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ON IDEAL CONVERGENCE OF DOUBLE SEQUENCES IN NEUTROSOPHIC NORMED SPACES

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Abstract. In this article, we present the concept of $\mathcal{I}_2$ and $\mathcal{I}_2^*-$convergence for double sequences in neutrosophic normed spaces. This concept serves as an extension of the idea of statistical convergence for double sequences in neutrosophic normed spaces. We explore several essential characteristics and analyze the connection between these two concepts. We also introduce and investigate the concept of $\mathcal{I}_2$ and $\mathcal{I}_2^*-$Cauchy sequence of double sequences and show that the condition $(AP2)$ plays a crucial role to study the interrelationship between them.

1. Introduction and Background

In 1951, the idea of statistical convergence emerged independently through the work of Fast [5] and Steinhaus [30]. This development aimed to provide deeper insights into summability theory. Subsequently, researchers, including Fridy [8, 9], Salat [27], and others like Nuray [24] and Savas and Gurdal [28], delved into this concept from the perspective of sequence spaces.

In 2003, Mursaleen and Edely [22] extended the concept of statistical convergence to double sequences and primarily examined its relationship with statistical Cauchy double sequences and strong Cesaro summable double sequences. Furthermore, Tripathy [31] explored various properties of sequence spaces formed by statistically convergent double sequences and established a decomposition theorem. Dundar and Cakan [4] developed the notion of rough convergence for double sequences.

Statistical convergence finds applications in numerous branches of mathematics, including number theory, mathematical analysis, probability theory, and more.

The concept of $\mathcal{I}$-convergence represents an extension of statistical convergence, originally introduced by Kostyrko et al. [18]. They employed the concept of an ideal $\mathcal{I}$ comprising subsets of the set $\mathbb{N}$ to formulate this notion. For a comprehensive understanding of this article, we recommend that readers refer to [3, 17, 33], where they can find additional references and explore this topic more extensively.

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In 2005, Tripathy and Tripathy [32] introduced the concept of $\mathcal{I}-$convergence and $\mathcal{I}-$Cauchy sequences for double sequences. They provided proofs for various properties, including solidity, symmetricity, completeness, and denseness within this framework. Additionally, Das [2] and Kumar [19] conducted research on the concepts of $\mathcal{I}$ and $\mathcal{I}^*-$convergence for double sequences, presenting results that offer a more natural perspective on these concepts.

The concept of fuzzy sets was initially introduced by Zadeh [34] in 1965, as an extension of the classical set-theoretical concept. Over time, it has found extensive applications in various fields of science and engineering. However, the theory of fuzzy sets sometimes struggles to handle situations with limited knowledge about membership degrees. In 1986, Atanassov [1] introduced intuitionistic fuzzy sets as an expansion of fuzzy sets, aiming to address these limitations. Intuitionistic fuzzy sets have since been widely employed to tackle a range of decision-making problems.

In decision-making, it's common for decision-makers to encounter uncertainty and hesitation, rather than simply choosing between "yes" or "no." Additionally, some real-world events yield outcomes with three components, such as sports results or voting processes. Recognizing these complexities, Smarandache [29] introduced the concept of Neutrosophic sets in 2005. Neutrosophic sets represent a generalization that encompasses both fuzzy sets and intuitionistic fuzzy sets.

A neutrosophic set consists of a triplet, which includes a truth-membership function (T), an indeterminacy-membership function (F), and a falsity-membership function (I). In essence, a neutrosophic set is defined as a set where every element of the universe has associated degrees of T, F, and I, reflecting the inherent uncertainty and indeterminacy in real-world situations.

The concept of fuzzy normed space was first introduced by Felbin [6] in 1992. Subsequently, in 2006, Saadati and Park [26] introduced the notion of intuitionistic fuzzy normed spaces. In 2008, the study of statistical convergence in intuitionistic fuzzy normed spaces was initiated and explored by Karakus et al. [10].

For a more comprehensive understanding of statistical convergence and its related generalizations within intuitionistic fuzzy normed spaces, interested readers can refer to references such as [14, 21, 23].

More recently, Kirisci and Simsek [15] introduced the concept of a neutrosophic normed linear space and investigated the idea of statistical convergence within this framework. Following their work, numerous researchers have delved into various notions of sequence convergence within neutrosophic normed spaces. Additional details on these developments can be found in references like [11, 12, 13].

In a recent development, Kisi [16] expanded the concept of statistical convergence to include $\mathcal{I}$ and $\mathcal{I}^*-$convergence of sequences within neutrosophic normed spaces.

In the present paper, our primary objective is to extend this concept further to $\mathcal{I}_2$ and $\mathcal{I}_2^*-$convergence for double sequences. This extension represents a broader generalization of statistical convergence for double sequences within the context of neutrosophic normed spaces, as initially developed by Granados and Dhital [7].

2. Definitions and Preliminaries

Throughout the paper, $\mathbb{N}$ and $\mathbb{R}$ denotes the set of all positive integers and the set of all real numbers respectively and by the convergence of a double sequence we mean the convergence in Pringsheim's [25] sense.

Definition 2.1. [25] A double sequence (x_{ij}) is said to be convergent to l if, for any $\varrho > 0$, there exists a positive integer $k_0 = k_0(\varrho)$ such that for all $i, j \geq k_0$,

$$|x_{ij} - l| < \varrho.$$

Definition 2.2. [22] Assume that $S \subseteq \mathbb{N}\times\mathbb{N}$ and the set $\{(i, j) \in S : i \leq m, j \leq n\}$ is indicated by $S_{m,n}$. If the following limit

$$\delta^2(S) = \lim_{m,n\to+\infty} \frac{|S_{m,n}|}{mn}.$$

exists then $\delta^2(S)$ is called the double natural density of S. Here, $|S_{m,n}|$ indicates the number of elements of the set $S_{m,n}$.

Definition 2.3. [22] Let (x_{ij}) be a double sequence of real numbers. If for each $\varrho > 0$,

$$\delta^2(\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - l| \geq \varrho\}) = 0$$

holds, then (x_{ij}) is said to be statistical convergent to l. In this context, we refer to l as the statistical limit of the double sequence (x_{ij}) and it is represented as $x_{ij} \xrightarrow{st} l$.

Definition 2.4. [31] Let (x_{ij}) be a double sequence of real numbers. If for each $\varrho > 0$, there exists two natural numbers $M = M(\varrho)$ and $N = N(\varrho)$ so that

$$\delta^2(\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - x_{MN}| \geq \varrho\}) = 0$$

holds, then (x_{ij}) is said to be statistical Cauchy.

Definition 2.5. [18] Let X is a non-empty set. A family of subsets $\mathcal{I} \subset P(X)$ is called an ideal on X if

(i) for each $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$;

(ii) for each $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$.

$\mathcal{I}$ is called non-trivial if $X \notin \mathcal{I}$.

Definition 2.6. [18] Let X be a non-empty set. A family of subsets $\mathcal{F} \subset P(X)$ is called a filter on X if

(i) for each $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$;

(ii) for each $A \in \mathcal{F}$ and $B \supset A$ implies $B \in \mathcal{F}$.

If $\mathcal{I}$ is a non-trivial ideal in X with $X \neq \emptyset$, then the class

$$\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{X - A : A \in \mathcal{I}\}$$

forms a filter on X, known as the filter associated with $\mathcal{I}$.

A non-trivial ideal $\mathcal{I} \subset P(X)$ is called an admissible ideal in X if and only if $\mathcal{I} \supset \{\{x\} : x \in X\}$.

A non-trivial ideal $\mathcal{I}$ in $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if for any $k \in \mathbb{N}$, $\mathbb{N} \times \{k\}$ and $\{k\} \times \mathbb{N}$ both belong to $\mathcal{I}$.

Throughout the paper, we consider $\mathcal{I}_2$ as a non-trivial admissible ideal in $\mathbb{N}\times\mathbb{N}$.

Definition 2.7. [32] Let (x_{ij}) be a double sequence of real numbers. If for each $\varrho > 0$,

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - l| \geq \varrho\} \in \mathcal{I}_2$$

holds then (x_{ij}) is said to be $\mathcal{I}_2-$convergent to l. In this scenario, l is termed the $\mathcal{I}_2-$limit of (x_{ij}) and it is represented as $x_{ij} \to l(\mathcal{I}_2)$.

Definition 2.8. [32] Let (x_{ij}) be a double sequence of real numbers. If for each $\varrho > 0$, there exists two natural numbers $M = M(\varrho)$ and $N = N(\varrho)$ so that

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - x_{MN}| \geq \varrho\} \in \mathcal{I}_2$$

holds, then (x_{ij}) is said to be $\mathcal{I}_2-$Cauchy.

Definition 2.9. [2] An admissible ideal $\mathcal{I}_2 \subset P(\mathbb{N} \times \mathbb{N})$ is said to satisfy the condition $(AP2)$ if for every countable family of mutually disjoint sets $\{D_1, D_2, ...\}$ belonging to $\mathcal{I}_2$, there exists a countable family of sets $\{E_1, E_2, ...\}$ such that for all $m \in \mathbb{N}$, the symmetric differences $D_m \triangle E_m$ is included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ (i.e., $D_m \triangle E_m \subseteq \{A \subseteq \mathbb{N} \times \mathbb{N} : (\text{there exists } k_0 \in \mathbb{N})(i,j \geq k_0 \Rightarrow (i,j) \notin A)\}$) and $E = \bigcup_{m=1}^{+\infty} E_m \in \mathcal{I}_2$.

Definition 2.10. [20] A binary operation denoted as $\odot : [0,1] \times [0,1] \to [0,1]$, is considered a continuous t-norm when it fulfills the following conditions:

i. $\odot$ is continuous,
ii. $\odot$ is commutative and associative,
iii. for all $s \in [0,1]$, $s \odot 1 = s$,
iv. for all $s, t, u, v \in [0,1]$, $s \odot t \leq u \odot v$ provided that $s \leq u$ and $t \leq v$.

Definition 2.11. [20] A binary operation denoted as $\circledast : [0,1] \times [0,1] \to [0,1]$, is considered a continuous t-conorm when it fulfills the following conditions:

i. $\circledast$ is continuous,
ii. $\circledast$ is commutative and associative,
iii. for all $s \in [0,1]$, $s \circledast 0 = s$,
iv. for all $s, t, u, v \in [0,1]$, $s \circledast t \leq u \circledast v$ provided that $s \leq u$ and $t \leq v$.

Definition 2.12. [29] Let X be the universe of discourse. In this context, the set $A_{NS} \subseteq X$, which is defined as:

$$A_{NS} = \{< u, \mathcal{L}_A(u), \mathcal{M}_A(u), \mathcal{V}_A(u) >: u \in X\}$$

is referred to as a neutrosophic set. In this definition, $\mathcal{L}_A(u), \mathcal{M}_A(u)$, and $\mathcal{V}_A(u)$ are functions from X to $[0,1]$. These functions represents the degree of truth-membership, degree of indeterminacy-membership, and degree of false-membership respectively. It's important to note that these membership degrees satisfy the constraint $0 \leq \mathcal{L}_A(u) + \mathcal{M}_A(u) + \mathcal{V}_A(u) \leq 3$.

Definition 2.13. [15] Let F denotes a vector space and let

$$\mathcal{N} = \{< u, \mathcal{L}(u), \mathcal{M}(u), \mathcal{V}(u) >: u \in F\}$$

indicates a normed space (NS) with functions $\mathcal{L}, \mathcal{M}, \mathcal{V} : F \times \mathbb{R}^+ \to [0,1]$. Let's suppose that $\odot$ signifies the continuous t-norm and $\circledast$ signifies the continuous t-conorm. Then the four-tuple $V = (F, \mathcal{N}, \odot, \circledast)$ is termed a neutrosophic normed

space (NNS) provided that for all $u, v \in F$ and $\psi, \nu > 0$ and for each $\sigma \neq 0$ the following conditions hold:

i. $\mathcal{L}(u,\psi) + \mathcal{M}(u,\psi) + \mathcal{V}(u,\psi) \leq 3$,
ii. $0 \leq \mathcal{L}(u,\psi) \leq 1$, $0 \leq \mathcal{M}(u,\psi) \leq 1$, $0 \leq \mathcal{V}(u,\psi) \leq 1$,
iii. $\mathcal{L}(\sigma u,\psi) = \mathcal{L}\left(u, \frac{\psi}{|\sigma|}\right)$,
iv. $\mathcal{L}(u,\psi) = 1$ (for $\psi > 0$) if and only if $u = 0$,
v. $\mathcal{L}(u,.)$ is a continuous non-decreasing function,
vi. $\mathcal{L}(u,\psi) \odot \mathcal{L}(v,\psi) \leq \mathcal{L}(u+v,\psi+\nu)$,
vii. $\mathcal{M}(u,\psi) = 0$ (for $\psi > 0$) if and only if $u = 0$,
viii. $\lim\limits_{\psi\to+\infty} \mathcal{L}(u,\psi) = 1$,
ix. $\mathcal{M}(u,\nu) \circledast \mathcal{M}(v,\psi) \geq \mathcal{M}(u+v,\psi+\nu)$,
x. $\mathcal{M}(\sigma u,\psi) = \mathcal{M}\left(u, \frac{\psi}{|\sigma|}\right)$,
xi. $\lim\limits_{\psi\to+\infty} \mathcal{M}(u,\psi) = 0$,
xii. $\mathcal{M}(u,.)$ is a continuous and non-increasing function,
xiii. $\mathcal{V}(u,\nu) \circledast \mathcal{V}(v,\psi) \geq \gamma(u+v,\psi+\nu)$,
xiv. $\mathcal{V}(\sigma u,\psi) = \mathcal{V}\left(u, \frac{\psi}{|\sigma|}\right)$,
xv. $\mathcal{V}(u,\psi) = 0$ (for $\psi > 0$) if and only if $u = 0$,
xvi. If $\psi \leq 0$, then $\mathcal{L}(u,\psi) = 0, \mathcal{M}(u,\psi) = 1$ and $\mathcal{V}(u,\psi) = 1$,
xvii. $\lim\limits_{\psi\to+\infty} \mathcal{V}(u,\psi) = 0$,
xviii. $\mathcal{V}(u,.)$ is a continuous non-increasing function.

Also, $\mathcal{N} = (\mathcal{L}, \mathcal{M}, \mathcal{V})$ is termed as Neutrosophic norm (NN).

Definition 2.14. [16] Let V be a NNS and $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. A sequence (x_k) is said to be $\mathcal{I}-$convergent to l with respect to the neutrosophic norm (NN), if for every $0 < \varrho < 1$, $K(\varrho) \in \mathcal{I}$, where

$$K(\varrho) = \{k \in \mathbb{N} : \mathcal{L}(x_k - l, \psi) \leq 1 - \varrho, \mathcal{M}(x_k - l, \psi) \geq \varrho \text{ and } \mathcal{V}(x_k - l, \psi) \geq \varrho\}.$$

Symbolically it is denoted as $\mathcal{I} - \mathcal{N} - \lim x_k = l$ or $x_k \to l(\mathcal{I} - \mathcal{N})$.

Definition 2.15. [16] Let V be a NNS and $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. A sequence (x_k) is said to be $\mathcal{I}^*-$convergent to l with respect to neutrosophic norm (NN), if there exists a set $M \in \mathcal{F}(\mathcal{I})$ (i.e., $\mathbb{N} \setminus M \in \mathcal{I}$) such that $x_{k,k\in M} \to l(\mathcal{N})$.

In this case, we write $\mathcal{I}^* - \mathcal{N} - \lim x_k = l$ or $x_k \to l(\mathcal{I}^* - \mathcal{N})$.

Definition 2.16. [16] Let (x_k) be a sequence in a NNS V and $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. Then, (x_k) is said to be $\mathcal{I}-$Cauchy with respect to neutrosophic norm (NN), if for any $0 < \varrho < 1$, there exists $M = M(\varrho)$ such that $KC(\varrho) \in \mathcal{I}$, where

$$KC(\varrho) = \{k \in \mathbb{N} : \mathcal{L}(x_k - x_M, \psi) \leq 1 - \varrho \text{ or } \mathcal{M}(x_k - x_M, \psi) \geq \varrho, \mathcal{V}(x_k - x_M, \psi) \geq \varrho\}.$$

Definition 2.17. [16] Let V be a NNS and $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. A sequence (x_k) is said to be $\mathcal{I}^*-$Cauchy with respect to neutrosophic norm (NN), if there exists a set $M \in \mathcal{F}(\mathcal{I})$ (i.e., $\mathbb{N} \setminus M \in \mathcal{I}$) such that $(x_{k,k\in M})$ is an ordinary Cauchy sequence with respect to neutrosophic norm (NN).

Example 2.18. [15] Suppose $(F, \|\cdot\|)$ be a NS. For $s, t \in [0,1]$, define the t-norm $\odot$ and the t-conorm $\circledast$ as $s \odot t = st$ and $s \circledast t = s + t - st$ respectively. For $\psi > \|u\|$, let

$$\mathcal{L}(u,\psi) = \frac{\psi}{\psi + \|u\|},\ \mathcal{M}(u,\psi) = \frac{\|u\|}{\psi + \|u\|},\ \mathcal{V}(u,\psi) = \frac{\|u\|}{\psi} \text{ for all } u \in F \text{ and } \psi > 0$$

and for $\psi \leq \|u\|$, let $\mathcal{L}(u,\psi) = 0, \mathcal{M}(u,\psi) = 1$ and $\mathcal{V}(u,\psi) = 1$. Then, $(F, \mathcal{N}, \odot, \circledast)$ is a NNS.

Definition 2.19. [7] Let V be a NNS. A double sequence (x_{ij}) is said to be statistical convergent to l with respect to the neutrosophic norm (NN), if for every $0 < \varrho < 1$,

$$K(\varrho) = \{(i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{L}(x_{ij} - l, \psi) \leq 1 - \varrho, \mathcal{M}(x_{ij} - l, \psi) \geq \varrho \text{ and } \mathcal{V}(x_{ij} - l, \psi) \geq \varrho\}.$$

Symbolically it is denoted as $st^2 - \mathcal{N} - \lim x_{ij} = l$ or $x_{ij} \to l(st^2 - \mathcal{N})$.

Definition 2.20. [7] Let (x_{ij}) be a double sequence in a NNS V. Then, (x_{ij}) is said to be statistical Cauchy if for any $0 < \varrho < 1$, there exists $M = M(\varrho), N = N(\varrho)$ such that $\delta^2(KC(\varrho)) = 0$, where

$$KC(\varrho) = \{(i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{L}(x_{ij} - x_{MN}, \psi) \leq 1 - \varrho \text{ or } \mathcal{M}(x_{ij} - x_{MN}, \psi) \geq \varrho, \mathcal{V}(x_{ij} - x_{MN}, \psi) \geq \varrho\}.$$

3. Main Results

In this section, we will outline the primary findings of the paper. We commence by introducing the subsequent definition:

Definition 3.1. Let V be a NNS and (x_{ij}) be a double sequence. If for every $0 < \varrho < 1$,

$$K(\varrho) = \{(i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{L}(x_{ij} - l, \psi) \leq 1 - \varrho, \mathcal{M}(x_{ij} - l, \psi) \geq \varrho \text{ and } \mathcal{V}(x_{ij} - l, \psi) \geq \varrho\} \in \mathcal{I}_2$$

holds, then (x_{ij}) is called $\mathcal{I}_2-$convergent to l with respect to neutrosophic norm (NN). In this scenario, we express this as, $\mathcal{I}_2 - \mathcal{N} - \lim x_{ij} = l$ or $x_{ij} \to l(\mathcal{I}_2 - \mathcal{N})$.

Especially, if we choose $\mathcal{I}_2 = \mathcal{I}_{\delta^2} = \{A \subseteq \mathbb{N} \times \mathbb{N} : \delta^2(A) = 0\}$, then the previously mentioned definition aligns with the concept of statistical convergence of double sequences in NNS, a concept that was recently explored by Granados and Dhital [7] in their work.

Example 3.2. Assume that $(F, \|\cdot\|)$ is a NS. For any s and t within the interval $[0,1]$, we define the continuous t-norm as $s \odot t = st$ and the continuous t-conorm as $s \circledast t = \min\{s + t, 1\}$. From Example 2.18, let us take $\mathcal{L}, \mathcal{M}, \mathcal{V}$ for all $\psi > 0$. Under these conditions, we can say that V is a NNS. Consider the double sequence (x_{ij}) defined by

$$x_{ij} = \begin{cases} 1, & i \text{ is a perfect cube, for all } j \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}.$$

Then, $x_{ij} \to 0(\mathcal{I}_{\delta^2} - \mathcal{N})$.

Justification: For any $0 < \varrho < 1$, $K_{m,n}(\varrho) = \{i \leq m, j \leq n : \mathcal{L}(x_{ij} - 0, \psi) \leq 1 - \varrho,\ \mathcal{M}(x_{ij} - 0, \psi) \geq \varrho \text{ and } \mathcal{V}(x_{ij} - 0, \psi) \geq \varrho\}$.This implies that,

$$\begin{aligned} K_{m,n}(\varrho) &= \left\{ i \leq m, j \leq n : \frac{\psi}{\psi + \|x_{ij}\|} \leq 1 - \varrho,\ \frac{\|x_{ij}\|}{\psi + \|x_{ij}\|} \geq \varrho \text{ and } \frac{\|x_{ij}\|}{\psi} \geq \varrho \right\} \\ &= \left\{ i \leq m, j \leq n : \|x_{ij}\| \geq \frac{\psi\varrho}{1 - \varrho} \text{ and } \|x_{ij}\| \geq \psi\varrho \right\} \\ &= \{i \leq m, j \leq n : x_{ij} = 1\}. \end{aligned}$$

Then we have, $\delta^2(K(\varrho)) = \lim\limits_{n \to +\infty} \frac{|K_{m,n}(\varrho)|}{mn} \leq \lim\limits_{n \to +\infty} \frac{\sqrt[3]{mn}}{mn} = 0$.

Hence, $x_{ij} \to 0(\mathcal{I}_{\delta^2} - \mathcal{N})$.

Lemma 3.3. *Assuming that V is a NNS, the following statements are equivalent for any value of ϱ within the interval $(0, 1)$:*

i. $\{(i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{L}(x_{ij} - l, \psi) \leq 1 - \varrho\} \in \mathcal{I}_2, \{(i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}(x_{ij} - l, \psi) \geq \varrho\} \in \mathcal{I}_2$, *and* $\{(i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{V}(x_{ij} - l, \psi) \geq \varrho\} \in \mathcal{I}_2$;

ii. $\{(i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{L}(x_{ij} - l, \psi) > 1 - \varrho,\ \mathcal{M}(x_{ij} - l, \psi) < \varrho$ *and* $\mathcal{V}(x_{ij} - l, \psi) < \varrho\} \in \mathcal{F}(\mathcal{I}_2)$;

iii. $\{(i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{L}(x_{ij} - l, \psi) > 1 - \varrho\} \in \mathcal{F}(\mathcal{I}_2), \{(i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}(x_{ij} - l, \psi) < \varrho\} \in \mathcal{F}(\mathcal{I}_2)$, *and* $\{(i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{V}(x_{ij} - l, \psi) < \varrho\} \in \mathcal{F}(\mathcal{I}_2)$;

iv. $\mathcal{L}(x_{ij} - l, \psi) \to 1(\mathcal{I}_2 - \mathcal{N})$, $\mathcal{M}(x_{ij} - l, \psi) \to 0(\mathcal{I}_2 - \mathcal{N})$ *and* $\mathcal{V}(x_{ij} - l, \psi) \to 0(\mathcal{I}_2 - \mathcal{N})$;

v. $x_{ij} \to l(\mathcal{I}_2 - \mathcal{N})$.

Theorem 3.4. *Suppose we have a NNS V and a double sequence (x_{ij}) so that $x_{ij} \to l(\mathcal{I}_2 - \mathcal{N})$. Then l is uniquely determined.*

Proof. If possible let $x_{ij} \to l_1(\mathcal{I}_2 - \mathcal{N})$ and $x_{ij} \to l_2(\mathcal{I}_2 - \mathcal{N})$ for $l_1 \neq l_2$. In that case, for any specified value $0 < \varrho < 1$, we can select a positive value ν such that $(1 - \varrho) \odot (1 - \varrho) > 1 - \nu$ and $\varrho \circledast \varrho < \nu$. Now, we establish the following sets for any $\psi > 0$:

$$\begin{aligned} K_{\mathcal{L}_1}(\varrho, \psi) &= \left\{ (i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{L}\left(x_{ij} - l_1, \frac{\psi}{2}\right) \leq 1 - \varrho \right\}, \\ K_{\mathcal{L}_2}(\varrho, \psi) &= \left\{ (i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{L}\left(x_{ij} - l_2, \frac{\psi}{2}\right) \leq 1 - \varrho \right\}, \\ K_{\mathcal{M}_1}(\varrho, \psi) &= \left\{ (i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}\left(x_{ij} - l_1, \frac{\psi}{2}\right) \geq \varrho \right\}, \\ K_{\mathcal{M}_2}(\varrho, \psi) &= \left\{ (i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}\left(x_{ij} - l_2, \frac{\psi}{2}\right) \geq \varrho \right\}, \\ K_{\mathcal{V}_1}(\varrho, \psi) &= \left\{ (i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{V}\left(x_{ij} - l_1, \frac{\psi}{2}\right) \geq \varrho \right\}, \\ K_{\mathcal{V}_2}(\varrho, \psi) &= \left\{ (i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{V}\left(x_{ij} - l_2, \frac{\psi}{2}\right) \geq \varrho \right\}. \end{aligned}$$

As $x_{ij} \to l_1(\mathcal{I}_2 - \mathcal{N})$, we can apply Lemma 3.3 to conclude that for any given $\psi > 0$,

$$K_{\mathcal{L}_1}(\varrho, \psi), K_{\mathcal{M}_1}(\varrho, \psi), \text{ and } K_{\mathcal{V}_1}(\varrho, \psi) \in \mathcal{I}_2.$$

Furthermore, as $x_{ij} \to l_2(\mathcal{I}_2 - \mathcal{N})$, we can apply Lemma 3.3 to conclude that for any given $\psi > 0$,

$$K_{\mathcal{L}_2}(\varrho, \psi), K_{\mathcal{M}_2}(\varrho, \psi), \text{ and } K_{\mathcal{V}_2}(\varrho, \psi) \in \mathcal{I}_2.$$

Let

$$K(\varrho, \psi) = (K_{\mathcal{L}_1}(\varrho, \psi) \cup K_{\mathcal{L}_2}(\varrho, \psi)) \cap (K_{\mathcal{M}_1}(\varrho, \psi) \cup K_{\mathcal{M}_2}(\varrho, \psi)) \cap (K_{\mathcal{V}_1}(\varrho, \psi) \cup K_{\mathcal{V}_2}(\varrho, \psi)).$$

Subsequently, it follows that $K(\varrho, \psi) \in \mathcal{I}_2$ and therefore, $(\mathbb{N} \times \mathbb{N}) \setminus K(\varrho, \psi) \in \mathcal{F}(\mathcal{I}_2)$. Consequently, $(\mathbb{N} \times \mathbb{N}) \setminus K(\varrho, \psi) \neq \emptyset$. Let's select $(s, t) \in \mathbb{N} \setminus K(\varrho, \psi)$. In this scenario, there are three possible situations:

i. $(s, t) \in ((\mathbb{N} \times \mathbb{N}) \setminus K_{\mathcal{L}_1}(\varrho, \psi)) \cap ((\mathbb{N} \times \mathbb{N}) \setminus K_{\mathcal{L}_2}(\varrho, \psi))$;
ii. $(s, t) \in ((\mathbb{N} \times \mathbb{N}) \setminus K_{\mathcal{M}_1}(\varrho, \psi)) \cap ((\mathbb{N} \times \mathbb{N}) \setminus K_{\mathcal{M}_2}(\varrho, \psi))$;
iii. $(s, t) \in ((\mathbb{N} \times \mathbb{N}) \setminus K_{\mathcal{V}_1}(\varrho, \psi)) \cap ((\mathbb{N} \times \mathbb{N}) \setminus K_{\mathcal{V}_2}(\varrho, \psi))$.

Taking into account (i), the following can be observed:

$$\mathcal{L}(l_1 - l_2, \psi) \geq \mathcal{L}\left(x_{st} - l_1, \frac{\psi}{2}\right) \odot \mathcal{L}\left(x_{st} - l_2, \frac{\psi}{2}\right) > (1-\varrho) \odot (1-\varrho) > 1-\nu. \quad (3.1)$$

As ν can take on any arbitrary value, we can deduce from Equation (3.1) that for any given $\psi > 0$, $\mathcal{L}(l_1 - l_2, \psi) = 1$ i.e., $l_1 = l_2$. Similarly, by examining equation (ii), we can deduce the following:

$$\mathcal{M}(l_1 - l_2, \psi) \leq \mathcal{M}\left(x_{st} - l_1, \frac{\psi}{2}\right) \circledast \mathcal{M}\left(x_{st} - l_2, \frac{\psi}{2}\right) < \varrho \circledast \varrho < \nu. \quad (3.2)$$

Once again, considering that ν can assume any value, we can conclude from Equation (3.2) that for any $\psi > 0$, $\mathcal{M}(l_1 - l_2, \psi) = 0$ i.e., $l_1 = l_2$. Lastly, when examining equation (iii), the following can be observed:

$$\mathcal{V}(l_1 - l_2, \psi) \leq \mathcal{V}\left(x_{st} - l_1, \frac{\psi}{2}\right) \circledast \mathcal{V}\left(x_{st} - l_2, \frac{\psi}{2}\right) < \varrho \circledast \varrho < \nu. \quad (3.3)$$

Since ν is arbitrary so for any $\psi > 0$, Equation (3.3) yields $\mathcal{V}(l_1 - l_2, \psi) = 0$ i.e., $l_1 = l_2$. Thus in all cases we obtain $l_1 = l_2$ and this completes the proof. □

Theorem 3.5. *Let (x_{ij}) and (y_{ij}) be two double sequences in the NNS V such that $x_{ij} \to l_1(\mathcal{I}_2 - \mathcal{N})$ and $y_{ij} \to l_2(\mathcal{I}_2 - \mathcal{N})$. Then,*
(i) $x_{ij} + y_{ij} \to l_1 + l_2(\mathcal{I}_2 - \mathcal{N})$ and (ii) $\alpha x_{ij} \to \alpha l_1(\mathcal{I}_2 - \mathcal{N})$ where $\alpha \in \mathbb{R}$.

Proof. (i) Suppose $x_{ij} \to l_1(\mathcal{I}_2 - \mathcal{N})$ and $y_{ij} \to l_2(\mathcal{I}_2 - \mathcal{N})$. Then, by definition for any $0 < \varrho < 1$,

$$K(\varrho), K'(\varrho) \in \mathcal{I}_2,$$

where

$$K(\varrho)=\left\{(i,j)\in\mathbb{N}\times\mathbb{N}:\mathcal{L}\left(x_{ij}-l_1,\frac{\psi}{2}\right)\leq 1-\varrho,\mathcal{M}\left(x_{ij}-l_1,\frac{\psi}{2}\right)\geq\varrho \right.$$
$$\left. \text{and } \mathcal{V}\left(x_{ij}-l_1,\frac{\psi}{2}\right)\geq\varrho\right\}.$$

and

$$K'(\varrho)=\left\{(i,j)\in\mathbb{N}\times\mathbb{N}:\mathcal{L}\left(y_{ij}-l_2,\frac{\psi}{2}\right)\leq 1-\varrho,\mathcal{M}\left(y_{ij}-l_2,\frac{\psi}{2}\right)\geq\varrho \right.$$
$$\left. \text{and } \mathcal{V}\left(y_{ij}-l_2,\frac{\psi}{2}\right)\geq\varrho\right\}.$$

Now as the inclusion

$$\begin{aligned}((\mathbb{N}\times\mathbb{N})\setminus K(\varrho))\cap((\mathbb{N}\times\mathbb{N})\setminus K'(\varrho))&\\ \subseteq\{(i,j)\in\mathbb{N}\times\mathbb{N}:\mathcal{L}(x_{ij}+y_{ij}-l_1-l_2,\psi)>1-\varrho,&\\ \mathcal{M}(x_{ij}+y_{ij}-l_1-l_2,\psi)<\varrho,\mathcal{V}(x_{ij}+y_{ij}-l_1-l_2,\psi)<\varrho\}&\end{aligned}$$

holds, so we must have

$$K''(\varrho)=\{(i,j)\in\mathbb{N}\times\mathbb{N}:\mathcal{L}(x_{ij}+y_{ij}-l_1-l_2,\psi)\leq 1-\varrho,\mathcal{M}(x_{ij}+y_{ij}-l_1-l_2,\psi)\geq\varrho$$
$$\text{and } \mathcal{V}(x_{ij}+y_{ij}-l_1-l_2,\psi)\geq\varrho\}\subseteq K(\varrho)\cup K'(\varrho)$$

and consequently, $K''(\varrho)\in\mathcal{I}_2$ i.e., $x_{ij}+y_{ij}\to l_1+l_2(\mathcal{I}_2-\mathcal{N})$.

(ii) If $\alpha=0$, there's no need for any demonstration. Hence, we presume that $\alpha\neq 0$. Given that $x_{ij}\to l_1(\mathcal{I}_2-\mathcal{N})$, it follows that for any $0<\varrho<1$, $K(\varrho)\in\mathcal{I}_2$, where

$$K(\varrho)=\left\{(i,j)\in\mathbb{N}\times\mathbb{N}:\mathcal{L}\left(x_{ij}-l,\frac{\psi}{|\alpha|}\right)\leq 1-\varrho,\mathcal{M}\left(x_{ij}-l,\frac{\psi}{|\alpha|}\right)\geq\varrho \right.$$
$$\left. \text{and } \mathcal{V}\left(x_{ij}-l,\frac{\psi}{|\alpha|}\right)\geq\varrho\right\}.$$

Suppose

$$K'(\varrho)=\{(i,j)\in\mathbb{N}\times\mathbb{N}:\mathcal{L}(\alpha x_{ij}-\alpha l,\psi)\leq 1-\varrho,\mathcal{M}(\alpha x_{ij}-\alpha l,\psi)\geq\varrho$$
$$\text{and } \mathcal{V}(\alpha x_{ij}-\alpha l,\psi)\geq\varrho\}.$$

Subsequently, the containment $K'(\varrho)\subseteq K(\varrho)$ is valid, and ultimately, $K'(\varrho)\in\mathcal{I}_2$. Hence, $\alpha x_{ij}\to\alpha l_1(\mathcal{I}_2-\mathcal{N})$. □

Theorem 3.6. *Consider two double sequences (x_{ij}) and (y_{ij}) in the NNS V with $y_{ij}\to l(\mathcal{N})$ and $\{(i,j)\in\mathbb{N}\times\mathbb{N}:x_{ij}\neq y_{ij}\}\in\mathcal{I}_2$. Then, $x_{ij}\to l(\mathcal{I}_2-\mathcal{N})$.*

Proof. Suppose $\{(i,j) \in \mathbb{N} \times \mathbb{N} : x_{ij} \neq y_{ij}\} \in \mathcal{I}_2$ holds and $y_{ij} \to l(\mathcal{N})$. Then, for any $0 < \varrho < 1$, the set

$$K(\varrho) = \{(i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{L}(y_{ij} - l, \psi) \leq 1 - \varrho, \mathcal{M}(y_{ij} - l, \psi) \geq \varrho \text{ and } \mathcal{V}(y_{ij} - l, \psi) \geq \varrho\}$$

contains mostly finitely many elements and consequently, it follows that $K(\varrho) \in \mathcal{I}_2$. Since the inclusion

$$K'(\varrho) = \{(i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{L}(x_{ij} - l, \psi) \leq 1 - \varrho, \mathcal{M}(x_{ij} - l, \psi) \geq \varrho \text{ and } \mathcal{V}(x_{ij} - l, \psi) \geq \varrho\}$$
$$\subseteq K(\varrho) \cap \{(i,j) \in \mathbb{N} \times \mathbb{N} : x_{ij} \neq y_{ij}\}$$

holds, so we have $K'(\varrho) \in \mathcal{I}_2$. This shows that $x_{ij} \to l(\mathcal{I}_2 - \mathcal{N})$. □

Definition 3.7. Let V be a NNS. A double sequence (x_{ij}) is said to be $\mathcal{I}_2^*-$convergent to l with respect to neutrosophic norm (NN), if there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $(\mathbb{N} \times \mathbb{N}) \setminus M \in \mathcal{I}_2$) such that $x_{ij,(i,j)\in M} \to l(\mathcal{N})$.

In this case, we write $\mathcal{I}_2^* - \mathcal{N} - \lim x_{ij} = l$ or $x_{ij} \to l(\mathcal{I}_2^* - \mathcal{N})$.

Theorem 3.8. *Let V be a NNS and $\mathcal{I}_2$ be a strong admissible ideal. If $x_{ij} \to l(\mathcal{I}_2^* - \mathcal{N})$ then $x_{ij} \to l(\mathcal{I}_2 - \mathcal{N})$.*

Proof. Since $x_{ij} \to l(\mathcal{I}_2^* - \mathcal{N})$, so by definition there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $(\mathbb{N} \times \mathbb{N}) \setminus M \in \mathcal{I}_2$) such that $x_{ij,(i,j)\in M} \to l(\mathcal{N})$. This means that for any $0 < \varrho < 1$, there exists $k_0 \in \mathbb{N}$ such that for all $i, j \geq k_0$,

$$\mathcal{L}(x_{ij} - l, \psi) > 1 - \varrho, \mathcal{M}(x_{ij} - l, \psi) < \varrho \text{ and } \mathcal{V}(x_{ij} - l, \psi) < \varrho.$$

Then we have,

$$K(\varrho) = \{(i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{L}(x_{ij} - l, \psi) \leq 1 - \varrho, \mathcal{M}(x_{ij} - l, \psi) \geq \varrho \text{ and } \mathcal{V}(x_{ij} - l, \psi) \geq \varrho\}$$
$$\subseteq ((\mathbb{N} \times \mathbb{N}) \setminus M) \cup (M \cap (((\{1, 2, ..., (k_0 - 1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, ..., (k_0 - 1)\}))) \in \mathcal{I}_2$$

Thus by hereditary property, $K(\varrho) \in \mathcal{I}_2$. Hence, $x_{ij} \to l(\mathcal{I}_2 - \mathcal{N})$. □

Remark 3.9. The reverse of Theorem 3.8 is not guaranteed to hold, as illustrated by the following example.

Example 3.10. Consider the NS $(\mathbb{R}, \|\cdot\|)$. Define the continuous t-norm as $s \odot t = st$ and the continuous t-conorm as $s \circledast t = \min\{s + t, 1\}$ for all $s, t \in [0, 1]$. From Example 2.18, we take $\mathcal{L}, \mathcal{M}, \mathcal{V}$ for all $\psi > 0$. Then, $(\mathbb{R}, \mathcal{N}, \odot, \circledast)$ is a NNS. Let $\mathbb{N} = \bigcup_{m=1}^{+\infty} D_m$ be a decomposition of $\mathbb{N}$ be a disjoint decomposition of $\mathbb{N}$ such that each D_p is an infinite set. Then it is obvious that

$$\mathbb{N} \times \mathbb{N} = \bigcup_{m=1}^{+\infty} \bigcup_{n=1}^{+\infty} (D_m \times D_n)$$

is a disjoint decomposition of $\mathbb{N}\times\mathbb{N}$. Let

$$\mathcal{I}_2=\left\{A\subseteq\mathbb{N}\times\mathbb{N}:A\subseteq\left(\mathbb{N}\times\left(\bigcup_{m=1}^{p}D_m\right)\right)\cup\left(\left(\bigcup_{m=1}^{q}D_m\right)\times\mathbb{N}\right),\text{for some }\ p,q\in\mathbb{N}\right\}$$

be an admissible ideal in $\mathbb{N}\times\mathbb{N}$. Clearly, for any set $H(\subseteq\mathbb{N})$ containing a finite number of elements we have, $\mathbb{N}\times H$, $H\times\mathbb{N}\in\mathcal{I}_2$. Define a double sequence (x_{ij}) as follows:

$$x_{ij}=\tfrac{1}{m}+\tfrac{1}{n},\text{ whenever }(i,j)\in D_m\times D_n.$$

Then,

$$\mathcal{L}(x_{ij}-0,\psi)=\frac{\psi}{\psi+\|x_{ij}\|}\to 1,$$
$$\mathcal{M}(x_{ij}-0,\psi)=\frac{\|x_{ij}\|}{\psi+\|x_{ij}\|}\to 0,$$
$$\text{and }\mathcal{V}(x_{ij}-0,\psi)=\frac{\|x_{ij}\|}{\psi}\to 0$$

as $i,j\to+\infty$. This shows that $x_{ij}\to 0(\mathcal{I}_2-\mathcal{N})$. But we claim that $x_{ij}\nrightarrow 0(\mathcal{I}_2^*-\mathcal{N})$. To prove this, let us assume the contrary. Then, there exists a set $M\in\mathcal{F}(\mathcal{I}_2)$ (i.e., $(\mathbb{N}\times\mathbb{N})\setminus M\in\mathcal{I}_2$) such that $x_{ij,(i,j)\in M}\to 0(\mathcal{N})$. Now by definition of $\mathcal{I}_2$, we can say that there exists $p,q\in\mathbb{N}$ such that

$$(\mathbb{N}\times\mathbb{N})\setminus M\subseteq\left(\mathbb{N}\times\left(\bigcup_{m=1}^{p}D_m\right)\right)\cup\left(\left(\bigcup_{m=1}^{q}D_m\right)\times\mathbb{N}\right).$$

But then, $D_{p+1}\times D_{q+1}\subseteq M$ holds and as a consequence $x_{ij}=\frac{1}{p+1}+\frac{1}{q+1}$ for infinitely many $(i,j)\in D_{p+1}\times D_{q+1}\subseteq M$, which is a contradiction to the fact that $x_{ij,(i,j)\in M}\to 0(\mathcal{N})$.

Theorem 3.11. *Let V be a NNS and $\mathcal{I}_2$ be an ideal in $\mathbb{N}\times\mathbb{N}$ with the property $(AP2)$. Then, if $x_{ij}\to l(\mathcal{I}_2-\mathcal{N})$ then $x_{ij}\to l(\mathcal{I}_2^*-\mathcal{N})$.*

Proof. Suppose $\mathcal{I}_2$ satisfies the condition $(AP)_2$ and $x_{ij}\to l(\mathcal{I}_2-\mathcal{N})$. Then, by definition, for every $0<\varrho<1$, $K(\varrho)\in\mathcal{I}_2$, where

$$K(\varrho)=\{(i,j)\in\mathbb{N}\times\mathbb{N}:\mathcal{L}(x_{ij}-l,\psi)\leq 1-\varrho,\mathcal{M}(x_{ij}-l,\psi)\geq\varrho \text{ and }\mathcal{V}(x_{ij}-l,\psi)\geq\varrho\}.$$

Suppose for $m\in\mathbb{N}$, D_m denotes the set

$$\left\{(i,j)\in\mathbb{N}\times\mathbb{N}:1-\frac{1}{m}\leq\mathcal{L}(x_{ij}-l,\psi)<1-\frac{1}{m+1},\right.$$
$$\left.\frac{1}{m+1}<\mathcal{M}(x_{ij}-l,\psi)\leq\frac{1}{m}\text{ and }\frac{1}{m+1}<\mathcal{V}(x_{ij}-l,\psi)\leq\frac{1}{m}\right\}.$$

Then, it is clear that for all $m\in\mathbb{N}$, $D_m\in\mathcal{I}_2$ and for $m\neq n$, $D_m\cap D_n=\emptyset$. By virtue of $(AP2)$, there exists a sequence of sets (E_m) such that for each $m\in\mathbb{N}$,

the symmetric differences $D_m \triangle E_m$ is contained in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ and $E = \bigcup_{m=1}^{+\infty} E_m \in \mathcal{I}_2$.

Now we will prove that for

$$M = (\mathbb{N} \times \mathbb{N}) \setminus E \in \mathcal{F}(\mathcal{I}_2),\ x_{ij,(i,j)\in M} \to l(\mathcal{N}).$$

Let $\kappa > 0$ be given. By Archimedean property, choose $m_0 \in \mathbb{N}$ such that $\frac{1}{m_0} < \kappa$. Then the following inclusion

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{L}(x_{ij} - l, \psi) \leq 1 - \kappa, \mathcal{M}(x_{ij} - l, \psi) \geq \kappa \text{ and } \mathcal{V}(x_{ij} - l, \psi) \geq \kappa\} \subseteq \bigcup_{m=1}^{m_0} D_m \tag{3.4}$$

holds. Since, $D_m \triangle E_m, m = 1, 2, ..., m_0$ are contained in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$, so there exists $k_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{m=1}^{m_0} E_m\right) \cap \{(i,j) \in \mathbb{N} \times \mathbb{N} : i \geq k_0, j \geq k_0\} = \left(\bigcup_{m=1}^{m_0} D_m\right) \cap \{(i,j) \in \mathbb{N} \times \mathbb{N} : i \geq k_0, j \geq k_0\}. \tag{3.5}$$

If $i, j \geq k_0$, and $(i,j) \in M$, then $(i,j) \notin \bigcup_{m=1}^{m_0} E_m$ and consequently from Equation (3.5), $(i,j) \notin \bigcup_{m=1}^{m_0} D_m$. From (3.4), it is clear that for any $i, j \geq k_0$ and $(i,j) \in \mathbb{N} \times \mathbb{N}$,

$$\mathcal{L}(x_{ij} - l, \psi) > 1 - \kappa, \mathcal{M}(x_{ij} - l, \psi) < \kappa \text{ and } \mathcal{V}(x_{ij} - l, \psi) < \kappa.$$

This means that $x_{ij} \to l(\mathcal{I}_2^* - \mathcal{N})$. Hence the theorem. □

Definition 3.12. Let (x_{ij}) be a double sequence in a NNS V. If for any $0 < \varrho < 1$, there exists two positive integers $M = M(\varrho), N = N(\varrho)$ such that $KC(\varrho) \in \mathcal{I}_2$, where

$$KC(\varrho) = \{(i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{L}(x_{ij} - x_{MN}, \psi) \leq 1 - \varrho \text{ or } \mathcal{M}(x_{ij} - x_{MN}, \psi) \geq \varrho, \mathcal{V}(x_{ij} - x_{MN}, \psi) \geq \varrho\},$$

then (x_{ij}) is said to be $\mathcal{I}_2-$Cauchy.

In a specific case, if we choose $\mathcal{I}_2 = \mathcal{I}_{\delta^2} = \{A \subseteq \mathbb{N} \times \mathbb{N} : \delta^2(A) = 0\}$, then the definition provided earlier aligns with the concept of statistical Cauchy sequences for double sequences in the context of NNS. This concept has been recently explored by Granados and Dhital [7] in their work.

Theorem 3.13. *Let (x_{ij}) be a double sequence in a NNS V. Then, (x_{ij}) is $\mathcal{I}_2-$convergent sequence if and only if it is $\mathcal{I}_2-$Cauchy sequence.*

Proof. Suppose $x_{ij} \to l(\mathcal{I}_2 - \mathcal{N})$. For a given $0 < \varrho < 1$, we choose $\nu > 0$ such that

$$(1-\varrho) \odot (1-\varrho) > 1-\nu \text{ and } \varrho \circledast \varrho < \nu.$$

Then by definition, for any $0 < \varrho < 1$, $(\mathbb{N} \times \mathbb{N}) \setminus K(\varrho) \in \mathcal{F}(\mathcal{I}_2)$, where

$$K(\varrho) = \Big\{(i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{L}\left(x_{ij} - l, \frac{\psi}{2}\right) \leq 1-\varrho, \mathcal{M}\left(x_{ij} - l, \frac{\psi}{2}\right) \geq \varrho \text{ and } \mathcal{V}\left(x_{ij} - l, \frac{\psi}{2}\right) \geq \varrho\Big\}.$$

Thus, the set $(\mathbb{N} \times \mathbb{N}) \setminus K(\varrho)$ is non-empty. Let $(M,N) \in (\mathbb{N} \times \mathbb{N}) \setminus K(\varrho)$.

Then we have,

$$\mathcal{L}\left(x_{MN} - l, \frac{\psi}{2}\right) > 1-\varrho,\ \mathcal{M}\left(x_{MN} - l, \frac{\psi}{2}\right) < \varrho \text{ and } \mathcal{V}\left(x_{MN} - l, \frac{\psi}{2}\right) < \varrho.$$

Now suppose

$$KC(\varrho) = \{(i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{L}(x_{ij} - x_{MN}, \psi) \leq 1-\nu, \mathcal{M}(x_{ij} - x_{MN}, \psi) \geq \nu \text{ and } \mathcal{V}(x_{ij} - x_{MN}, \psi) \geq \nu\}.$$

We claim that $KC(\varrho) \subseteq K(\varrho)$ because if the inclusion does not hold then we must have some $(M_0, N_0) \in KC(\varrho) \setminus K(\varrho)$ which immediately yields

$$\mathcal{L}(x_{M_0N_0} - x_{MN}, \psi) \leq 1-\nu \text{ and } \mathcal{L}\left(x_{M_0N_0} - l, \tfrac{\psi}{2}\right) > 1-\varrho.$$

In particular, $\mathcal{L}\left(x_{MN} - l, \frac{\psi}{2}\right) > 1-\varrho$. But then,

$$\begin{aligned} 1-\nu &\geq \mathcal{L}(x_{M_0N_0} - x_{MN}, \psi) \\ &\geq \mathcal{L}\left(x_{M_0N_0} - l, \frac{\psi}{2}\right) \odot \mathcal{L}\left(x_{MN} - l, \frac{\psi}{2}\right) \\ &> (1-\varrho) \odot (1-\varrho) > 1-\nu, \end{aligned}$$

which is a contradiction. Further, we have,

$$\mathcal{M}(x_{M_0N_0} - x_{MN}, \psi) \geq \nu \text{ and } \mathcal{M}\left(x_{M_0N_0} - l, \tfrac{\psi}{2}\right) < \varrho.$$

In particular, $\mathcal{M}(x_{MN} - l, \frac{\psi}{2}) < \varrho$. But then,

$$\nu \leq \mathcal{M}(x_{M_0N_0} - x_{MN}, \psi) \leq \mathcal{M}\left(x_{M_0N_0} - l, \frac{\psi}{2}\right) \circledast \mathcal{M}\left(x_{MN} - l, \frac{\psi}{2}\right) < \varrho \circledast \varrho < \nu,$$

which is a contradiction. Finally, we have,

$$\mathcal{V}(x_{M_0N_0} - x_{MN}, \psi) \geq \nu \text{ and } \mathcal{V}(x_{M_0N_0} - l, \tfrac{\psi}{2}) < \varrho.$$

In particular, $\mathcal{V}(x_{MN} - l, \frac{\psi}{2}) < \varrho$. But then,

$$\nu \leq \mathcal{V}(x_{M_0N_0} - x_{MN}, \psi) \leq \mathcal{V}\left(x_{M_0N_0} - l, \frac{\psi}{2}\right) \circledast \mathcal{V}\left(x_{MN} - l, \frac{\psi}{2}\right) < \varrho \circledast \varrho < \nu,$$

which is a contradiction. Thus all possibilities contradict the existence of an element $(M_0, N_0) \in KC(\varrho) \setminus K(\varrho)$. Therefore, we must have $KC(\varrho) \subseteq K(\varrho)$ and as a consequence $KC(\varrho) \in \mathcal{I}_2$. Hence (x_{ij}) is $\mathcal{I}_2-$Cauchy.

To prove the converse part, we assume that (x_{ij}) is a $\mathcal{I}_2-$Cauchy sequence but not $\mathcal{I}_2-$convergent. For a given $0 < \varrho < 1$, we choose $\nu > 0$ such that $(1-\varrho) \odot (1-\varrho) > 1-\nu$ and $\varrho \circledast \varrho < \nu$. Then, since (x_{ij}) is not $\mathcal{I}_2-$convergent, so

$$\mathcal{L}(x_{ij} - x_{MN}, \psi) \geq \mathcal{L}\left(x_{ij} - l, \frac{\psi}{2}\right) \odot \mathcal{L}\left(x_{MN} - l, \frac{\psi}{2}\right) > (1-\varrho) \odot (1-\varrho) > 1-\nu,$$

$$\mathcal{M}(x_{ij} - x_{MN}, \psi) \leq \mathcal{M}\left(x_{ij} - l, \frac{\psi}{2}\right) \circledast \mathcal{M}\left(x_{MN} - l, \frac{\psi}{2}\right) < \varrho \circledast \varrho < \nu,$$

$$\mathcal{V}(x_{ij} - x_{MN}, \psi) \leq \mathcal{V}\left(x_{ij} - l, \frac{\psi}{2}\right) \circledast \mathcal{V}\left(x_{MN} - l, \frac{\psi}{2}\right) < \varrho \circledast \varrho < \nu,$$

holds for $P(\varrho, \nu) = \{i \leq M, j \leq N : \mathcal{M}(x_{ij} - x_{MN}, \psi) \leq 1-\nu\}$. Therefore, $P(\varrho, \nu) \in \mathcal{F}(\mathcal{I}_2)$, which is a contradiction to the fact that (x_{ij}) is $\mathcal{I}_2-$Cauchy. Hence, (x_{ij}) must be a $\mathcal{I}_2-$convergent sequence. This completes the proof. □

Definition 3.14. Let (x_{ij}) be a double sequence in a NNS V. Then (x_{ij}) is said to be $\mathcal{I}_2^*-$Cauchy with respect to neutrosophic norm (NN), if there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $(\mathbb{N} \times \mathbb{N}) \setminus M \in \mathcal{I}_2$) such that $x_{ij,(i,j)\in M}$ is an ordinary Cauchy sequence with respect to neutrosophic norm (NN).

Theorem 3.15. *Let (x_{ij}) be a double sequence in a NNS V. If (x_{ij}) is $\mathcal{I}_2^*-$Cauchy with respect to neutrosophic norm (NN), then it is $\mathcal{I}_2-$Cauchy with respect to neutrosophic norm (NN).*

Proof. We omit the proof as it essentially follows from the same argument for convergence. □

Theorem 3.16. *Let V be a NNS and $\mathcal{I}_2$ be an ideal in $\mathbb{N} \times \mathbb{N}$ with the property $(AP2)$. If (x_{ij}) is $\mathcal{I}_2-$Cauchy with respect to neutrosophic norm (NN), then it is $\mathcal{I}_2^*-$Cauchy with respect to neutrosophic norm (NN).*

Proof. We omit the proof as it essentially follows from the same argument for convergence. □

4. Conclusion

In this paper, we primarily focused on exploring fundamental properties of $\mathcal{I}_2$ and $\mathcal{I}_2^*-$convergence of double sequences within neutrosophic normed spaces. Theorem 3.8 and Theorem 3.11 were established to show the relationship between $\mathcal{I}_2$ and $\mathcal{I}_2^*-$convergence of double sequences. Additionally, Theorem 3.15 and Theorem 3.16 were presented to investigate the implication relationship between $\mathcal{I}_2$ and $\mathcal{I}_2^*-$Cauchy double sequences. Moreover, Theorem 3.11 and Theorem 3.16 shed light on the significance of condition $(AP2)$ in our study. As a natural extension of this research, one could explore various properties such as solidity, symmetry, and monotonicity within the sequence spaces formed by the collection of all $\mathcal{I}_2$-convergent double sequences in NNS.

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