpha\omega(x^3y)(xt)(xz) + 16\alpha\omega(x^3y)x^2(tz) + 48\alpha\omega(x^2tz)x^2(xy) \\ &+ 32\alpha\omega(x^3z)(xt)(xy) + 16\alpha\omega(x^3z)x^2(ty) + 32\alpha\omega(x^3t)(xz)(xy) \\ &+ 8\alpha\omega(x)^4(tz)(xy) + 8\alpha\omega(x)^4(xz)(ty) + 16\alpha\omega(x^3t)x^2(zy) \\ &+ 8\alpha\omega(x)^4(xt)(zy) + 120\beta\omega(x^3tzy)x^2 + 60\beta\omega(x^4zy)xt + 60\beta\omega(x^4ty)xz \\ &+ 12\beta\omega(x^5y)tz + 60\beta\omega(x^4tz)xy + 12\beta\omega(x^5z)ty + 12\beta\omega(x^5t)yz \\ &+ 210\gamma\omega(x^4tyz)x + 42\gamma\omega(x^5yz)t + 42\gamma\omega(x^5ty)z + 42\gamma\omega(x^5tz)y\end{aligned} \tag{2.4}$$

3. Peirce's decomposition

Let e be an idempotent element of A. For all $y \in \ker\omega$ and $x = e$ in (2.2), $P(L_e)(y) = 0$ with $P = 8X^3 - 4\alpha X^2 - 2\beta X - \gamma$. The polynomial P can be written $P = (2X-1)Q$ with $Q = 4X^2 + 2(1-\alpha)X + \gamma$. In some convenable extension of K, $Q = 4(X-r)(X-s)$ with r, s in the extension, $\alpha = 1 + 2r + 2s$, $\beta = -2(r + s + 2rs)$ and $\gamma = 4rs$. We assume throughout that the polynomial Q is split on the field K.

Theorem 3.1. *If e is an idempotent element of A and if the scalars $1/2, r$ and s are mutually different, then we have the Peirce decomposition of A attached to e:*

$$A = Ke \oplus A_{1/2}(e) \oplus A_r(e) \oplus A_s(e),$$

where $A_\mu(e) = \{x \in \ker\omega | ex = \mu x\}$. Moreover

$$\begin{cases} A_{1/2}(e)^2 & \subset A_r(e) \oplus A_s(e), \\ A_{1/2}(e)A_r(e) & \subset A_{1/2}(e) \oplus A_s(e), \\ A_{1/2}(e)A_s(e) & \subset A_{1/2}(e) \oplus A_r(e); \end{cases} \tag{3.1}$$

$$A_r(e)^2 \subset \begin{cases} A_{1/2}(e) \oplus A_s(e) & \text{if } r = 0, \\ A_r(e) \oplus A_s(e) & \text{if } 2r + 1 = 0, \\ A_{1/2}(e) \oplus A_r(e) & \text{if } 2r^2 - s = 0, \\ 0 & \text{if } r(2r^2 - s)(2r+1) \neq 0; \end{cases} \tag{3.2}$$

$$A_s(e)^2 \subset \begin{cases} A_{1/2}(e) \oplus A_r(e) & \textit{if } s = 0, \\ A_r(e) \oplus A_s(e) & \textit{if } 2s+1 = 0, \\ A_{1/2}(e) \oplus A_s(e) & \textit{if } 2s^2 - r = 0, \\ \quad 0 & \textit{if } s(2s^2-r)(2s+1) \neq 0; \end{cases} \tag{3.3}$$

$$A_r(e)A_s(e) \subset \begin{cases} A_r(e) \oplus A_s(e) & \textit{if } 4rs - 1 = 0, \\ A_{1/2}(e) \oplus A_s(e) & \textit{if } r = 0, \\ A_{1/2}(e) \oplus A_r(e) & \textit{if } s = 0, \\ \quad 0 & \textit{if } rs(4rs-1) \neq 0. \end{cases} \tag{3.4}$$

These inclusions are called Peirce's dictionary of A.

Proof. Take $x = e$ and y, z in $\ker \omega$. By using (2.3), we have

$$16(e(ez))(e(ey)) + 8e((ez)(ey)) + 4e(e(zy)) = 4\alpha(ez)(ey) + 2\alpha e(zy) + \beta zy \tag{3.5}$$

Let $y \in A_\mu(e)$ and $z \in A_\delta(e)$ with $\mu, \delta \in \{1/2, r, s\}$ in (3.5). We obtain

$$4e(eyz) + (8\mu\delta - 2\alpha)e(yz) + (16\mu^2\delta^2 - 4\mu\delta\alpha - \beta)yz = 0.$$

Let $yz = (yz)_{1/2} + (yz)_r + (yz)_s$ be the decomposition of yz in A. The previous identity become

$$C_{1/2}(\mu,\delta)(yz)_{1/2} + C_r(\mu,\delta)(yz)_r + C_s(\mu,\delta)(yz)_s = 0, \tag{3.6}$$

with

$$\begin{cases} C_{1/2}(\mu,\delta) &=& 16\mu^2\delta^2 - 4\mu\delta\alpha - \beta + 1 + 4\mu\delta - \alpha, \\ C_r(\mu,\delta) &=& 16\mu^2\delta^2 - 4\mu\delta\alpha - \beta + 4r^2 + (8\mu\delta - 2\alpha)r, \\ C_s(\mu,\delta) &=& 16\mu^2\delta^2 - 4\mu\delta\alpha - \beta + 4s^2 + (8\mu\delta - 2\alpha)s. \end{cases}$$

From (3.6) we observe that $C_\tau(\mu,\delta) \neq 0$ led $(yz)_\tau = 0$ for all $\tau \in \{1/2, r, s\}$. It is know that $\alpha = 1 + 2r + 2s$, $\beta = -2(r + s + 2rs)$ and $\gamma = 4rs$. The previous system becomes

$$\begin{cases} C_{1/2}(1/2,1/2) &=& (2r-1)(2s-1), \\ C_r(1/2,r) &=& 2(2r-1)(r-s), \\ C_s(1/2,s) &=& 2(2s-1)(s-r). \end{cases} \tag{3.7}$$

As the scalars $1/2, r$ and s are two by two distinct then the coefficients of (3.7) are all non-zero. So, (3.1) is satisfied.

We consider the system:

$$\begin{cases} C_{1/2}(r,r) &=& 4r(2r-1)(2r^2-s), \\ C_r(r,r) &=& 2(2r-1)(2r+1)(2r^2-s), \\ C_s(r,r) &=& 2r(2r-1)^2(2r+1). \end{cases}$$

The relations (3.2) result from this. The relations (3.3) are similarly obtained. Finally, the following system gives us the inclusions (3.4).

$$\begin{cases} C_{1/2}(r,s) &=& 4rs(2r-1)(2s-1, \\ C_r(r,s) &=& 2s(2r-1)(4rs-1), \\ C_s(r,s) &=& 2r(2s-1)(4rs-1). \end{cases}$$

□

Corollary 3.2. *If $rs(2r+1)(2s+1)(2r^2-s)(2s^2-r)(4rs-1) \neq 0$ then $(A_r(e) \oplus A_s(e))^2 = 0$.*

For $x = e$ and $y, z, t \in \ker\omega$, the identity (2.4) give the following one:

$$\begin{aligned} &8\big((et)(ez)\big)\big(e(ey)\big) + 8\big((et)(ey)\big)\big(e(ez)\big) + 8\big(e(et)\big)\big((ez)(ey)\big)\\ &+ 4\big(e(ez)\big)\big(e(ty)\big) + 4\big(e(tz)\big)\big(e(ey)\big) + 4\big(e(et)\big)\big(e(zy)\big)\\ &+ 2e\big((ez)(ty)\big) + 2e\big((tz)(ey)\big) + 2e\big((et)(zy)\big)\\ =&\alpha[(tz)(ey) + (ez)(ty) + (et)(zy)] \end{aligned} \tag{3.8}$$

Proposition 3.3. *If the characteristic of K is different from 3 then for all $x_{1/2} \in A_{1/2}(e)$:*

(i) $(r+s)x_{1/2}^3 - x_{1/2}(ex_{1/2}^2) - ex_{1/2}^3 = 0$;

(ii) $\big[(1-2s)x_{1/2}(x_{1/2}^2)_r + (1-2r)x_{1/2}(x_{1/2}^2)_s\big]_{1/2} = 0$.

Proof. We take $y = z = t = x_{1/2}$ in (3.8), because the field is of characteristic different from 3, we obtain : $(1-\alpha)x_{1/2}^3 + 2(ex_{1/2}^2)x_{1/2} + 2ex_{1/2}^3 = 0$. This last identity give (i) because $\alpha = 1+2r+2s$. Identity (ii) is an immediate consequence of (i). □

Proposition 3.4. *Assume that $(A_r(e) \oplus A_s(e))^2 = 0$. Then for all $x_{1/2}, x_r$ and x_s respectively in $A_{1/2}(e)$, $A_r(e)$ and $A_s(e)$ we obtain the following identities:*

(i) $2(r-s)[(x_{1/2}x_r)_{1/2}x_{1/2}]_r + (2r-1)[(x_{1/2}x_r)_s x_{1/2}]_r = 0$,
$2(s-r)[(x_{1/2}x_s)_{1/2}x_{1/2}]_s + (2s-1)[(x_{1/2}x_s)_r x_{1/2}]_s = 0$;

(ii) $r(2r^2-s)[(x_{1/2}x_r)x_r]_{1/2} = 0 = r(4r^2-1)[(x_{1/2}x_r)x_r]_s$,
$s(2s^2-r)[(x_{1/2}x_s)x_s]_{1/2} = 0 = s(4s^2-1)[(x_{1/2}x_s)x_s]_r$;

(iii) $rs(2r-1)[(x_{1/2}x_s)x_r]_{1/2} + rs(2s-1)[(x_{1/2}x_r)x_s]_{1/2} = 0$,
$s(4rs-1)[(x_{1/2}x_s)x_r]_r = 0$,
$r(4rs-1)[(x_{1/2}x_r)x_s]_s = 0$.

(iv) $x_{1/2}^4 = 0$ *if the characteristic of K is different from 3.*

Proof. All the identities (i), (ii) and (iii) follow from (3.8) by an appropriate choice of the variables y, z and t using Peirce's dictionary of A. The linearization of the identity (2.4) gives the following one:

$$xJ(y,z,t) + yJ(z,t,x) + zJ(t,x,y) + tJ(x,y,z) = 0 \tag{3.9}$$

with $x, y, z, t \in A_{1/2}(e)$ and $J(a,b,c) = (ab)c + b(ca) + c(ab)$. Hence the result. □

Proposition 3.5. *Assume that $(A_r(e) \oplus A_s(e))^2 = 0$. Then*

(i) *The baric algebra (A, ω) is a principal train algebra of rank 3 and satisfies $x^3 = (1+r)\omega(x)x^2 - r\omega(x)^2x$ if and only if $A_r(e) \neq 0$ and $A_s(e) = 0$;*

(ii) *The baric algebra (A, ω) A is a principal train algebra of rank 3 and satisfies $x^3 = (1+s)\omega(x)x^2 - s\omega(x)^2x$ if and only if $A_r(e) = 0$ and $A_s(e) \neq 0$;*

(iii) *The rank of A is 2 if and only if $A_r(e) = 0$ and $A_s(e) = 0$.*

Proof. (i) Suppose that $A_s(e) = 0$. Let $x = e + x_{1/2} + x_r$. Then $x^2 = e + x_{1/2} + 2x_{1/2}x_r + 2rx_r + x_{1/2}^2$, and $x^3 = e + x_{1/2} + 2(1+r)x_{1/2}x_r + r(1+2r)x_r + (1+r)x_{1/2}^2$. Therefore $x^3 = (1+r)\omega(x)x^2 - r\omega(x)^2x$ for all $x \in A$. If $A_r(e) \neq 0$ then for

$y = e + x_r$ with $x_r \neq 0$, the family $\{y, y^2\}$ of vectors is linearly independent. If $A_r(e) = 0$ then $A = Ke \oplus A_{1/2}(e)$. For any $u \in A_{1/2}(e)$ we have $u^2 = 0$. So, for x in H we have $x^2 = x$, which is a contradiction. The assertion (ii) is similarly proved. For (iii), if $A_r(e) = 0$ and $A_s(e) = 0$ then $A = Ke \oplus A_{1/2}(e)$. For any $u \in A_{1/2}(e)$ we have $u^2 = 0$. So, for $x = e + u \in H$ we have $x^2 = e + u = x$: A is of rank 2. If $A_r(e) \neq 0$ or $A_s(e) \neq 0$ then, from (i) or from (ii), the rank of A is more than 3. □

Corollary 3.6. *If $(A_r(e) \oplus A_s(e))^2 = 0$ then the following assertions are equivalent:*

(i) *The baric algebra (A, ω) is of rank 4;*
(ii) *$A_r(e) \neq 0$ and $A_s(e) \neq 0$.*

4. Change of idempotent

In this section the scalars $1/2, r, s$ are mutually different and $(A_r(e) \oplus A_s(e))^2 = 0$. We study the change of idempotent for Peirce's decomposition under some assumptions.

Definition 4.1. ([1]) We say that A is e-stable if

$$A_{1/2}(e)A_r(e) \subset A_{1/2}(e) \text{ and } A_{1/2}(e)A_s(e) \subset A_{1/2}(e).$$

Proposition 4.2. *If A is e-stable then for all integer k, $A_{1/2}(e)^{2k+1} \subset A_{1/2}(e)$.*

Proof. It is immediate for $k = 0$. Assume the property to be true up to $k \geq 1$: $A_{1/2}(e)^{2k-1} \subset A_{1/2}(e)$. Then from Theorem 3.1 we have $A_{1/2}(e)^{2k} \subset A_r(e) \oplus A_s(e)$. And the fact that A is e-stable we obtain $A_{1/2}(e)^{2k+1} \subset A_{1/2}(e)$. □

Lemma 4.3. *If A is e-stable, then for any $x_{1/2}, x_r$ and x_s respectively in $A_{1/2}(e)$, $A_r(e)$ and $A_s(e)$:*

$$[x_{1/2}(x_{1/2}x_r)]_r = 0, \tag{4.1}$$
$$[x_{1/2}(x_{1/2}x_s)_s] = 0. \tag{4.2}$$

Proof. We use the identity (ii) of Proposition 3.4. □

Proposition 4.4. *If A is e-stable then the set of idempotent is*

$$I_p(A) = \{e + x_{1/2} + (1-2r)^{-1}(x_{1/2})_r^2 + (1-2s)^{-1}(x_{1/2})_s^2, x_{1/2} \in A_{1/2}(e)\}.$$

Proof. Let $e' = e + x_{1/2} + x_r + x_s$ be an idempotent of A. then

$$e'^2 = e' \iff \begin{cases} x_r = (1-2r)^{-1}(x_{1/2})_r^2 \\ x_s = (1-2s)^{-1}(x_{1/2})_s^2 \\ x_{1/2}x_r + x_{1/2}x_s = 0. \end{cases}$$

Indeed, $x_{1/2}x_r + x_{1/2}x_s = (1-2r)^{-1}(x_{1/2})_r^2 x_{1/2}x_r + (1-2s)^{-1}(x_{1/2})_s^2 x_{1/2}x_s = 0$ by using (ii) of Proposition 3.3. □

Let e' be an idempotent of A, then $A = Ke' \oplus A_{1/2}(e') \oplus A_r(e') \oplus A_s(e')$.

Proposition 4.5. *Let A be e-stable, $e' = e+u+(1-2r)^{-1}(u^2)_r+(1-2s)^{-1}(u^2)_s$ be an idempotent, where $u \in A_{1/2}(e)$ such as for all $x \in A_r(e) \oplus A_s(e)$, $u(u(u(ux))) = 0$. Then we have*

$$\begin{aligned} A_{1/2}(e') =& \big\{x_{1/2} + 2(1-2r)^{-1}(ux_{1/2})_r + 2(1-2s)^{-1}(ux_{1/2})_s,\ x_{1/2} \in A_{1/2}(e)\big\}, \\ A_r(e') =& \big\{x_r - 2(1-2r)^{-1}ux_r - 2(1-2r)^{-1}(r-s)^{-1}u(ux_r) \\ &+ 4(1-2r)^{-1}(1-2s)^{-1}(r-s)^{-1}u(u(ux_r)), x_r \in A_r(e)\big\}, \\ A_s(e') =& \big\{x_s - 2(1-2s)^{-1}ux_s - 2(1-2s)^{-1}(s-r)^{-1}u(ux_s) \\ &+ 4(1-2r)^{-1}(1-2s)^{-1}(s-r)^{-1}u(u(ux_s)), x_s \in A_s(e)\big\}. \end{aligned}$$

Proof. Suppose A is e-stable, then the identity (ii) of Proposition 3.3 is equivalent to

$$(1-2s)(x_{1/2}^2)_r x_{1/2} + (1-2r)(x_{1/2}^2)_s x_{1/2} = 0. \tag{4.3}$$

Linearising (4.3) and multiplying by $(1-2r)^{-1}(1-2s)^{-1}$, we obtain that

$$\begin{aligned} &(1-2r)^{-1}[(x_{1/2})_r^2 y_{1/2} + (x_{1/2}y_{1/2})_r x_{1/2}] \\ &+(1-2s)^{-1}[(x_{1/2})_s^2 y_{1/2} + 2(x_{1/2}y_{1/2})_s x_{1/2}] = 0. \end{aligned} \tag{4.4}$$

We know that if A is $e-$stable then for all $u \in A_{1/2}(e)$, $e' = e + u + (1-2r)^{-1}(u^2)_r + (1-2s)^{-1}(u^2)_s$ is an idempotent. Let $x = x_{1/2} + x_r + x_s$ be an element of A such that $x_\lambda \in A_\lambda(e')$ with $\lambda \in \{1/2, r, s\}$; then $e'x = \lambda x$ if and only if $\big(e+u+(1-2r)^{-1}(u^2)_r+(1-2s)^{-1}(u^2)_s\big)(x_{1/2}+x_r+x_s) = \lambda x_{1/2}+\lambda x_r+\lambda x_s$.

We obtain the following system:

$$\begin{cases} \left(\frac{1}{2}-\lambda\right)x_{1/2} + \big((1-2r)^{-1}(u^2)_r + (1-2s)^{-1}(u^2)_s\big)x_{1/2} + ux_r + ux_s &= 0 \\ (r-\lambda)x_r + (ux_{1/2})_r &= 0 \\ (s-\lambda)x_s + (ux_{1/2})_s &= 0. \end{cases}$$

So, for $\lambda = 1/2$, we have

$$x_r = 2(1-2r)^{-1}(ux_{1/2})_r \text{ and } x_s = 2(1-2s)^{-1}(ux_{1/2})_s.$$

Using (4.4) and replacing $y_{1/2}$ by $x_{1/2}$ and $x_{1/2}$ by u, we obtain.

$$\begin{aligned} &(1-2r)^{-1}(u^2)_r x_{1/2} + (1-2s)^{-1}(u^2)_s x_{1/2} \\ &+2(1-2r)^{-1}u(ux_{1/2})_r + 2(1-2s)^{-1}u(ux_{1/2})_s \quad = \quad 0 \end{aligned}$$

So,

$$A_{1/2}(e') = \big\{x_{1/2} + (1-2r)^{-1}(ux_{1/2})_r + (1-2s)^{-1}(ux_{1/2})_s,\ x_{1/2} \in A_{1/2}(e)\big\}.$$

For $\lambda = r$ we have $(ux_{1/2})_r = 0$, $(s-r)x_s + (ux_{1/2})_s = 0$ and

$$\left(\frac{1}{2}-r\right)x_{1/2} + (1-2r)^{-1}(u^2)_r x_{1/2} + (1-2s)^{-1}(u^2)_s x_{1/2} + ux_r + ux_s = 0.$$

From (4.4) we obtain,

$$(1-2r)^{-1}(u^2)_r x_{1/2} + (1-2s)^{-1}(u^2)_s x_{1/2} = -2(1-2s)^{-1}u(ux_{1/2}).$$

Therefore

$$\left(\frac{1}{2}-r\right)x_{1/2}-2(1-2s)^{-1}u(ux_{1/2})+ux_r+ux_s=0.$$

Because $(r-s)x_s = ux_{1/2}$, we get $x_s = 2(1-2r)^{-1}(s-r)^{-1}u(ux_r)$ and $ux_s = (r-s)^{-1}u(ux_{1/2})$. We obtain

$$\left(\frac{1}{2}-r\right)x_{1/2}+(1-2r)(1-2s)^{-1}(r-s)^{-1}u(ux_{1/2})+ux_r=0.$$

By Lemma 4.3, we have $[u(ux_r)]_r = 0$ and $[u(u(ux_{1/2}))]_s = 0$. Then

$$\left(\frac{1}{2}-r\right)ux_{1/2}+u(ux_r)=0.$$

So, $ux_{1/2} = -2(1-2r)^{-1}u(ux_r)$ and $u(ux_{1/2}) = -2(1-2r)^{-1}u(u(ux_r))$. Then we have

$$\left(\frac{1}{2}-r\right)x_{1/2}+2(1-2s)^{-1}(s-r)^{-1}u(u(ux_r))+ux_r=0,$$

and $x_{1/2} = 4(1-2r)^{-1}(1-2s)^{-1}(r-s)^{-1}u(u(ux_r)) - 2(1-2r)^{-1}ux_r$. Moreover, since for all $x \in A_r(e) \oplus A_s(e)$, $u(u(u(ux))) = 0$, we have $(ux_{1/2})_r = 0$.

So,

$$\begin{aligned} x =& x_r - 2(1-2r)^{-1}ux_r - 2(1-2r)^{-1}(r-s)^{-1}u(ux_r) \\ &+ 4(1-2r)^{-1}(1-2s)^{-1}(r-s)^{-1}u(u(ux_r)). \end{aligned}$$

We conclude that

$$\begin{aligned} A_r(e') =& \{x_r - 2(1-2r)^{-1}ux_r - 2(1-2r)^{-1}(r-s)^{-1}u(ux_r) \\ &+ 4(1-2r)^{-1}(1-2s)^{-1}(r-s)^{-1}u(u(ux_r)), x_r \in A_r(e)\}. \end{aligned}$$

The subspace $A_s(e')$ is obtained in the same way. □

Proposition 4.6. *Let A be a finite-dimensional e-stable algebra. Then the dimensions of the subspaces of Peirce's decomposition of A are independent of the choice of the idempotent.*

Proof. We consider the following maps : $\phi_\lambda : A_\lambda(e) \longrightarrow A_\lambda(e')$, $x_\lambda \longmapsto x'_\lambda$, $\lambda \in \{1/2, r, s\}$. We observe that these are bijective linear maps. □

Proposition 4.7. *If $rs(2r+1)(2s+1)(2r^2-s)(2s^2-r)(4rs-1) \neq 0$ and $A_{1/2}(e) = 0$ then (A, ω) is a principal train algebra of rank 4 and satisfies*

$$x^4 = (1+r+s)\omega(x)x^3 - (r+s+rs)\omega(x)^2x^2 + rs\omega(x)^3x.$$

In addition e is the only idempotent of A.

Proof. By Corollary 3.2, we know that $(A_r(e) \oplus A_s(e))^2 = 0$. For $x = e + x_r + x_s$, we have $x^2 = e + 2rx_r + 2sx_s$, $x^3 = e + (r+2r^2)x_r + (s+2s^2)x_s$, $x^4 = e + (r + r^2 + 2r^3)x_r + (s + s^2 + 2s^3)x_s$. The result follows. □

5. A subclass

In this section, we are interested in the subclass of algebras verifying the identity $(x^2-\omega(x)x)^2=0$. These algebras are called back-crossing algebras [9].

Proposition 5.1. *Let (A,ω) be a weighted algebra verifying $(x^2-\omega(x)x)^2=0$. Then A satisfies the identity (2.1) if and only if A satisfies the following principal train identity of rank 4 :*

$$x^4=\frac{1}{2}(1+\alpha)\omega(x)x^3+\frac{1}{4}(1+\beta-\alpha)\omega(x)^2x^2+\frac{1}{4}\gamma\omega(x)^3x. \tag{5.1}$$

Proof. Using the identity $(x^2-\omega(x)x)^2=0$ on H. We obtain

$$x^2x^2=2x^3-x^2, \tag{5.2}$$

$$2x^2(xy)=x^3+2x(xy)+x^2y-x^2-xy, \tag{5.3}$$

$$\begin{aligned}4(xy)(xz)+2x^2(yz)=&2x(xz)+x^2z+2x(xy)+x^2y\\&+2[(xy)z+(xz)y+(yz)x]\\&-x^2-2xz-2xy-yz.\end{aligned} \tag{5.4}$$

From (5.2), $x^{[3]}=2x^3-x^2$. From (5.3) and (5.4), $x^3x^2=x^4+x^3-x^2$ and $x^3x^3=2x^4-x^2$. So, $x^{[4]}=(2x^3-x^2)^2=4x^4-2x^3-x^2$. Replacing $x^{[3]}$ et $x^{[4]}$ in (2.1), we obtain (5.1). □

Corollary 5.2. *If A satisfies $(x^2-\omega(x)x)^2=0$ then the following properties are equivalents :*

(i) *A is a principal train algebra of rank 4 and principal train roots $1,\lambda_1,\lambda_2$;*
(ii) *A is a plenary train algebra of rank 4 and plenary train roots $1,2\lambda_1,2\lambda_2$.*

Proof. We note P_1 the principal train polynomial and P_2 the plenary one. Using (2.1) and (5.1), we know that 0 and 1 are roots of P_1 and P_2. We get so, $P_1(X)=X(X-1)h_1(X)$ and $P_2(X)=X(X-1)h_2(X)$ with $h_1(X)=4X^2+2(1-\alpha)X+\gamma$ and $h_2(X)=X^2+(1-\alpha)X+\gamma$. Let us observe that $h_1(X)=h_2(2X)$. □

Proposition 5.3. *Let K be an infinite field. The algebra $A=Ke\oplus A_{1/2}(e)\oplus A_r(e)\oplus A_s(e)$ satisfies the identity $\left(x^2-\omega(x)x\right)^2=0$ if and only if the following identities are satisfied:*

$$\begin{gathered}(A_r(e)\oplus A_s(e))^2=0,\\(x_{1/2}x_r)x_r=0=(x_{1/2}x_s)x_s,\\(x_{1/2}x_r)x_s=0=(x_{1/2}x_s)x_r,\\(x_{1/2}x_r)^2=0=(x_{1/2}x_s)^2,\\(x_{1/2}x_r)x_{1/2}^2=0=(x_{1/2}x_s)x_{1/2}^2.\end{gathered}$$

Proof. Assume that $\left(x^2-\omega(x)x\right)^2=0$. Then, for $x=e+\lambda x_r$, $\lambda\in K$, we have $(x^2-\omega(x)x)^2=0$. Then $\lambda^2(2r-1)^2x_r^2+2(2r-1)\lambda^3x_r^3+\lambda^4(x_r^2)^2=0$ for all $\lambda\in K$. We conclude that $x_r^2=0$ and $A_r(e)^2=0$.

By the same way we obtain $A_s(e)^2=0$.

For $x = e + \lambda x_r + \delta x_s,\ \lambda, \delta \in K$ we have $(x^2 - \omega(x)x)^2 = 0$. Thus $(2r-1)(2s-1)\lambda\delta x_r x_s + 2\lambda^2\delta^2(x_r x_s)^2 + 2(2r-1)\lambda\delta x_r(x_r x_s) + 2(2s-1)\lambda\delta x_s(x_r x_s) = 0$ $\forall\lambda, \delta \in K$. Therefore $x_r x_s = 0$. So $(A_r(e) + A_s(e))^2 = 0$.

Let $x = e + \lambda x_{1/2} + \delta x_r + \tau x_s,\ \lambda, \delta, \tau \in K$.

$$\begin{aligned}(x^2 - \omega(x)x)^2 = {} & (2r-1)\lambda\delta^2(x_{1/2}x_r)x_r + 2(2s-1)\lambda\tau^2(x_{1/2}x_s)x_s \\ & + 4(2s-1)\lambda\delta\tau(x_{1/2}x_r)x_s + 4(2r-1)\lambda\delta\tau(x_{1/2}x_s)x_r \\ & + 4\lambda^2\delta^2(x_{1/2}x_r)^2 + 4\lambda^2\tau^2(x_{1/2}x_s)^2 \\ & + 4\lambda^3\delta(x_{1/2}x_r)x_{1/2}^2 + 4\lambda^3\tau(x_{1/2}x_s)x_{1/2}^2 = 0.\end{aligned}$$

This calculation completes the proof. □

Proposition 5.4. *If $2 - \alpha + \gamma \neq 0$ then the set of idempotents of A is*

$$Ip(A) = \left\{(2-\alpha+\gamma)^{-1}(2a^3 - \alpha a^2 + \gamma a) \mid a \in A, \omega(a) = 1\right\}.$$

Proof. Let $a \in A$ with $\omega(a) = 1$ and $e_a = (2-\alpha+\gamma)^{-1}(2a^3 - \alpha a^2 + \gamma a)$. We have $e_a = a + (2-\alpha+\gamma)^{-1}[2(a^3 - a) - \alpha(a^2 - a)]$. Then $e_a^2 = a^2 + 2(2-\alpha+\gamma)^{-1}[2(a^4 - a^2) - \alpha(a^3 - a^2)]$. From Proposition 5.1, $a^4 = \frac{1}{2}(1+\alpha)a^3 + \frac{1}{4}(1+\beta-\alpha)a^2 + \frac{1}{4}\gamma a$. So $e_a^2 = e_a$. Finally, if a is an idempotent then $e_a = a$. □

Proposition 5.5. *Let $a \in H$ and $K[a]$ be the subalgebra of A generated by a. We suppose $2 - \alpha + \gamma \neq 0$ and $K[a]$ is a train algebra of rank 4. Then the Peirce decomposition of $K[a]$ attached to its unique idempotent $e_a = (2-\alpha+\gamma)^{-1}(2a^3 - \alpha a^2 + \gamma a)$ satisfy:*

$$\begin{aligned}K[a]_{1/2} &= 0, \\ K[a]_r &= \langle a^3 - (1+s)a^2 + sa\rangle, \\ K[a]_s &= \langle a^3 - (1+r)a^2 + ra\rangle, \\ \left(K[a]_r \oplus K[a]_s\right)^2 &= 0.\end{aligned}$$

Proof. We know that $\alpha = 1 + 2r + 2s, \beta = -2(r + s + 2rs), \gamma = 4rs$. So, $\gamma - \alpha + 2 = (1-2r)(1-2s)$. Let $e_a = (2-\alpha+\gamma)^{-1}(2a^3 - \alpha a^2 + \gamma a)$ be an idempotent of $K[a]$ and $x = \eta_1 a^3 + \eta_2 a^2 - (\eta_1 + \eta_2)a \in \ker\omega$, $\eta_1, \eta_2 \in K$. We have $e_a x = \eta_1(a^4 - a^2) + \eta_2(a^3 - a^2)$. From Proposition 5.1 we have

$$\begin{aligned}a^4 &= \frac{1}{2}(1+\alpha)a^3 + \frac{1}{4}(1+\beta-\alpha)a^2 + \frac{1}{4}\gamma a, \\ a^4 - a^2 &= \frac{1}{2}(1+\alpha)a^3 + \frac{1}{4}(-3+\beta-\alpha)a^2 + \frac{1}{4}\gamma a \\ &= \frac{1}{2}(1+\alpha)(a^3 - a) + \frac{1}{4}(-3+\beta-\alpha)(a^2 - a).\end{aligned}$$

So,

$$\begin{aligned} e_a x &= \eta_1(a^4 - a^2) + \eta_2(a^3 - a^2) \\ &= \eta_1[\frac{1}{2}(1+\alpha)(a^3 - a) + \frac{1}{4}(-3+\beta-\alpha)(a^2 - a)] + \eta_2(a^3 - a^2) \\ &= \frac{1}{2}\eta_1(1+\alpha)(a^3 - a) + \frac{1}{4}\eta_1(-3+\beta-\alpha)(a^2 - a) + \eta_2(a^3 - a^2) \\ &= \big[\frac{1}{2}\eta_1(1+\alpha) + \eta_2\big](a^3 - a) + \big[\frac{1}{4}\eta_1(-3+\beta-\alpha) - \eta_2\big](a^2 - a) \end{aligned}$$

For x in $K[a]_\lambda$ we have:

$$e_a x = \lambda[\eta_1(a^3 - a) + \eta_2(a^2 - a)].$$

Since $K[a]$ is train of rank 4, the family $\{a, a^2, a^3\}$ is independent and we can consider the following system :

$$\begin{cases} \frac{1}{2}\eta_1(1+\alpha) + \eta_2 & = \lambda\eta_1 \\ \frac{1}{4}\eta_1(-3+\beta-\alpha) - \eta_2 & = \lambda\eta_2. \end{cases}$$

For $\lambda = 1/2$ we have

$$\begin{cases} \eta_2 & = -\frac{\alpha}{2}\eta_1 \\ (3 - 2\alpha - \beta)\eta_1 & = 0. \end{cases}$$

$3 - 2\alpha - \beta = 3 - 2\alpha - 1 + \alpha + \gamma = 2 - \alpha + \gamma \neq 0$. We obtain $\eta_1 = 0$ and $\eta_2 = 0$. Therefore $K[a]_{1/2} = 0$.

For $\lambda = r$ we have

$$\begin{cases} \frac{1}{2}\eta_1(1+\alpha) + \eta_2 & = r\eta_1 \\ \frac{1}{4}\eta_1(-3+\beta-\alpha) - \eta_2 & = r\eta_2. \end{cases}$$

$\frac{1}{2}\eta_1(1+\alpha) + \eta_2 = r\eta_1 \Longrightarrow (1+s)\eta_1 + \eta_2 = 0 \Longrightarrow \eta_2 = -(1+s)\eta_1$.
We obtain $x = \eta_1(a^3 - a) + \eta_2(a^2 - a) = \eta_1[(a^3 - a) - (1+s)(a^2 - a)]$.
Therefore

$$K[a]_r = \langle a^3 - (1+s)a^2 + sa\rangle.$$

By the same way,

$$K[a]_s = \langle a^3 - (1+r)a^2 + ra\rangle.$$

□

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