Gulf Journal of Mathematics
Vol 15, Issue 1 (2023) 158-172

ISSN: 2309-4966
https://doi.org/10.56947/gjom.v15i1.1387

EVOLUTION ALGEBRAS SATISFYING TRAIN IDENTITY OF DEGREE 2 AND EXPONENT 3

JOSEPH TENKODOGO[1], SOULEYMANE SAVADOGO [2], ANDRÉ CONSEIBO[3*] AND ABDOULAYE DEMBEGA [4]

Abstract. In this paper, we give the necessary and sufficient conditions for an evolution algebra to satisfy a train identity of degree 2 and exponent 3. We show that this class of algebras is a subclass of the Bernstein algebras of order 2. Then, we study the relations existing between these algebras and the Bernstein algebras as well as the power associative algebras. Moreover we give a classification in dimension ≤ 4 of evolution algebras satisfying strictly a train identity of degree 2 and exponent 3. Finally, we describe the derivations and automorphisms of these algebras.

1. Introduction

In the algebra theory of population genetics, several classes of algebras are used to describe the process of inheritance in genetic. Some of these algebras are baric and satisfy polynomial identities. This is how Philip Holgate in 1975 ([9]) defines the Bernstein algebras, i.e. commutative and not necessarily associative algebras verifying the equation $(x^2)^2 = \omega(x)^2x^2$. These algebras were defined to model Hardy Weinberg's principle which states that the genetic heritage of a population stabilizes in the second generation. In [2] and [15], the authors studied the algebras verifying a train identity of degree 2 and exponent n for $n = 3$ or 4, that is, such that $(x^n)^2 = \omega(x)^n x^n$ ($n = 3$ or $n = 4$). They showed that this last class of algebras contains strictly that of Bernstein algebras. As for evolution algebras, they have been studied for two decades (see[14]). Unlike previous algebras, evolution algebras concern non-Mendelian genetics. The purpose of this paper is to describe the baric evolution algebras that verifies the identity $(x^3)^2 = \omega(x)^3x^3$. Section 2 is devoted to preliminary results. In section 3, we characterize the evolution algebras satisfying a train identity of degree 2 and exponent 3. We show that these algebras are Bernstein algebras of order 2. The link between evolution algebras satisfying a train identity of degree 2 and exponent 3 with Bernstein algebras and power-associative algebras is studied. Incidentally, we show that in dimension 2, the evolution algebras verifying the identity (2.1) are Bernstein

Date: Received: Dec 28, 2022; Accepted: May 18, 2023.

* Corresponding author.

2010 *Mathematics Subject Classification.* Primary 17D92, 17A30; Secondary 17D99, 17A60.

Key words and phrases. Evolution algebra, train identity of degree 2 and exponent 3, Bernstein algebra of order n, Automorphism, Derivation, power associative, idempotent, Natural basis.

algebras. Then in section 4 we are interested in the classification in dimension 3 and 4 of these algebras. Finally, since derivations ([5]) and automorphisms play an important role in the study of algebraic structures, we proceeded to their calculations in section 5.

2. Preliminaries

Let K be a commutative field and A a commutative non-necessarily associative K-algebra. For any x in A, the *principal powers* of x are defined by $x^1 = x$ and $x^{n+1} = x^n x$ and those of A are defined by $A^1 = A$ and $A^{n+1} = A^n A$ for any integer $n \geq 1$. We also define the plenary powers of $x \in A$ by $x^{[1]} = x$ and $x^{[n+1]} = x^{[n]}x^{[n]}$ with $n \geq 1$.

Let K be an infinite commutative field of characteristic different from 2 and 3, and (A, ω) be a baric K-algebra, i.e. $\omega : A \longrightarrow K$ is a surjective morphism of K-algebras called weight function. We will say that (A, ω) verifies a *train identity of degree* 2 *and exponent* 3 if for any x in A we have

$$(x^3)^2 = \omega(x)^3 x^3. \tag{2.1}$$

Since ω is a surjective morphism of algebras, then the set $\mathcal{I} = \{x^3 \mid \omega(x) = 1\}$ of nonzero idempotents is not empty. The Peirce decomposition of A is given by $A = Ke \oplus A_0(e) \oplus A_{1/2}(e) \oplus A_{-1/2}(e)$, where e is a nonzero idempotent and $A_\lambda = \{x \in \ker\omega \mid ex = \lambda x\}$ for $\lambda \in \{0,\ 1/2,\ -1/2\}$. The relations between the Peirce subspaces are given by: $A^2_{1/2}(e) \subset A_0(e) \oplus A_{-1/2}(e)$, $A^2_0(e) \subset A_{1/2}(e) \oplus A_0(e)$, $A_0(e)A_{1/2}(e) \subset A_{1/2}(e) \oplus A_{-1/2}(e)$, $A_{1/2}(e)A_{-1/2}(e) \subset A_{1/2}(e) \oplus A_0(e)$ and $A^2_{-1/2}(e) \subset A_{1/2}(e)$ [2, Theorem 3.4].

We will say that a baric algebra (A, ω) is *Bernstein of order* $n \geq 0$ if for any element x in A we have

$$x^{[n+2]} = \omega(x)^{2^n} x^{[n+1]}.$$

For $n = 1$, we simply say that A is a Bernstein algebra. In [10], the authors show that in a Bernstein algebra of order n, (A, ω), the weight function ω is unique and A has a nonzero idempotent. In characteristic different from 2, a Bernstein algebra of order 2 (A, ω) admits the Peirce decomposition, relatively to the following nonzero idempotent e:

$$A = Ke \oplus U_e \oplus V_e \tag{2.2}$$

with $U_e = \{x \in \ker\omega \mid ex = \frac{1}{2}x\}$ and $V_e = \{x \in \ker\omega \mid e(ex) = 0\}$ [7, Proposition 2]. Bernstein algebras of order 2 satisfying $U_e^2 = 0$, for a given idempotent e, are studied in [8]. The authors characterize in particular the set of nonzero idempotents and show that U_e^2 is an invariant. They also characterize the subspaces of $U_{e'}$ and $V_{e'}$ when e' is another nonzero idempotent of algebra. The classification in dimension 3 and 4 of Bernstein algebras of order 2 is given in [6, 10].

We say that a finite dimensional K-algebra of dimension n is an evolution algebra if it admits a basis $B = \{e_1, \cdots, e_n\}$ such that

$$e_ie_j = 0 \text{ for } 1 \leq i \neq j \leq n \text{ and } e_i^2 = \sum_{k=1}^{n} a_{ik}e_k \tag{2.3}$$

Such a basis is called the natural basis of A and the matrix $M = (a_{ik})_{1\leq i,k\leq n}$ is the matrix of the structure constants of A relatively to the natural basis B. Evolution algebras are commutative, they are not associative in general[14].

Let K be a commutative field of characteristic different from 2 and 3. In [12, Corollary 3.4], the authors show that a baric evolution algebra (A, ω) of finite dimension n admits a natural basis $B = \{e_1, \cdots, e_n\}$ whose multiplication table is defined by:

$$e_1^2 = e_1 + \sum_{k=2}^{n} a_{1k}e_k,\ e_j^2 = \sum_{k=2}^{n} a_{jk}e_k \text{ with } \omega(e_1) = 1,\ \omega(e_j) = 0 \text{ for } 2 \leq j \leq n. \tag{2.4}$$

In the rest of the paper, K denotes a commutative field of characteristic different from 2 and 3 and any finite dimensional baric evolution algebra will be endowed with such a natural basis.

3. Evolution algebras verifying the identity $x^3x^3 = \omega(x)^3x^3$

The following result gives a characterization of baric evolution algebras satisfying the train identity of degree 2 and exponent 3.

3.1. Characterization theorem.

Theorem 3.1 (Characterization)**.** *Let (A, ω) be a baric evolution K-algebra of finite dimension. Then the algebra A verifies the identity (2.1) if and only if the following assertions hold:*

(1) $(e_1^2)^2 = e_1^2$;
(2) $e_1^2(e_1^2e_j) = \frac{1}{2}e_1^2e_j$ *for* $2 \leq j \leq n$;
(3) $(e_1^2e_j)(e_1^2e_k) = 0$ *for* $2 \leq j \leq k \leq n$;
(4) $e_1^2(e_j^2e_k) = \frac{1}{2}e_j^2e_k$ *for* $2 \leq k, j \leq n$.

Proof. We assume that (A, ω) is a baric evolution algebra verifying the identity (2.1). Since $e_1^3 = e_1^2$, the assertion (1) is obtained by setting $x = e_1$ in the identity (2.1). By partially linearizing the identity (2.1), we obtain:

$$4(x(xy))x^3 + 2x^3(x^2y) = 3\omega(x^2y)x^3 + \omega(x^3)x^2y + 2\omega(x^3)x(xy). \tag{3.1}$$

The assertion (2) is obtained by setting $x = e_1$ and $y = e_j$ with $j \neq 1$ in the identity (3.1). A partial linearization of the identity (3.1) gives the following equation:

$$\begin{aligned}
&8(t(xz))(x(xy)) + 8(x(tz))(x(xy)) + 8(x(xz))(t(xy)) + 8(x(xz))(x(ty))+\\
&8(z(xt))(x(xy)) + 4(x^2z)(t(xy)) + 4(x^2z)(x(ty)) + 8(x(tx))(z(xy))+\\
&4(x^2t)(z(xy)) + 4x^3(z(ty)) + 4(x(tx))(x(zy)) + 4(x^2t)(x(zy))+\\
&4x^3(t(zy)) + 4(t(xz))(x^2y) + 4(x(tz))(x^2y) + 8(x(xz))(y(tx))+\\
&4(z(tx))(x^2y) + 4(x^2z)(y(tx)) + 8(x(tx))(y(xz)) + 4(x^2t)(y(xz))+\\
&4x^3(y(tz)) = 6\omega(tyz)x^3 + 6\omega(xyz)(x^2t) + 12\omega(xyz)(x(xt))+\\
&6\omega(xty)(x^2z) + 6\omega(x^2y)(z(xt)) + 12\omega(xty)(x(xz)) + 6\omega(x^2y)(t(xz))+\\
&6\omega(x^2y)(x(tz)) + 6\omega(xtz)(x^2y) + 6\omega(x^2z)(y(xt)) + 6\omega(x^2t)(y(xz)) + 2\omega(x)^3(y(tz))+\\
&12\omega(xtz)(x(xy)) + 6\omega(x^2z)(t(xy)) + 6\omega(x^2z)(x(ty)) + 6\omega(x^2t)(z(xy))+\\
&2\omega(x)^3(z(ty)) + 6\omega(x^2t)(x(zy)) + 2\omega(x)^3(t(zy))
\end{aligned} \tag{3.2}$$

By setting $x = y = e_1,\ z = t = e_j$ with $j \neq 1$ in the identity (3.2), we obtain:

$$(e_1^2e_j)(e_1^2e_j) = 0 \tag{3.3}$$

By setting in the same identity $x = y = e_1,\ z = e_j,\ t = e_k$ with $2 \leq j \neq k \leq n$ we get:

$$(e_1^2e_j)(e_1^2e_k) = 0 \tag{3.4}$$

Thus the identities (3.3) and (3.4) allow us to obtain assertion (3). By setting in the identity (3.2), $x = e_1$, $y = z = t = e_j$ with $j \neq 1$ we obtain:

$$e_1^2(e_j^2e_j) = \frac{1}{2}(e_j^2e_j) \tag{3.5}$$

Let us now set in the identity (3.2) $x = e_1,\ y = t = e_j,\ z = e_k$ with $2 \leq j \neq k \leq n$ we obtain:

$$e_1^2(e_j^2e_k) = \frac{1}{2}(e_j^2e_k) \tag{3.6}$$

Assertion (4) is therefore obtained from the identities (3.5) and (3.6). For the converse, we assume that the assertions (1), (2), (3) and (4) hold. Let $x = \alpha e_1 + \sum\limits_{j=2}^{n} x_je_j \in A$. We have:

$$x^2 = \alpha^2e_1^2 + \sum_{j=2}^{n} x_j^2e_j^2\ ;\ x^3 = \alpha^3e_1^2 + \alpha^2\sum_{j=2}^{n} x_je_1^2e_j + \sum_{k,j=2}^{n} x_kx_j^2e_j^2e_k$$

and

$$\begin{aligned}
x^3x^3 &= \alpha^6 e_1^2 + \sum_{k,j,t,l=2}^{n} (x_kx_j^2)(x_lx_t^2)(e_j^2e_k)(e_t^2e_l) + \alpha^5 \sum_{j=2}^{n} x_j(e_1^2e_j) + \alpha^3 \sum_{k,j=2}^{n} x_kx_j^2(e_j^2e_k) \\
&\quad +2\alpha^2 \sum_{k,j,l=2}^{n} x_lx_kx_j^2(e_1^2e_l)(e_j^2e_k) \\
&= \alpha^6 e_1^2 + 4 \sum_{k,j,t,l,p,q=2}^{n} (x_kx_j^2)(x_lx_t^2)(a_{jk}a_{tl})(a_{kp}a_{lq})(e_1^2e_p)(e_1^2e_q) + \alpha^5 \sum_{j=2}^{n} x_j(e_1^2e_j) \\
&\quad +\alpha^3 \sum_{k,j=2}^{n} x_kx_j^2(e_j^2e_k) + 4\alpha^2 \sum_{k,j,l,t=2}^{n} x_lx_kx_j^2a_{jk}a_{kt}(e_1^2e_l)(e_1^2e_t) \\
&= \alpha^6 e_1^2 + \alpha^5 \sum_{j=2}^{n} x_je_1^2e_j + \alpha^3 \sum_{k,j=2}^{n} x_kx_j^2e_j^2e_k \\
x^3x^3 &= \omega(x)^3x^3.
\end{aligned}$$

So the algebra A verifies the identity (2.1). □

We characterize in the following result the Peirce subspaces of an evolution algebra verifying the identity (2.1).
Let $\Lambda_B = Support(e_1^2) \setminus \{1\} = \{j \in \{2, \cdots, n\}, a_{1j} \neq 0\}$.

Theorem 3.2. *Let (A, ω) be a baric evolution K-algebra of finite dimension verifying the identity (2.1). Then:*

(1) $A_{1/2} = \langle e_j^2, j \in \Lambda_B \rangle$;
(2) $A_{-1/2} = 0$;
(3) $A_0 = \langle e_j - 2e_1^2e_j, j \in \{2, \cdots, n\}\rangle$.

Proof. Since A verifies the identity (2.1), then its Peirce decomposition relative to the idempotent e_1^2 is given by $A = Ke_1^2 \oplus A_{1/2} \oplus A_0 \oplus A_{-1/2}$.

(1) Let $x = \sum_{j=2}^{n} x_je_j \in A_{1/2}$. The equality $\frac{1}{2}x = e_1^2x = \sum_{j=2}^{n} x_je_1^2e_j = \sum_{j\in\Lambda_B} (x_ja_{1j})e_j^2$ implies that $x = \sum_{j\in\Lambda_B} (2x_ja_{1j})e_j^2$ and therefore $A_{1/2} \subset \langle e_j^2 \mid j \in \Lambda_B \rangle$. Assertion (2) from Theorem 3.1 shows that $e_j^2 \in A_{1/2}$ for $j \in \Lambda_B$. Therefore, $A_{1/2} = \langle e_j^2 \mid j \in \Lambda_B \rangle$.

(2) For the second assertion we set $x = \sum_{j=2}^{n} x_je_j \in A_{-1/2}$. The equalities $-\frac{1}{2}x = e_1^2x = \sum_{j\in\Lambda_B} (x_ja_{1j})e_j^2$ implies that $x = -\sum_{j\in\Lambda_B} (2x_ja_{1j})e_j^2 \in A_{1/2}$. So $x = 0$ and $A_{-1/2} = 0$.

(3) For $x = \sum_{j=2}^{n} x_je_j \in A_0$, we have $0 = e_1^2x = \sum_{j=2}^{n} x_je_1^2e_j$, which gives $x = \sum_{j=2}^{n} x_j(e_j - 2e_1^2e_j)$. Therefore $A_0 \subset \langle e_j - 2e_1^2e_j \mid j \in \{2, \cdots, n\}\rangle$. According to the assertion (2) of Theorem 3.1, we have $(e_j - 2e_1^2e_j) \in A_0$ for

any $j \in \{2, \cdots, n\}$, which implies $\langle e_j - 2e_1^2 e_j \mid j \in \{2, \cdots, n\} \rangle \subset A_0$. Thus $A_0 = \langle e_j - 2e_1^2 e_j \mid j \in \{2, \cdots, n\} \rangle$.

□

Corollary 3.3. *Let (A, ω) be a baric evolution K-algebra of finite dimension verifying the identity (2.1). So:*

(1) *The Peirce decomposition of A relative to the idempotent e_1^2 is:* $A = Ke_1^2 \oplus A_{1/2} \oplus A_0$ *such that* $A_{1/2}^2 = 0$, $A_0^2 \subset A_{1/2} \oplus A_0$, $A_0 A_{1/2} \subset A_{1/2}$;

(2) $(\ker \omega)^3 = \left\langle e_j^2 e_k \; ; k, j = \overline{2, n} \right\rangle \subset A_{1/2}$.

Proof. The first assertion follows from the Peirce decomposition of an algebra verifying the identity (2.1), from the assertion (3) of Theorem 3.1 and from the assertion (2) of Theorem 3.2. The second assertion is due to (4) of Theorem 3.1. □

Theorem 3.4. *Let (A, ω) be a finite dimensional K-baric evolution algebra satisfying the identity (2.1). Then (A, ω) is a Bernstein algebra of order 2.*

Proof. We consider a K-baric evolution algebra (A, ω) of finite dimension satisfying the identity (2.1). Let $x = \alpha e_1 + y \in A$ with $\alpha \in K$ and $y \in ker\omega$. We have: $x^2 = \alpha^2 e_1^2 + y^2$ and $(x^2)^2 = \alpha^4 e_1^2 + 2\alpha^2 e_1^2 y^2 + y^2 y^2$. Since $e_1^2 y^2 \in A_{1/2}$ and $y^2 y^2 \in (ker\omega)^{[3]} \subset (ker\omega)^3 \subset A_{1/2}$, then $(x^2 x^2)^2 = \alpha^8 e_1^2 + 4\alpha^6 e_1^2 (e_1^2 y^2) + 2\alpha^4 e_1^2 (y^2 y^2) = \alpha^4(\alpha^4 e_1^2 + 2\alpha^2 e_1^2 y^2 + y^2 y^2) = \omega(x)^4 x^2 x^2$. Hence the algebra is Bernstein algebra of order 2. □

As the example below shows, there exist Bernstein algebras of order 2 which are evolution algebras but which do not satisfy the identity (2.1) and therefore cannot be Bernstein.

Example 3.5. Let A be a 3-dimensional K-algebra whose multiplication table in the basis (e, v_1, v_2) is given by: $e^2 = e$, $v_1 v_2 = v_1$, $v_2^2 = v_1$ and the other products are zero. A is a Bernstein algebra of order 2 [10, Page 21]. Moreover A is an evolution algebra in the natural basis $(e, \; v_2, \; v_1 - v_2)$. Let $x = e + v_2$, we have $x^2 = e + v_1$, $x^3 = (e + v_2)(e + v_1) = e + v_1$, $(x^2)^2 = e$ and $(x^3)^2 = e$. Since $\omega(x) = 1$, then $(x^2)^2 - \omega(x)^2 x^2 = -v_1 \neq 0$ and $(x^3)^2 - \omega(x)^3 x^3 = -v_1 \neq 0$. It follows that A is not Bernstein and does not verify the identity (2.1).

Let (A, ω) be a baric evolution K-algebra of finite dimension verifying the identity (2.1). Since the algebra A is Bernstein of order 2 and $A_{1/2}^2 = 0$, the set of its nonzero idempotents is defined by $\mathcal{I} = \{e_1^2 + u \mid u \in A_{1/2}(e_1^2)\}$. Moreover, if $e' = e_1^2 + u$ is another nonzero idempotent of A, then $A_{1/2}(e') = A_{1/2}(e_1^2)$ and $A_0(e') = \{v - 2uv \mid v \in A_0(e_1^2)\}$ [8, Theorem 2, 3 and 4].

3.2. **Relation with Bernstein algebras.** In[4], the authors give the following result which characterizes the baric evolution algebras which are Bernstein.

Proposition 3.6 ([4, Theorem 4.2])**.** *A K-baric evolution algebra (A, ω) of finite dimension n is Bernstein if, and only if, the following conditions hold:*

(1) $(e_1^2)^2 = e_1^2$;

(2) $e_i^2 e_j^2 = 0$, *for* $2 \leq i, j \leq n$;
(3) $e_1^2 e_i^2 = \frac{1}{2} e_i^2$ *for* $2 \leq i \leq n$.

As the following example shows, there are baric evolution algebras satisfying the identity (2.1) but which are not Bernstein algebras.

Example 3.7. Let (A, ω) be a K-baric evolution algebra of dimension 5 whose table of multiplication in the natural basis $B = (e_1, e_2, e_3, e_4, e_5)$ is given by: $e_1^2 = e_1 + e_2 + e_4$, $e_2^2 = \frac{1}{4}(e_2 - e_4)$, $e_3^2 = \alpha e_2 + \beta e_4$, $e_4^2 = -e_2^2 = -\frac{1}{4}e_2 + \frac{1}{4}e_4$, $e_5^2 = \gamma_2 e_2^2 + \gamma_3 e_3^2 = (\frac{1}{4}\gamma_2 + \alpha\gamma_3)e_2 + (-\frac{1}{4}\gamma_2 + \beta\gamma_3)e_4$ with $\alpha, \beta, \gamma_2, \gamma_3 \in K$ and $\alpha + \beta \neq 0$. Since $e_2^2 e_3^2 = \frac{1}{4}(\alpha + \beta)e_2^2 \neq 0$, then the algebra A is not Bernstein algebra. Let $x = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5$. We have $x^2 = x_1^2 e_1^2 + (x_2^2 - x_4^2)e_2^2 + x_3^2 e_3^2 + x_5^2 e_5^2$; $x^3 = x_1^3 e_1^2 + ((x_2 + x_4)(x_1^2 + \frac{1}{4}(x_2^2 - x_4^2)) + x_3^2(\alpha x_2 - \beta x_4) + (\frac{1}{4}\gamma_2 + \alpha\gamma_3)x_2x_5^2 - (-\frac{1}{4}\gamma_2 + \beta\gamma_3)x_4x_5^2)e_2^2$ et $(x^3)^2 = x_1^6 e_1^2 + x_1^3(x_1^2(x_2 + x_4) + \frac{1}{4}(x_2 + x_4)(x_2^2 - x_4^2) + x_3^2(\alpha x_2 - \beta x_4) + (\frac{1}{4}\gamma_2 + \alpha\gamma_3)x_2x_5^2 - (-\frac{1}{4}\gamma_2 + \beta\gamma_3)x_4x_5^2)e_2^2$. Since $\omega(x^3) = x_1^3$ then $(x^3)^2 - \omega(x^3)x^3 =0$. The algebra A thus verifies the identity (2.1).

Proposition 3.8 ([2, Theorem 4.1])**.** *Let* $A = Ke \oplus A_{1/2} \oplus A_0 \oplus A_{-1/2}$ *be the Peirce decomposition relative to a nonzero idempotent* e *of a* K*-algebra satisfying the identity* (2.1)*. Then* A *is a Bernstein algebra if, and only if,* $A_{-1/2} = 0$, $A_0^2 \subset A_{1/2}$, $A_0^2 A_{1/2} = 0$ *and* $(x_{1/2}x_0)^2 = 0$*, for all* $x_{1/2} \in A_{1/2}$ *and* $x_0 \in A_0$.

Proposition 3.9. *Let* $A = Ke_1^2 \oplus A_{1/2} \oplus A_0$ *be the Peirce decomposition of a finite dimensional baric evolution algebra satisfying the identity* (2.1)*. Then* A *is Bernstein if and only if* $A_0^2 \subset A_{1/2}$.

Proof. The algebra A verifying the identity (2.1) and being of evolution then we have $A_{-1/2} = 0$, $A_0 A_{1/2} \subset A_{1/2}$, $A_0^2 \subset A_{1/2} \oplus A_0$ and $A_{1/2}^2 = 0$. If $A_0^2 \subset A_{1/2}$, then $A_0^2 A_{1/2} = 0$ and $(x_{1/2}x_0)^2 = 0$ for all $x_{1/2} \in A_{1/2}$, $x_0 \in A_0$. The Proposition 3.8 gives that A is a Bernstein algebra. Conversely, if A is a Bernstein algebra, thus $A_0^2 \subset A_{1/2}$. □

Corollary 3.10. *Let* $A = Ke_1^2 \oplus A_{1/2} \oplus A_0$ *be the Peirce decomposition of a finite dimensional baric evolution algebra satisfying the identity* (2.1) *such that* $A_{1/2} \neq 0$ *and* $dim(ker\omega)^2 = 1$*. Then* A *is a Bernstein algebra.*

Proof. Assume $(\ker\omega)^2 = \langle e_2^2 \rangle$. Since $A_{1/2} \neq 0$ and $A_{1/2} \subset (\ker\omega)^2$, then $dim A_{1/2} = dim(\ker\omega)^2 = 1$ and $A_{1/2} = \langle e_2^2 \rangle$. We have $A_0^2 \subset (\ker\omega)^2 = A_{1/2}$ and Proposition 3.9 gives us that the algebra is Bernstein. □

Corollary 3.11. *Let* $A = Ke_1^2 \oplus A_{1/2} \oplus A_0$ *be a Peirce decomposition of a finite dimensional baric evolution algebra satisfying the identity* (2.1) *such that* $Card(Support(e_1^2)) = n$*. Then* A *is a Bernstein algebra.*

Proof. We assume that $Card(Support(e_1^2)) = n$. Then $a_{1j} \neq 0$ for $2 \leq j \leq n$. Hence $A_{1/2} = (\ker\omega)^2$. Since $A_0 \subset \ker\omega$, then $A_0^2 \subset (\ker\omega)^2 = A_{1/2}$. According to Proposition 3.9, A is a Bernstein algebra. □

Proposition 3.12. *Let* $A = Ke_1^2 \oplus \ker\omega$ *be a finite dimensional baric evolution algebra satisfying the identity* (2.1) *such that* $(\ker\omega)^2 = 0$*. Then* A *is a Bernstein algebra.*

Proof. Suppose $(\ker\omega)^2 = 0$. Then $A_{1/2} = 0$ because $A_{1/2} \subset (\ker\omega)^2$ and $A_0 = \ker\omega$. We deduce that $A_0^2 = (\ker\omega)^2 = 0 = A_{1/2}$ and A is a Bernstein algebra. □

Corollary 3.13. *Let (A, ω) be a baric evolution algebra of dimension 2, satisfying the identity (2.1). Then A is a Bernstein algebra.*

Proof. We have $\ker\omega = Ke_2$ and $e_2^2 = \alpha e_2$ with $\alpha \in K$. The relation $0 = (e_2^3)^2 = \alpha^5 e_2$ implies $\alpha = 0$, so $e_2^2 = 0$. It follows that $(\ker\omega)^2 = 0$ and A is a Bernstein algebra. □

3.3. Relation with power associative algebras.

Definition 3.14. Let A be a K-algebra.

(1) We say that the algebra A is *power associative* if any subalgebra generated by an element is associative, i.e for any $x \in A$, $x^i x^j = x^{i+j}$ for all integers $i, j > 1$.
(2) The algebra A is said to be *Jordan* algebra if, it is commutative and $x^2(yx) = (x^2y)x$ for all $x, y \in A$.
(3) The annihilator of A is defined by $ann(A) = \{x \in A \mid xA = 0\}$.

Definition 3.15. A baric K-algebra (A, ω) is said to be a train algebra of rank $r \geq 2$ if there exist scalars $\gamma_i \in K$ such that

$$x^r + \gamma_1\omega(x)x^{r-1} + \cdots + \gamma_{r-1}\omega(x)^{r-1}x = 0$$

for any $x \in A$ and r is the smallest such integer.

Lemma 3.16 ([12, Proposition 3.18])**.** *Let $A = Ke_1 \oplus \ker\omega$ be an evolution train algebra over a field K of characteristic different from 2, 3, 5 where $e_1^2 = e_1 + z$ with $z \in \ker\omega$. Then A is power associative if and only if $\ker\omega$ is power associative and $z \in ann(\ker\omega)$.*

Lemma 3.17 ([12, Remark 3.20])**.** *Let K be a commutative field with characteristic different from 2 and $(A,\ \omega)$ a finite dimensional power associative evolution train algebra. Then the Peirce decomposition of A is given by $A = Ke_1^2 \oplus A_0$, where e_1^2 is an idempotent.*

Lemma 3.18 ([2, Théorème 4.2])**.** *Let $A = Ke \oplus A_{1/2} \oplus A_0 \oplus A_{-1/2}$ be a Peirce decomposition of an algebra verifying the identity (2.1), where e is an idempotent. The following assertions are equivalent:*

(1) *A is power associative algebra;*
(2) *A is power associative train algebra of rank at most 4, satisfying the equation $x^4 - \omega(x)x^3 = 0$;*
(3) *A is Jordan algebra;*
(4) *$A_{-1/2} = 0$, $A_0^2 \subset A_0$, $2x_0(x_0x_{1/2}) = x_0^2 x_{1/2}$ and $x_{1/2}(x_{1/2}^2 x_0) = x_0(x_0^2 x_{1/2}) = 0$, for all $x_{1/2} \in A_{1/2}$ and $x_0 \in A_0$.*

If one of these assertions is satisfied, then A is a Bernstein algebra of order 2.

We then deduce from Lemma 3.17 and from Lemma 3.18 the following proposition:

Proposition 3.19. *Let $A = Ke_1^2 \oplus A_{1/2} \oplus A_0$ be a Peirce decomposition of a finite dimensional baric evolution algebra satisfying the identity* (2.1)*. The algebra A is power associative if and only if $A_{1/2} = 0$. Moreover, if A is power associative, then* $(\ker \omega)^3 = 0$.

Remark 3.20. The classification in dimension ≤ 4 of power associative evolution algebras [13, Proposition 4], Lemma 3.16 and Lemma 3.17 imply that in dimension 3 and in dimension 4, there does not exist a baric power associative evolution algebra (A, ω) such that $\dim(\ker \omega)^2 > 1$.

4. Classification

We start by stating some general results that will be useful for the classification.

Lemma 4.1. *Let $A = Ke_1^2 \oplus A_{1/2} \oplus A_0$ be the Peirce decomposition of a K-baric evolution algebra of finite dimension $n > 1$ verifying identity* (2.1)*. Then, the family $\{e_j^2 \mid 2 \leq j \leq n\}$ is linearly dependent.*

Proof. Suppose the family $\{e_j^2 \mid 2 \leq j \leq n\}$ is free. Since $(ker\omega)^2 \subset ker\omega$, then $(ker\omega)^2 = ker\omega$. Thus $(ker\omega)^3 = (ker\omega)^2 = ker\omega$ and $[(ker\omega)^3]^2 = (ker\omega)^2 = ker\omega$. The relations $(ker\omega)^3 \subset A_{1/2}$ and $A_{1/2}^2 = 0$ lead to $[(ker\omega)^3]^2 = 0$ and we deduce that $\ker \omega = 0$, which is absurd. So the family $\{e_j^2 \mid 2 \leq j \leq n\}$ is linearly dependent. □

Theorem 4.2. *Let $A = Ke_1^2 \oplus A_{1/2} \oplus A_0$ be a Peirce decomposition of a n-dimensional ($n > 2$) K-baric evolution algebra verifying identity* (2.1)*. If the algebra A is not power associative, then $1 \leq \dim(A_{1/2}) \leq \frac{n-1}{2}$ and $\dim(A_0) \geq \dim(A_{1/2})$.*

Proof. We assume that A is not power associative. We have $A_{1/2} \neq 0$ and we set $\dim(A_{1/2}) = p \geq 1$. Without losing generality, we can take $A_{1/2} = \langle e_2^2, e_3^2, \cdots, e_{p+1}^2 \rangle$. As $A_{1/2}^2 = 0$, [4, Proof of Theorem 4.10], tells us that the family $(e_2, e_3, \cdots, e_{p+1}, e_2^2, e_3^2, \cdots, e_{p+1}^2)$ is free. We deduce that $1 \leq \dim(A_{1/2}) \leq \frac{n-1}{2}$ and [4, proof of corollary 4.11] shows that $\dim(A_0) \geq \dim(A_{1/2})$. □

Definition 4.3. We say that a baric algebra (A, ω) strictly verifies the identity (2.1), when it verifies this identity and does not verify an identity of degree strictly less than 6.

In [4], the authors give a classification up to isomorphism and in dimension ≤ 4 of evolution algebras which are Bernstein. Since a Bernstein algebra satisfies the identity (2.1), we give here, up to isomorphism and in dimension ≤ 4, the classification of evolution algebras strictly verifying the identity (2.1).

Proposition 4.4 (3-dimensional classification)**.** *There is no baric evolution algebra of dimension 3 strictly satisfying the identity (2.1).*

Proof. We suppose that $A = Ke_1^2 \oplus A_{1/2} \oplus A_0$ is the Peirce decomposition of a K-algebra of baric evolution of dimension 3 verifying the identity (2.1) and which is not power associative. The relation $1 \leq \dim(A_{1/2}) \leq \frac{3-1}{2}$ implies that $\dim(A_{1/2}) = 1$. Without losing generalities, we suppose $A_{1/2} = Ke_2^2$. Then $a_{12} \neq 0$

and $e_2^2 \neq 0$. The equality $e_1^2 = e_1^2 e_1^2 = e_1^2 + a_{12}^2 e_2^2 + a_{13}^2 e_3^2$ leads to $a_{12}^2 e_2^2 + a_{13}^2 e_3^2 = 0$. This implies that $a_{13} \neq 0$ and $Card(Support(e_1^2)) = 3$. We deduce from the corollary 3.11 that A is a Bernstein algebra. □

Proposition 4.5 (Classification in dimension 4)**.** *Let $A = Ke \oplus A_{1/2} \oplus A_0$ the Peirce decomposition of a baric evolution K-algebra of dimension 4, verifying strictly the identity (2.1) and let $B = \{e, u, v_1, v_2\}$ a canonical basis of A. Then A is isomorphic to one and only one of the following algebras:*

(1) $A_1(\alpha)\ :\ e^2 = e,\ eu = \frac{1}{2}u,\ uv_1 = u,\ v_2^2 = u + \alpha v_1$ *with* $\alpha \in K^*$*;*
(2) $A_2(\alpha)\ :\ e^2 = e,\ eu = \frac{1}{2}u,\ uv_1 = u,\ v_2^2 = \alpha v_1$ *with* $\alpha \in K^*$*.*

Other products being zero. Moreover, for $i \in \{1,\ 2\}$, the algebra $A_i(\alpha) \simeq A_i(\alpha')$ if and only if $\alpha = k^2\alpha'$ with $k \in K^*$.

Proof. The algebra A is not power associative. Then, $1 \leq \dim(A_{1/2}) \leq \frac{4-1}{2}$ yields $\dim(A_{1/2}) = 1$. Without losing generality, suppose that $A_{1/2} = Ke_2^2$. Then $a_{12}e_2^2 \neq 0$ and since A is not Bernstein, we have $a_{13}a_{14} = 0$. We distinguish three cases:

(1) $a_{13} \neq 0$ and $a_{14} = 0$.
  (a) The relation $e_1^2 = e_1^2 e_1^2 = e_1^2 + a_{12}^2 e_2^2 + a_{13}^2 e_3^2$ leads to $a_{12}^2 e_2^2 + a_{13}^2 e_3^2 = 0$; so $e_3^2 = -(a_{12}a_{13}^{-1})^2 e_2^2$. Since A is not a Bernstein algebra then $\dim(\ker\omega)^2 \neq 1$. We deduce that $\dim(\ker\omega)^2 = 2$ and $(\ker\omega)^2 = Ke_2^2 \oplus Ke_4^2$.
  (b) The relation $\frac{1}{2}e_2^2 = e_1^2 e_2^2 = a_{12}a_{22}e_2^2 + a_{13}a_{23}e_3^2 = a_{12}(a_{22} - a_{12}a_{13}^{-1}a_{23})e_2^2$ gives

$$\frac{1}{2} = a_{12}(a_{22} - a_{12}a_{13}^{-1}a_{23}). \tag{4.1}$$

  (c) The equality $0 = e_2^2 e_2^2 = a_{22}^2 e_2^2 + a_{23}^2 e_3^2 + a_{24}^2 e_4^2 = (a_{22} - a_{12}a_{13}^{-1}a_{23})(a_{22} + a_{12}a_{13}^{-1}a_{23})e_2^2 + a_{24}^2 e_4^2 = \frac{1}{2}a_{12}^{-1}(a_{22} + a_{12}a_{13}^{-1}a_{23})e_2^2 + a_{24}^2 e_4^2$ gives

$$a_{22} = -a_{12}a_{13}^{-1}a_{23} \text{ and } a_{24} = 0 \tag{4.2}$$

  The identities (4.1) and (4.2) gives that $a_{23} = -\frac{1}{4}a_{12}^{-2}a_{13}$ and $a_{22} = \frac{1}{4}a_{12}^{-1}$.
  (d) Since $e_4^2 e_4 = a_{44}e_4^2 \in A_{1/2}$ and the family $\{e_2^2,\ e_4^2\}$ is free, then $a_{44} = 0$.

  Let $u = a_{12}e_2$ and $v = a_{13}e_3$. The family $B_1 = \{e_1, u, v, e_4\}$ is a natural basis of the algebra A and the multiplication table of A in the basis B_1 is given by: $e_1^2 = e_1 + u + v$; $u^2 = \frac{1}{4}(u - v)$; $v^2 = -u^2$ and $e_4^2 = \alpha u + \beta v$ with $\alpha + \beta \neq 0$. We have: $A_{1/2} = Ku^2$, $A_0 = K(u + v) \oplus Ke_4$. The multiplication table of A in the base $B_2 = \{e_1^2, u^2, u + v, e_4\}$ is: $(e_1^2)^2 = e_1^2$, $u^2(u + v) = \frac{1}{2}u^2$, $e_1^2 u^2 = \frac{1}{2}u^2$, $e_4^2 = 2(\alpha - \beta)u^2 + \frac{1}{2}(\alpha + \beta)(u + v)$ and other unmentioned products are zero. We distinguish the following two cases:

  - $\alpha - \beta \neq 0$. Let $u' = 2(\alpha - \beta)u^2$ and $v' = 2(u + v)$. The multiplication table of A in the basis $B_3 = \{e_1^2, u', v', e_4\}$ is given by: $(e_1^2)^2 = e_1^2$, $e_1^2 u' = \frac{1}{2}u'$, $u'v' = u'$, $e_4^2 = u' + \alpha' v'$ with $\alpha' = \frac{1}{4}(\alpha + \beta) \neq 0$ and the other products not mentioned are zero. Then the algebra A is isomorphic to $A_1(\alpha')$. The algebra $A_1(\alpha) \simeq A_1(\alpha')$ if and only if $\alpha = k^2\alpha'$ with $k \in K^*$ [6, Theorem 2.5.5].

- $\alpha = \beta$, we get that the algebra A is isomorphic to $A_2(\alpha)$. The algebra $A_2(\alpha) \simeq A_2(\alpha')$ if and only if $\alpha = k^2\alpha'$ with $k \in K^*$ [6, Theorem 2.5.5].

(2) $a_{13} = 0$ and $a_{14} \neq 0$. By setting $e'_3 = e_4$ and $e'_4 = e_3$, we find the previous case. A is isomorphic to the algebra $A_1(\alpha)$ or the algebra $A_2(\alpha)$.

(3) $a_{13} = a_{14} = 0$. The relation $e_1^2 = (e_1^2)^2 = e_1^2 + a_{12}^2 e_2^2$ implies $a_{12}^2 e_2^2 = 0$. So $a_{12} = 0$ or $e_2^2 = 0$. Which is absurd because $a_{12} \neq 0$ and $e_2^2 \neq 0$.

□

5. Derivations and automorphisms

In this section we study derivations and automorphisms of a finite dimensional evolution algebra satisfying the identity (2.1).

5.1. **Derivations.** Let K be a commutative field and A a commutative, non-associative K-algebra of finite dimension over K. We will say that an endomorphism d of A is a derivation of A if for all $x, y \in A$, $d(xy) = d(x)y + xd(y)$. The set $Der_K(A)$ of all derivations of A is a Lie algebra for the Lie bracket $[d,\ d'] = dod' - d'od$ where $d, d' \in Der_K(A)$.

Remark 5.1. Let (A, ω) be a baric evolution algebra satisfying an identity (2.1). Then $\omega \circ d = 0$.

Lemma 5.2. *Let A be a finite dimensional evolution algebra satisfying the identity (2.1) and d a derivation of A. Then $d(e_1) \in ann(A)$ where $ann(A) = \{e_j \in B \mid e_j^2 = 0\}$.*

Proof. Let $d(e_i) = \sum_{k=1}^{n} d_{ik} e_k$ with $i \in \{1, \cdots, n\}$. We have

$$0 = d(e_1 e_j) = e_1 d(e_j) + d(e_1) e_j \text{ with } j > 1 \tag{5.1}$$

Since $d(e_j) \in \ker \omega$, then the relation (5.1) leads to $0 = d(e_1)e_j = d_{1j} e_j^2$. For $e_j^2 \neq 0$ we will have $d_{1j} = 0$. Hence $d(e_1) = \sum_{j \in \Lambda} d_{1j} e_j$ with $\Lambda = \{j \mid e_j^2 = 0\}$. So $d(e_1) \in ann(A)$. □

Proposition 5.3. *Let A be a finite dimensional evolution algebra satisfying the identity (2.1). The map $d \in End(A)$ is a derivation of A if and only if the following assertions hold:*

(1) $d(e_1^2) = 0$;
(2) *the derivation d stabilizes $A_{1/2}$ and A_0;*
We note respectively $d_{1/2}$ and d_0 the endomorphisms induced by d on $A_{1/2}$ and A_0;
(3) $d_{1/2}(x_{1/2} x_0) = d_{1/2}(x_{1/2}) x_0 + x_{1/2} d_0(x_0)$;
(4) $d_{1/2}((x_0 y_0)_{1/2}) = (x_0 d_0(y_0) + d_0(x_0) y_0)_{1/2}$;
(5) $d_0((x_0 y_0)_0) = (x_0 d_0(y_0) + d_0(x_0) y_0)_0$.

Proof. For the assertion (1), we have $d(e_1^2) = 2e_1 d(e_1) = 0$ because $d(e_1) \in ann(A)$. The assertions (2) to (5) are obtained by setting $e_1^2 = e,\ f_d = d_{1/2},\ g_d = d_0\ h_d = 0$ in the [3, Théorème 5.1.1] and taking into account $d(e_1^2) = 0$. For the reciprocal we assume that d is an endomorphism of A satisfying the assertions (1) to (5). Let us show that d is a derivation. Let $x = \alpha e_1^2 + x_{1/2} + x_0$ and $y = \beta e_1^2 + y_{1/2} + y_0$ with $\alpha, \beta \in K$. We have $xy = \alpha\beta e_1^2 + \frac{1}{2}\alpha y_{1/2} + \frac{1}{2}\beta x_{1/2} + x_{1/2}y_0 + x_0 y_{1/2} + x_0 y_0$. So $d(xy) = \frac{1}{2}\alpha d(y_{1/2}) + \frac{1}{2}\beta d(x_{1/2}) + d(x_{1/2}y_0) + d(x_0 y_{1/2}) + d(x_0 y_0) = \frac{1}{2}(\alpha d_{1/2}(y_{1/2}) + \beta d_{1/2}(x_{1/2})) + d_{1/2}(x_{1/2})y_0 + x_{1/2}d_0(y_0) + d_0(x_0)y_{1/2} + x_0 d_{1/2}(y_{1/2}) + x_0 d_0(y_0) + d_0(x_0)y_0$ and $xd(y) + d(x)y = \frac{1}{2}\alpha d_{1/2}(y_{1/2}) + x_{1/2}d_0(y_0) + x_0 d_{1/2}(y_{1/2}) + x_0 d_0(y_0) + \frac{1}{2}\beta d_{1/2}(x_{1/2}) + d_{1/2}(x_{1/2})y_0 + d_0(x_0)y_{1/2} + d_0(x_0)y_0 = d(xy)$. So d is a derivation of A. □

Corollary 5.4. *Let A be a finite dimensional evolution algebra verifying the identity (2.1) and d a derivation of A. Then the induced endomorphism $d_{1/2}$ is a derivation of $A_{1/2}$.*

Proof. Let $x, y \in A_{1/2}$ and d be a derivation of A. Then $d(xy) = d(x)y + xd(y) = d_{1/2}(x)y + xd_{1/2}(y)$. Since $xy = 0$ then $d(xy) = d_{1/2}(xy)$, therefore $d_{1/2}(xy) = d_{1/2}(x)y + xd_{1/2}(y)$ and $d_{1/2}$ is therefore a derivation of $A_{1/2}$. □

Proposition 5.5. *Let A be an evolution algebra of dimension 4 satisfying strictly the identity (2.1). Then the Lie algebra of derivations of A is abelian algebra.*

Proof. In dimension 4, an evolution algebra A satisfying the identity (2.1) is isomorphic to one of the algebras $A_1(\alpha)$ or $A_2(\alpha)$ of Proposition 4.5. Let be $d \in Der_K(A)$.

(1) For $A \simeq A_1(\alpha)$: $e^2 = e,\ eu = \frac{1}{2}u,\ uv_1 = u,\ v_2^2 = u + \alpha v_1$ with $\alpha \in K^*$. We have $d(e) = 0,\ d_{1/2}(u) = \beta u$; $d_0(v_1) = \gamma_1 v_1 + \gamma_2 v_2$ and $d_0(v_2) = \lambda_1 v_1 + \lambda_2 v_2$ with $\beta, \gamma_1, \gamma_2, \lambda_1, \lambda_2$ in K.
   (a) The relation $d_{1/2}(u) = d_{1/2}(uv_1) = d_{1/2}(u)v_1 + ud_0(v_1)$ implies $\beta u = \beta u + \gamma_1 u$. Thus $\gamma_1 = 0$ and $d_0(v_1) = \gamma_2 v_2$.
   (b) The equality $0 = d_{1/2}(uv_2) = d_{1/2}(u)v_2 + ud_0(v_2)$ implies $0 = \lambda_1 u$. So $\lambda_1 = 0$ and $d(v_2) = \lambda_2 v_2$.
   (c) The equality $d_{1/2}(u) = d_{1/2}((v_2^2)_{1/2}) = (2v_2 d_0(v_2))_{1/2} = (2\lambda_2 v_2^2)_{1/2} = 2\lambda_2 u$ implies $\beta = 2\lambda_2$.
   (d) The relation $\alpha d_0(v_1) = d_0((v_2^2)_0) = (2v_2 d_0(v_2))_0 = 2\lambda_2\alpha v_1$ implies $\alpha\gamma_2 v_2 = 2\lambda_2\alpha v_1$. Since $\alpha \neq 0$ then $\gamma_2 = \lambda_2 = 0$. So $\beta = 0$. Therefore $d = 0$ and $Der_K(A_1(\alpha))$ is an abelian algebra.

(2) For $A \simeq A_2(\alpha)$: $e^2 = e,\ eu = \frac{1}{2}u,\ uv_1 = u,\ v_2^2 = \alpha v_1$ with $\alpha \in K^*$. We got $d(e) = 0,\ d_{1/2}(u) = \beta u$; $d_0(v_1) = \gamma_1 v_1 + \gamma_2 v_2$ and $d_0(v_2) = \lambda_1 v_1 + \lambda_2 v_2$ with $\beta, \gamma_1, \gamma_2, \lambda_1, \lambda_2$ in K. Since $uv_1 = u$ and $uv_2 = 0$, point 1.) tells us that $d_0(v_1) = \gamma_2 v_2$ and $d_0(v_2) = \lambda_2 v_2$. The relation $\alpha d_0(v_1) = d_0((v_2^2)_0) = (2d_0(v_2)v_2)_0 = (2\lambda_2 v_2^2)_0 = 2\alpha\lambda_2 v_1$ implies $\alpha\gamma_2 v_2 = 2\alpha\lambda_2 v_1$. So $\gamma_2 = \lambda_2 = 0$. So we get $d(\gamma e + u + v_1 + v_2) = d_{1/2}(u) = \beta u$ with $\gamma \in K$.
   Let d and d' two derivations of $A_2(\alpha)$. We have $[d, d'](e) = [d, d'](v_1) = [d, d'](v_2) = 0$ and $[d, d'](u) = d \circ d'(u) - d' \circ d(u) = \beta'\beta u - \beta\beta' u = 0$. So $Der_K(A_2(\alpha))$ is the abelian algebra.

□

5.2. **Automorphisms.** In this subsection, $A = Ke_1^2 \oplus A_{1/2} \oplus A_0$ is a Peirce decomposition of an evolution algebra satisfying the identity (2.1), where e_1^2 is a nonzero idempotent. Since $A_{-1/2} = 0$ and $A_{1/2}^2 = 0$, [3, Théorème 5.3.1] gives us a characterization of the automorphisms of A.

Theorem 5.6. *A bijective map σ is an automorphism of the algebra A if and only if the following conditions hold:*

(1) $\sigma(e_1^2) = e_1^2 + u_\sigma$, $u_\sigma \in A_{1/2}$;
(2) $\sigma(x_{1/2}) = f_\sigma(x_{1/2})$, *for any* $x_{1/2} \in A_{1/2}$ *with* $f_\sigma \in GL_K(A_{1/2})$;
(3) $\sigma(x_0) = g_\sigma(x_0) - 2u_\sigma g_\sigma(x_0)$, *for any* $x_0 \in A_0$ *with* $g_\sigma \in GL_K(A_0)$;
(4) $f_\sigma(x_{1/2}x_0) = f_\sigma(x_{1/2})g_\sigma(x_0)$;
(5) $g_\sigma((x_0y_0)_0) = (g_\sigma(x_0)g_\sigma(y_0))_0$;
(6) $f_\sigma((x_0y_0)_{1/2}) - 2u_\sigma g_\sigma((x_0y_0)_0) = (g_\sigma(x_0)g_\sigma(y_0))_{1/2} - 2g_\sigma(x_0)(u_\sigma g_\sigma(y_0)) - 2g_\sigma(y_0)(u_\sigma g_\sigma(x_0))$.

Proposition 5.7. *If the algebra A has dimension 4, then the group of automorphisms of A denoted $Aut(A)$ is given by:*

$$Aut(A) = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix} \mid \gamma \in K^* \ and \ \epsilon = -1 \ or \ \epsilon = 1 \right\}.$$

Proof. In dimension 4, an evolution algebra A satisfying the identity (2.1) is isomorphic to one of the algebras $A_1(\alpha)$ or $A_2(\alpha)$ of the Proposition 4.5. Let $\sigma \in Aut(A)$.

(1) For $A \simeq A_1(\alpha)\ :\ e^2 = e,\ eu = \frac{1}{2}u,\ uv_1 = u,\ v_2^2 = u + \alpha v_1$ with $\alpha \in K^*$. We have $\sigma(e) = e+\beta u,\ \sigma(u) = \gamma u = f_\sigma(uu)$; $\sigma(v_1) = \lambda_1 v_1+\lambda_2 v_2-2\beta\lambda_1 u$; $\sigma(v_2) = \gamma_1 v_1+\gamma_2 v_2-2\beta\gamma_1 u$ with $g_\sigma(v_1) = \lambda_1 v_1+\lambda_2 v_2,\ g_\sigma(v_2) = \gamma_1 v_1+\gamma_2 v_2$ and $\det(\sigma) = \gamma(\lambda_1\gamma_2 - \lambda_2\gamma_1) \neq 0$ where $\beta, \gamma, \gamma_1, \gamma_2, \lambda_1, \lambda_2 \in K$.
  (a) The equality $f_\sigma(u) = f_\sigma(uv_1) = f_\sigma(u)g_\sigma(v_1)$ implies $\gamma u = \gamma\lambda_1 u$. Which implies $\lambda_1 = 1$ and $g_\sigma(v_1) = v_1 + \lambda_2 v_2$.
  (b) The relation $0 = f_\sigma(uv_2) = f_\sigma(u)g_\sigma(v_2) = \gamma\gamma_1 u$ leads to $\gamma_1 = 0$ and $g_\sigma(v_2) = \gamma_2 v_2$, so $\det(\sigma) = \gamma\gamma_2 \neq 0$.
  (c) The equality $\alpha g_\sigma(v_1) = g_\sigma((v_2^2)_0) = ((g_\sigma(v_2))^2)_0 = (\gamma_2^2 v_2^2)_0 = \alpha\gamma_2^2 v_1$ causes $\alpha v_1 + \alpha\lambda_2 v_2 = \alpha\gamma_2^2 v_1$. Which implies $\gamma_2^2 = 1$ and $\lambda_2 = 0$. Thus $g_\sigma(v_1) = v_1$ and $g_\sigma(v_2) = \epsilon v_2$ with $\epsilon = 1$ or $\epsilon = -1$.
  (d) The relation $f_\sigma(u) - 2\beta\alpha u g_\sigma(v_1) = f_\sigma((v_2^2)_{1/2}) - 2\beta u g_\sigma((v_2^2)_0) = ((g_\sigma(v_2))^2)_{1/2} - 4g_\sigma(v_2)(\beta u g_\sigma(v_2))$ leads to $\gamma u - 2\beta\alpha u = u$. So $\gamma = 1 + 2\beta\alpha$.
  (e) The equality $0 = -4g_\sigma(v_1)(\beta u g_\sigma(v_1)) = -4v_1(\beta u) = -4\beta u$ implies $\beta = 0$. We finally get $\sigma(e) = e$; $\sigma(u) = u$; $\sigma(v_1) = v_1$; $\sigma(v_2) = \epsilon v_2$ with $\epsilon = 1$ or $\epsilon = -1$. The set of automorphisms of $A_1(\alpha)$ is:

$$Aut(A_1(\alpha)) = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix} \mid \epsilon = 1 \text{ or } \epsilon = -1 \right\}.$$

(2) For $A \simeq A_2(\alpha)$: $e^2 = e,\ eu = \frac{1}{2}u,\ uv_1 = u,\ v_2^2 = \alpha v_1$ with $\alpha \in K^*$. We get $\sigma(e) = e + \beta u,\ \sigma(u) = \gamma u = f_\sigma(u),\ \sigma(v_1) = \lambda_1 v_1 + \lambda_2 v_2 - 2\beta\lambda_1 u$ and $\sigma(v_2) = \gamma_1 v_1 + \gamma_2 v_2 - 2\beta\gamma_1 u$ with $g_\sigma(v_1) = \lambda_1 v_1 + \lambda_2 v_2,\ g_\sigma(v_2) = \gamma_1 v_1 + \gamma_2 v_2$ and $det(\sigma) = \gamma(\lambda_1\gamma_2 - \lambda_2\gamma_1) \neq 0$ où $\beta, \gamma, \gamma_1, \gamma_2, \lambda_1, \lambda_2 \in K$. Since $uv_1 = u$, $uv_2 = 0$ and $(v_2^2)_0 = v_1$, 1.) tells us that $g_\sigma(v_1) = v_1 + \lambda_2 v_2$ and $g_\sigma(v_2) = \epsilon v_2$ with $\epsilon = 1$ or $\epsilon = -1$. The relation $-2\beta u g_\sigma(\alpha v_1) = f_\sigma((v_2^2)_{1/2}) - 2\beta u g_\sigma((v_2^2)_0) = ((g_\sigma(v_2))^2)_{1/2} - 4g_\sigma(v_2)(\beta u g_\sigma(v_2))$ implies $-2\beta u = (\alpha v_1)_{1/2}$. Thus $-2\beta u = 0$ and $\beta = 0$. We finally get $\sigma(e) = e,\ \sigma(u) = \gamma u,\ \sigma(v_1) = v_1$ and $\sigma(v_2) = \epsilon v_2$ with $\epsilon = 1$ or $\epsilon = -1$. The set of automorphisms of $A_2(\alpha)$ is therefore

$$Aut(A_2(\alpha)) = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix} \mid \gamma \in K^* \text{ and } \epsilon = -1 \text{ or } \epsilon = 1 \right\}.$$

Hence the result. □

Acknowledgement. The authors thank the referee whose suggestions have helped to improve the paper.

References

1. Albert, A.A . (1950). A theory of power-associative commutative algebras. Transactions of the American Mathematical Society, 69, 503-527. https://doi.org/10.1090/S0002-9947-1950-0038959-X
2. Bayara, J. Conseibo, A. and Ouattara, M. and Micali, A. (2011). Train algebras of degree 2 and exponent 3. Discrete and Continuous Dynamical Systems. Series S, 4(6), 1371-1386. https://doi.org/10.3934/dcdss.2011.4.1371
3. Conseibo, A. Algèbres Train de dégré 2 et d'exposent 3, Thèse de doctorat unique, Thèse de doctorat unique de l'Université de Ouagadougou, PP. 93,2013
4. Conseibo, A. Savadogo, S. and Ouattara, M. (2021). Duplicate, Bernstein algebras and evolution algebras. Le Matematiche, 76(1), 193-209. https://doi.org/10.4418/2021.76.1.11
5. Dembega, A. Konkobo, A. and Ouattara, M. (2019). Derivations and dimensionally nilpotent derivations in Lie triple algebras. Gulf Journal pf mathematics, 7(2). https://doi.org/10.56947/gjom.v7i2.192
6. González, S. and Gutiérrez, J. C. and Martínez, C. (1996). Second order Bernstein algebras of dimension 4. Linear Algebra and its Applications, 233, 243-273. https://doi.org/10.1016/0024-3795(94)00085-9
7. González, S. and Martínez, C. (1990). Idempotent elements in a Bernstein algebra. Journal of the London Mathematical Society. Second Series,42(3), 430-436. https://doi.org/10.1112/jlms/s2-42.3.430
8. González, S. Martínez, C. and Vicente, P. (1994). Special classes of second-order Bernstein algebra. Journal of Algebra,167(3), 855-868. https://doi.org/10.1006/jabr.1994.1215
9. P Holgate. Genetic algebras satisfying bernstein's stationarity principle. J. Lond. Math. Soc., II. Ser, 9(1),51–68, 1975. https://doi.org/10.1112/jlms/s2-9.4.613
10. Mallol, C. Micali, A. and Ouattara, M. (1991). Sur les algèbres de Bernstein. IV. Linear Algebra and its Applications,158, 1-26. https://doi.org/10.1016/0024-3795(91)90048-2
11. Ouattara, M. and Savadogo, S (2019). Derivations in power-associative evolution algebras. JP Journal of Algebra, Number Theory and Applications,43(1), 23-52. http://dx.doi.org/10.1007/978-3-030-35256-1_2

12. Ouattara, M. and Savadogo, S (2020). Evolution train algebras. Gulf Journal of Mathematics,8(1), 37-51. http://dx.doi.org/10.56947/gjom.v8i1.299
13. Ouattara, M. and Savadogo, S (2020). Power-associative evolution algebras. Associative and non-associative algebras and applications. Proceedings of the 3rd Moroccan Andalusian meeting on algebras and their applications, MAMAA 2018, Chefchaouen, Morocco, April, Cham: Springer, 23-49. http://dx.doi.org/1007/978-3-030-35256-1_2
14. Tian, J. P. and Vojtěchovský, P (2006). Mathematical concepts of evolution algebras in non-Mendelian genetics. Quasigroups and Related Systems,14(1), 111-122.
15. W.A Zangré, W.A. and Conseibo, A (2022). On train algebras of degree 2 and exponent 4. Gulf Journal of Mathematics, 13((1)), 41–53. https://doi.org/10.56947/gjom.v13i1.926

[1] Départment de Mathematiques, Université Norbert ZONGO, Burkina Faso
Email address: josephtenkodogo603@gmail.com

[2] Départment de Mathematiques, Université Norbert ZONGO, Burkina Faso
Email address: sara01souley@yahoo.fr

[3] Départment de Mathematiques, Université Norbert ZONGO, Burkina Faso
Email address: andreconsebo@yahoo.fr

[4] Départment de Mathematiques, Université Norbert ZONGO, Burkina Faso
Email address: doulaydem@yahoo.fr