ON THE NON-SPECIAL PART OF THE WEIERSTRASS SEMIGROUOS OF \( n \)-POINTS OF A SMOOTH CURVE

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Abstract. Let \( X \) be a smooth curve of genus \( g \geq 3 \). For any \( n \geq 2 \) and any \( n \) distinct points \( P_1, \ldots, P_n \in X \) let \( H(P_1, \ldots, P_n)^+ \) be the set of all \( (a_1, \ldots, a_n) \in \mathbb{N}^n \) such that \( \mathcal{O}_X(a_1 P_1 + \cdots + a_n P_n) \) is spanned and non-special. This set is closed by the operation \( \preccurlyeq \) where \( (a_1, \ldots, a_n) \preccurlyeq (b_1, \ldots, b_n) \) if and only if \( a_i \leq b_i \) for all \( i \). We raise some questions on the minimal cardinality of a subset of \( H(P_1, \ldots, P_n)^+ \) generating it using \( \preccurlyeq \) and compute this number in a few examples.

1. Introduction

Let \( X \) be a smooth curve of genus \( g \geq 3 \). Fix an integer \( n > 0 \) and \( P_1, \ldots, P_n \in X \) with \( P_i \neq P_j \) for all \( i \neq j \). Let \( H(P_1, \ldots, P_n) \subset \mathbb{N}^n \) be the set of all \( (a_1, \ldots, a_n) \in \mathbb{N}^n \) such that \( \mathcal{O}_X(a_1 P_1 + \cdots + a_n P_n) \) is spanned, i.e. such that there is rational function on \( X \) with \( a_1 P_1 + \cdots + a_n P_n \) as its divisors of poles. The set \( H(P_1, \ldots, P_n) \) is a semigroup for the componentwise addition \( + : \mathbb{N}^n \times \mathbb{N}^n \to \mathbb{N}^n \) and \( G(P_1, \ldots, P_n) := \mathbb{N}^n \setminus H(P_1, \ldots, P_n) \) is a finite set. The semigroup \( H(P_1, \ldots, P_n) \) is called the Weierstrass semigroup of the points \( P_1, \ldots, P_n \), while the elements of \( G(P_1, \ldots, P_n) \) are called the gaps of \( P_1, \ldots, P_n \) ([1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [13], [14]). Define the following partial ordering \( \preccurlyeq \) on \( \mathbb{N}^n \). Write \( (a_1, \ldots, a_n) \preccurlyeq (b_1, \ldots, b_n) \) if and only if \( a_i \leq b_i \) for all \( i \). Set \( H(P_1, \ldots, P_n)^+ := \{(a_1, \ldots, a_n) \in H(P_1, \ldots, P_n) : h^1(\mathcal{O}_X(a_1 P_1 + \cdots + a_n P_n)) = 0 \} \). We have \( a \in H(P_1, \ldots, P_n)^+ \) if and only if \( b \in H(P_1, \ldots, P_n) \) for all \( b \in \mathbb{N}^n \) such that \( b \succeq a \). Therefore \( H(P_1, \ldots, P_n)^+ \) is a set closed for \( \preccurlyeq \). The set \( \mathbb{N}^n \setminus H(P_1, \ldots, P_n)^+ \) is finite, because every line bundle on \( g \) with degree \( \geq 2g \) is spanned and non-special. If \( n = 1 \), then \( H(P_1)^+ \) is the set of all integers \( \geq n_g + 1 \), where \( n_g \) is the largest element of \( G(P_1) \). A non-empty subset \( A \) is closed for \( \preccurlyeq \) if and only \( a + b \in A \) for every \( a \in A \) and every \( b \in \mathbb{N}^n \). Hence the union of \( \{0\} \) and a set closed with respect to \( \preccurlyeq \) is a semigroup. For each \( S \subseteq \mathbb{N}^n \), \( S \neq \emptyset \), there is a minimal subset \( A[S] \) containing \( S \) and closed with respect to \( \preccurlyeq \). Let \( A \subseteq \mathbb{N}^n \) be a non-empty subset closed with respect to \( \preccurlyeq \) and with \( \mathbb{N}^n \setminus A \) finite. Let \( C(A) \) be a minimal set \( S \subset A \) such that \( A = A[S] \). The set \( C(A) \) exists, it is finite.
and it is unique, because the intersection of any family of subsets of \( \mathbb{N}^n \) closed with respect to \( \preceq \) is either empty or a set closed with respect to \( \preceq \). We will say that \( C(A) \) is the set of all corners of \( A \). If \( A = H(P_1, \ldots, P_n)^+ \) for some curve \( X \) and some \( P_1, \ldots, P_n \in X \), then set \( C(P_1, \ldots, P_n)^+ := C(H(P_1, \ldots, P_n)^+) \). Let \( \Delta(n, g) \) (resp. \( \eta(n, g) \)) denote the maximal (resp. minimal) cardinality of \( C(P_1, \ldots, P_n)^+ \) among all \( X, P_1, \ldots, P_n \) with \( X \) a smooth curve of genus \( g \) and \( P_1, \ldots, P_n \) distinct points of \( X \).

Now assume that \( X \) is a hyperelliptic curve of genus \( g \geq 2 \) and that \( P_1, \ldots, P_n \) are distinct Weierstrass points of \( X \). It is easy to check that the integer \( \delta(n, g) := \sharp(C(P_1, \ldots, P_n)^+) \) depends only from \( n \) and \( g \), not even on the characteristic of the algebraically closed base field \( K \) (Remark 3.2). Of course, the integer \( \delta(n, g) \) is defined only if a genus \( g \) hyperelliptic curve has at least \( n \) Weierstrass points, i.e. if \( n \leq 2g + 2 \) (case \( \text{char}(\mathbb{K}) \neq 2 \)) or \( n \leq g + 1 \) (case \( \text{char}(\mathbb{K}) = 2 \)).

**Question 1.** Is \( \Delta(n, g) = \delta(n, g) \) if \( g \gg n \)?

**Question 1** is true (for any \( g \geq 2 \)) when \( n = 2 \). Indeed we prove the following easy result.

**Theorem 1.1.** For any smooth curve \( X \) of genus \( g \geq 2 \) and \( n \) distinct points \( P_1, \ldots, P_n \) we have \( \sharp(C(P_1, \ldots, P_n)^+) \leq \binom{n+2g-1}{n-1} \) and equality holds if and only if \( n = 2 \), \( X \) is a hyperelliptic curve and \( P_1, P_2 \) are distinct Weierstrass points of \( X \).

Let \( X \) be an integral projective curve with arithmetic genus \( g := p_a(X) \geq 2 \). We may define without any modification \( H(P_1, \ldots, P_n) \) and \( H(P_1, \ldots, P_n)^+ \) and hence \( C(P_1, \ldots, P_n)^+ \) if we only take distinct smooth points of \( X \), i.e. we impose that \( P_i \in X_{\text{reg}} \) for all \( i \). Let \( \Delta'(n, g) \) (resp. \( \eta'(n, g) \)) be the the maximal (minimal) integer \( \sharp(C(P_1, \ldots, P_n)) \) among all integra curves \( X \) of genus \( g \) and any \( n \)-ple of distinct point \( P_1, \ldots, P_n \in X_{\text{reg}} \). Obviously \( \Delta'(n, g) \geq \Delta(n, g) \).

**Question 2.** Is \( \Delta'(n, g) = \Delta(n, g) \)? Is \( \eta'(n, g) = \eta(n, g) \)?

**Remark 1.2.** Theorem 1.1 works verbatim for smooth points of singular curves and hence \( \Delta'(2, g) = \Delta(2, g) = 2g + 1 \).

2. **General results**

For any \( a = (a_1, \ldots, a_n) \in \mathbb{N}^n \) set \( \|a\| := a_1 + \cdots + a_n \). For any integer \( r \geq 0 \) set \( S(n, r) := \{a \in \mathbb{N}^n : \|a\| = r\} \) and \( D(n, r) := \{a \in \mathbb{N}^n : \|a\| \leq r\} \)

**Remark 2.1.** Let \( A \subset \mathbb{N}^n \) be a set closed with respect to \( \preceq \) such that \( A \supseteq \mathbb{N}^n \setminus D(n, r) \) for some \( r > 0 \). Then \( C(A) \subseteq D(n, r+1) \). In particular if \( P_1, \ldots, P_n \) are points on a smooth curve of genus \( g \), then \( \|a\| \leq 2g \) for all \( a \in C(P_1, \ldots, P_n) \). Since \( H(P_1, \ldots, P_n)^+ \subseteq \mathbb{N}^n \setminus D(n, g) \), we also have \( \|a\| \geq g + 1 \).

**Lemma 2.2.** Let \( A \subset \mathbb{N}^n \) be a subset closed with respect to \( \preceq \) and containing \( \mathbb{N}^n \setminus D(n, d-1) \). Then \( \sharp(C(A)) \leq \binom{n+d-1}{n-1} \) and \( \|a\| \leq d \) for each \( a \in C(A) \).

**Proof.** We have \( \sharp(S(n, d)) = \binom{n+d-1}{n-1} \). Therefore it is sufficient to find an injective map \( u : C(A) \to S(n, d) \). We have \( C(A) \subset D(n, d) \setminus \{(0, \ldots, 0)\} \). Fix \( a =
Proof. Riemann-Roch gives $\mathbb{N}^n \setminus D(n, 2g - 1) \subseteq H(P_1, \ldots, P_n)^+$. Apply Lemma 2.2. □

Proof of Theorem 1.1. The inequality follows from Lemma 2.2, because every degree $2g$ line bundle on $X$ is spanned and non-special. If $X$ is hyperelliptic and $P_1, P_2$ are distinct Weierstrass points, then $\sharp(H(P_1, P_2)^+) = 2g + 1$ by part (i) of Section 3. Take arbitrary $X, P_1, \ldots, P_n$ such that $\sharp(C(P_1, \ldots, P_n^+)) = (n+2g-1)$. Let $u: C(P_1, \ldots, P_n^+) \to R(n, 2g)$ be the map defined in the proof of Lemma 2.2. Since $u$ is injective, we get that $u$ is bijective. If $a = (a_1, \ldots, a_n) \in D(r, 2g)$ and $a_n = 0$ and $a = u(b)$, then $a = b$, i.e. $H(P_1, \ldots, P_n)^+ \cap \mathbb{N}^{n-1} \times \{0\} = \mathbb{N}^{n-1} \setminus D(n - 1, 2g - 1)$. In particular we get $2g - 1 \notin H(P_i)$, i.e. $\omega_X \cong O_X((2g - 2)P_1)$. We may exchange the role of the indices $1, \ldots, n$ and get in this way that $O_X((2g - 2)P_i) \cong \omega_X$ for all $i$.

(a) Assume $n = 2$. Take $a = ((2g - 2)P_1, 2P_2)$ and fix $b \in C(P_1, \ldots, P_n^+)$ such that $a = u(b)$. By the definition of the map $u$ either $b = (2g - 2, 0)$ or $b = (2g - 2, 1)$ or $b = (2g - 2, 2)$. Since $\omega_X \cong O_X((2g - 2)P_1)$ we have $(2g - 2, 0) \notin H(P_1, P_2)^+$. Riemann-Roch gives that $\omega_X(P_1)$ and $\omega_X(P_2)$ are not spanned and hence $(2g - 1, 0) \notin H(P_1, P_2)$ and $(2g - 2, 1) \notin H(P_1, P_2)^+$. Therefore $b = (2g - 2, 2)$. Take $a' = ((2g - 1)P_1, P_2)$ and fix $b' \in C(P_1, \ldots, P_n^+)$ such that $a' = u(b')$. By the definition of the map $u$ either $b' = (2g - 1, 0)$ or $b = (2g - 1, 1)$. Since $(2g - 1, 0) \notin H(P_1, P_2)$, then $b' = (2g - 1, 1)$. Since $(2g - 2, 2) \in C(P_1, P_2^+)$, we have $(2g - 3, 2) \notin H(P_1, P_2)^+$. Since $h^1(O_X((2g - 3)P_1 + 2P_2)) = 0$ we get that $O_X((2g - 3)P_1 + 2P_2)$ is not spanned. Therefore the line bundle $O_X((2g - 3)P_1 + 2P_2)$ is not spanned either at $P_1$ or at $P_2$, i.e. (Riemann-Roch) either $O_X((2g - 4)P_1 + 2P_2) \cong \omega_X$ or $O_X((2g - 3)P_1 + P_2) \cong \omega_X$. First assume $O_X((2g - 3)P_1 + P_2) \cong \omega_X$. Since $\omega_X \cong O_X((2g - 2)P_1)$, we get $O_X(P_1) \cong O_X(P_2)$, contradicting the assumptions $P_1 \neq P_2$ and $g > 0$. Now assume $O_X((2g - 4)P_1 + 2P_2) \cong \omega_X$, i.e. $O_X((2g - 4)P_1 + 2P_2) \cong O_X((2g - 2)P_1)$, i.e. $O_X(2P_1) \cong O_X(2P_2)$. Since $P_1 \neq P_2$, we get that $X$ is hyperelliptic and that $P_1, P_2$ are Weierstrass points of $X$.

(b) Now assume $n > 2$. We use induction on $n$. Since $H(P_1, \ldots, P_n)^+ \cap \mathbb{N}^{n-1} \times \{0\} = \mathbb{N}^{n-1} \setminus D(n - 1, 2g - 1)$, we get that $X$ is hyperelliptic and that $P_1, \ldots, P_{n-1}$ are Weierstrass points of $X$. Exchanging the role of $n$ and $n-1$ we get that $P_1$ is a Weierstrass point of $X$. If $n = 3$, then part (ii) of section 3 gives a contradiction. If $n \geq 4$, we get the contradiction taking $H(P_1, \ldots, P_n)^+ \cap \mathbb{N}^3 \times \{0, \ldots, 0\}$ and then quoting again part (ii) of section 3. □

The following example is called the non-special case, because it is the case $H(P_1, \ldots, P_n)^+ = H(P_1, \ldots, P_n)$. 
Example 2.4. Let $X$ be a smooth curve of genus $g$ and $P_1, \ldots, P_n \in X$ (we allow the case $P_i = P_j$ for some $i = j$). Assume that for all $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$ we have $h^0(\mathcal{O}_X(a_1 P_1 + \cdots + a_n P_n)) = \max\{|a| + 1 - g, 0\}$, i.e. $h^0(\mathcal{O}_X(a_1 P_1 + \cdots + a_n P_n)) = \max\{g - 1 - |a|, 0\}$. We have $H(P_1, \ldots, P_n) = H(P_1, \ldots, P_n) = \mathbb{N}^n \setminus D(n, g)$ and hence $C(P_1, \ldots, P_n, +) = S(n, g + 1)$. Therefore $\sharp(\mathcal{C}(P_1, \ldots, P_n, +)) = (\frac{g + n}{n - 1})$.

Proposition 2.5. We have $\eta(2, 2) = 3$ and the triples $(X, P_1, P_2)$ such that $\sharp(\mathcal{C}(P_1, P_2, +)) = 3$ are the ones with neither $P_1$ nor $P_2$ a Weierstrass point of $X$ and $\omega_X \cong \mathcal{O}_X(P_1 + P_2)$.

Proof. Fix any $X, P_1, P_2$ with $P_1 \neq P_2$. The set $\mathcal{C}(P_1, P_2, +)$ contains exactly one element $(a, 0)$ and one element $(0, b)$. Since $g = 2$, we have $3 \leq a \leq b$ and $3 \leq b \leq 4$. Since $(2, 2)$ cannot be obtained from these two elements using $\leq$, we get $\sharp(\mathcal{C}(P_1, P_2)) \geq 3$. Now assume $\sharp(\mathcal{C}(P_1, P_2)) \leq 3$. Since $\{(3, 1), (1, 3)\} \in H(P_1, P_2)$, we get $a = b = 3$ (i.e. neither $P_1$ nor $P_2$ is a Weierstrass point of $X$) and that $(1, 2) \notin H(P_1, P_2)$, i.e. $\mathcal{O}_X(2P_1 + P_2)$ is not spanned. Since $P_1$ is not a Weierstrass point, we have $h^0(\mathcal{O}_X(2P_1)) = 1$. Since $h^0(\mathcal{O}_X(2P_1 + P_2)) = 3$ and $\mathcal{O}_X(2P_1 + P_2)$ is not spanned, Riemann-Roch gives $\omega_X \cong \mathcal{O}_X(P_1 + P_2)$. Conversely, if neither $P_1$ nor $P_2$ a Weierstrass point of $X$ and $\omega_X \cong \mathcal{O}_X(P_1 + P_2)$, then $(3, 0) \in H(P_1, P_2)$, $(0, 3) \in H(P_1, P_2)$, $(2, 1) \notin H(P_1, P_2)$, $(1, 2) \notin H(P_1, P_2)$, and hence $\mathcal{C}(P_1, P_2, +) = \{(3, 0), (2, 2), (0, 3)\}$. \hfill \Box

3. Hyperelliptic curves and their Weierstrass points

Let $X$ be a smooth hyperelliptic curve of genus $g$ and $P_1, \ldots, P_n$ Weierstrass points of $X$ with $P_i \neq P_j$ for all $i \neq j$. Set $\delta(g, n) := \sharp(\mathcal{C}(P_1, \ldots, P_n, +))$. In this section we first show that the integer $\delta(n, g)$ does not depend from $X, P_1, \ldots, P_n$ (not even from the characteristic of the base field), but only from $g$ and $n$. But of course, we need to assume the existence of $X, P_1, \ldots, P_n$. In characteristic $\neq 2$ for any fixed genus $g \geq 2$ we may take as $n$ any integer $\leq 2g + 2$. In characteristic $2$ we need the assumption $n \leq g + 1$ and for some hyperelliptic curve even this bound is not achieved. We are in the set-up of \cite[§3]{1} with $a = 0$ and $b = c = 0$. In parts (i), (ii), (iii), (iv), (vii), (viii) we compute $\delta(n, g)$ when $2 \leq n \leq 4$. We also check $\delta(n, 2)$ for any $n \leq 6$ (parts (v) and (vi)). In each of these cases we also compute the integers $\sharp(\mathcal{C}(P_1, \ldots, P_n, +)) \cap S(n, x)$ for all $x$ and describe the sets $H(P_1, \ldots, P_n) \cap S(n, x)$ and $\mathcal{C}(P_1, \ldots, P_n, +) \cap S(n, x)$.

Remark 3.1. Let $X$ be a hyperelliptic curve of genus $g$ and $P_1, \ldots, P_n$ Weierstrass points of $X$. Fix $(a_1, \ldots, a_n) \in \mathbb{N}^n$. We have $h^1(\mathcal{O}_X(a_1 P_1 + \cdots + a_n P_n)) = \max\{0, g - \sum i |(a_i + 1)/2|\}$ and $h^0(\mathcal{O}_X(a_1 P_1 + \cdots + a_n P_n)) = a_1 + \cdots + a_n + 1 - g + h^1(\mathcal{O}_X(a_1 P_1 + \cdots + a_n P_n))$. Therefore $h^1(\mathcal{O}_X(a_1 P_1 + \cdots + a_n P_n)) = 0$ if and only if $\sum i |(a_i + 1)/2| \geq g$. Since $\mathcal{O}_X(a_1 P_1 + \cdots + a_n P_n)$ is spanned outside $P_1, \ldots, P_n$ we also get that $(a_1, \ldots, a_n) \in H(P_1, \ldots, P_n)$ if and only if either $\sum i |(a_i + 1)/2| \geq g + 1$ or $\sum i |(a_i + 1)/2| = g$ and each $a_i$ is even. Note that in the latter case $\|a\|$ is even and $\|a\| \geq 2g$. By Remark 2.1 we need to test only $\sum i |(a_i + 1)/2| \geq g + 1$, except in the case $\|a\| 2g$ and all $a_i$ even.
Claim: Assume that each $a_i$ is even and that $\|a\| = 2g$. Then we have $a \in \mathcal{C}(P_1, \ldots, P_n, +)$.

Proof of the Claim: Remark 3.1 gives $a \in H(P_1, \ldots, P_n)^+$. Assume the existence of $b = (b_1, \ldots, b_n) \in H(P_1, \ldots, P_n)^+$ such that $a$ is obtained from $b$ using $\leq$, i.e. $a_i \geq b_i$ for all $i$ and there is $h \in \{1, \ldots, n\}$ with $a_h > b_h$. Since $H(P_1, \ldots, P_n)^+$ is closed for $\leq$, using $\|a\| - \|b\| - 1$ intermediate steps (if necessary) we reduce to the case $\|b\| = 2g - 1$, i.e. $b_i = a_i$ for all $i \neq h$ and $b_h = a_h - 1$. Since $b_h \in \mathbb{N}$ and $b_h$ is odd, then $b_h > 0$. Set $c_i := a_i$ if $i \neq h$ and $c_h = b_h - 1$. Set $c := (c_1, \ldots, c_n)$. Since $\|c\| = 2g - 2$ and each $c_j$ is even, then $c_1 P_1 + \cdots + c_n P_n \leq 1$. Riemann-Roch gives that $P_h$ is a base point of the line bundle $\mathcal{O}_X(b_1 P_1 + \cdots + b_n P_n)$, a contradiction.

Remark 3.2. By induction on $n$ using Remark 3.1 we first get the explicit description of the sets $H(P_1, \ldots, P_n)^+ \cap S(n, r)$ for any $r$ and then that the integer $\sharp(\mathcal{C}(P_1, \ldots, P_n, +))$ depends only on $n$ and $g$. We didn’t found a closed formula for it, but we list it (with proofs) when $n \leq 4$.

(i) We have $\delta(2, g) = 2g + 1$.

Proof. By Remark 3.1 we have $(a_1, a_2) \in H(P_1, P_2)^+$ if and only if $a_1 + a_2 \geq 2g$. Hence $\mathcal{C}(P_1, P_2) = \{(c, 2g - c)\}_{0 \leq c \leq 2g}$. Therefore $\delta(g, 2) = 2g + 1$. □

(ii) We have $\delta(3, g) = g^2 + 4g + 1$ and $(a_1, a_2, a_3) \in \mathcal{C}(P_1, P_2, P_3)$ if and only if either each $a_i$ is odd and $a_1 + a_2 + a_3 = 2g - 1$ or each $a_i$ is even and $a_1 + a_2 + a_3 = 2g$ or $a_h = 0$ for one index, the other two $a_i$ are odd and $a_1 + a_2 + a_3 = 2g$.

Proof. Fix $(a_1, a_2, a_3) \in \mathcal{C}(P_1, P_2, P_3)$. Remark 3.1 gives $a_1 + a_2 + a_3 \leq 2g$. Remark 3.1 gives $a_1 + a_2 + a_3 \geq 2g - 1$ and that equality holds if and only if each $a_i$ is odd (note that this is true even in the case $g = 2$ in which $g + 1 = 2g - 1$). We have $a_i = 2b_i + 1$ and $a_1 + a_2 + a_3 = 2g - 1$ with $b_i \in \mathbb{N}$ if and only if $b_1 + b_2 + b_3 = g - 2$. Therefore $\sharp(\mathcal{C}(P_1, P_2, P_3) \cap S(3, 2g - 1)) = \binom{3}{2}$. Fix $a = (a_1, a_2, a_3) \in \mathcal{C}(P_1, P_2, P_3) \cap S(3, 2g - 1)$. Using $\leq$ it gives the following elements $(a_1 + 1, a_2, a_3), (a_1, a_2 + 1, a_3)$ and $(a_1, a_2, a_3 + 1)$. Any such element, say $(c_1, c_2, c_3) = (a_1, a_2, a_3 + 1)$, uniquely recovers $a$, because its odd elements (here $a_1$ and $a_2$) are coordinates of $a$, while two of the coordinates of $a$ uniquely determine the third one, since $a_1 + a_2 + a_3 = 2g - 1$. Since $H(P_1, P_2, P_3)^+ \supset S(3, 2g)$ and $\sharp(S(3, 2g)) = \binom{2g + 2}{2}$, we get $\sharp(\mathcal{C}(P_1, P_2, P_3) \cap S(3, 2g)) = \binom{2g + 2}{2} - 3\binom{g}{2}$ and hence $\sharp(\mathcal{C}(P_1, P_2, P_3)) = \binom{2g + 2}{2} - 2\binom{g}{2} = g^2 + 4g + 1$. □

(iii) We have $\delta(4, g) = (2g^3 + 3g^2 + 13g + 6)/6$ for all $g \geq 5$.

Proof. Since $g \geq 5$, we have $g + 1 \leq 2g - 4$. Fix any $b = (b_1, b_2, b_3, b_4) \in \mathbb{N}^4$. By Remark 2.2 we have $b \notin \mathcal{C}(P_1, P_2, P_3, P_4, +)$ if $\|b\| > 2g$. Now assume $\|b\| = 2g - 3$. Since $n = 4$ is even, at least one entry of $b$ is even. Remark 3.1 gives $b \notin H(P_1, P_2, P_3, P_4)^+$. Therefore $H(P_1, P_2, P_3, P_4)^+ \cap S(4, x) = \emptyset$ for all $x < 2g - 2$. Now assume $\|b\| = 2g - 2$. Hence an even number of entries of $b$ is odd. By Remark 3.1 we have $b \in H(P_1, P_2, P_3, P_4)^+$ if and only if all $b_i$’s are odd. Writing $b_i = 2c_i + 1$ with $c_i \in \mathbb{N}$ and $\sum (2c_i + 1) = 2g - 2$ we get $\sharp(\mathcal{C}(P_1, P_2, P_3, P_4, +) \cap S(4, 2g - 2)) = \binom{g}{3}$. By Remark 3.1 $b \in H(P_1, P_2, P_3, P_4)^+ \cap$
$S(4, 2g - 1)$ if and only if $b_1 + b_2 + b_3 + b_4 = 2g - 1$ and exactly one entry of $b$ is even. Therefore a quadruple $b \in H(P_1, P_2, P_3, P_4) \cap S(4, 2g - 1)$ does not come from an element of $H(P_1, P_2, P_3, P_4)^+ \cap S(4, 2g - 2)$ using $\preceq$, i.e. adding $+1$ to exactly one entry of an element of $H(P_1, P_2, P_3, P_4)^+ \cap S(4, 2g - 2)$, if and only if its even entry is zero. Hence $\sharp(C(P_1, P_2, P_3, P_4^+) \cap S(4, 2g - 2)) = 4(\frac{g}{3}) = 2g^2 - 2g$. Now assume $b \in S(4, 2g)$. We have $S(4, 2g) \subset H(P_1, P_2, P_3, P_4)^+$. The quadruple $b$ has an even number of odd entries. The quadruple $b$ does not come from an element of $H(P_1, P_2, P_3, P_4)^+ \cap S(4, 2g - 1)$ using $\preceq$ if and only if either its 4 entries are even or if it has two entries odd and two entries zero. The first case happens for $(\frac{g+3}{3})$ $b$’s, while the second one happens for $6g$ $b$’s. Therefore $\delta(4, g) = (\frac{g}{3}) + 2g^2 - 2g + (\frac{g+3}{3}) + 6g = (2g^3 + 3g^2 + 13g + 6)/6$. 

(iv) We have $\delta(4, 2) = 26$ (this case does not exists in characteristic 2).

Proof. We have $g + 1 = 3$, $2g = 4$, $(2, 1, 0, 0) \notin H(P_1, P_2, P_3, P_4)^+$, $(3, 0, 0, 0) \notin H(P_1, P_2, P_3, P_4)^+$ and $(1, 1, 1, 0) \in H(P_1, P_2, P_3, P_4)^+$. Permuting the indices we get $\sharp(C(P_1, P_2, P_3, P_4^+) \cap S(4, 3)) = 4$. Take $a = (a_1, a_2, a_3, a_4) \in S(4, 4)$. We have $a \in H(P_1, P_2, P_3, P_4)^+$. The quadruple $a$ comes from $H(P_1, P_2, P_3, P_4)^+ \cap S(4, 3)$ using $\preceq$ if and only if at most one $a_i$ is zero. We get 4 elements of $C(P_1, P_2, P_3, P_4^+) \cap S(4, 4)$ with exactly 1 non-zero entry and $6 \times 3$ elements of $C(P_1, P_2, P_3, P_4^+) \cap S(4, 4)$ with exactly 2 non-zero entries. Therefore we have $\sharp(C(P_1, P_2, P_3, P_4^+) \cap S(4, 4)) = 22$. Hence $\delta(4, 2) = 26$. 

(v) We have $\delta(5, 2) = 45$ (this case does not exists in characteristic 2).

Proof. The set $H(P_1, \ldots, P_5)^+ \cap S(5, 3)$ is formed by all quintuples with exactly 3 ones and 2 zeroes as entries. Hence $a \in H(P_1, \ldots, P_5) \cap S(5, 4)$ does not come from $H(P_1, \ldots, P_5)^+ \cap S(5, 3)$ if and only if at least 3 of its entries are zero; we have $3(\frac{5}{3}) = 30$ elements of $S(5, 4)$ with exactly 3 zeroes as entries and 5 elements of $S(5, 4)$ with 4 zeroes. Therefore $\delta(5, 2) = 10 + 30 + 5 = 45$. 

(vi) We have $\delta(6, 2) = 71$ (this case does not exists in characteristic 2).

Proof. The set $H(P_1, \ldots, P_6)^+ \cap S(6, 3)$ is formed by all sextuples with exactly 3 ones and 3 zeroes as entries. Hence $a \in H(P_1, \ldots, P_6)^+ \cap S(6, 4)$ does not come from $H(P_1, \ldots, P_6)^+ \cap S(6, 3)$ if and only if at least 4 of its entries are zero; there are 6 elements of $S(6, 4)$ with 5 zero entries and $3(\frac{6}{2})$ elements of $S(6, 4)$ with exactly 4 zero entries. Therefore $\delta(6, 2) = 20 + 6 + 45 = 71$. 

(vii) We have $\delta(4, 3) = 51$.

Proof. We have $g + 1 = 4$ and $2g = 6$. By Remark 3.1 we have $(1, 1, 1, 1) \in H(P_1, P_2, P_3, P_4)^+$, $(4, 0, 0, 0) \notin H(P_1, P_2, P_3, P_4)^+$, $(3, 1, 0, 0) \notin H(P_1, P_2, P_3, P_4)^+$, $(2, 1, 1, 0) \notin H(P_1, P_2, P_3, P_4)^+$ and similar statements are true permuting the indices. Hence $H(P_1, P_2, P_3, P_4)^+ \cap S(4, 4) = \{(1, 1, 1, 1)\}$. By Remark 3.1 we have $a \in H(P_1, P_2, P_3, P_4)^+ \cap S(4, 5)$ if and only if $a$ has 3 odd entries (since $n$ is even, it cannot have all odd entries). If $a \in H(P_1, P_2, P_3, P_4)^+ \cap S(4, 5)$, then it comes from $(1, 1, 1, 1)$ using $\preceq$ if and only if it has no non non-zero entry. Therefore each element of $C(P_1, P_2, P_3, P_4^+) \cap S(4, 5)$ has as entries 1 zero, 2 ones and 1 three. Hence $\sharp(C(P_1, P_2, P_3, P_4^+) \cap S(4, 5)) = 12$. An element of $S(4, 6)$ comes
from an element of $H(P_1, P_2, P_3, P_4)^+ \cap S(4, 5)$ using $\leq$ if and only if it has either 4 odd entries or 2 odd entries and at least an even non-zero entry. Hence it is a corner if and only if either all its entries are even or it has two zero entries and two odd entries. Hence $\sharp(\mathcal{C}(P_1, P_2, P_3, P_4, ^+) \cap S(4, 6)) = \binom{6}{3} + 18$. Therefore $\sharp(\mathcal{C}(P_1, P_2, P_3, P_4, ^+)) = 51$. □

(viii) We have $\delta(4, 4) = 87$.

**Proof.** We have $g + 1 = 5$ and $2g = 8$. Fix $a = (a_1, a_2, a_3, a_4) \in \mathbb{N}^4$. Since $n = 4$ is even, every element of $S(4, 5)$ has at least one even entry. Remark 3.1 gives $H(P_1, P_2, P_3, P_4)^+ \cap S(4, 5) = \emptyset$. Remark 3.1 gives that $a \in H(P_1, P_2, P_3, P_4)^+ \cap S(4, 6)$ if and only if every $a_i$ is odd; writing $a_i = 2b_i + 1$ with $b_i \in \mathbb{N}$ we get that there are 4 such elements. $a \in H(P_1, P_2, P_3, P_4)^+ \cap S(4, 7)$ if and only if exactly one entry of $a$ is even; such a quadruple $a \in H(P_1, P_2, P_3, P_4)^+ \cap S(4, 7)$ does not come from $H(P_1, P_2, P_3, P_4)^+ \cap S(4, 6)$ using $\leq$ if and only if the even entry is zero; therefore $\sharp(\mathcal{C}(P_1, P_2, P_3, P_4, ^+) \cap S(4, 7)) = 4 \binom{1}{2} = 24$. An element of $S(4, 8)$ comes from $H(P_1, P_2, P_3, P_4, ^+) \cap S(4, 7)$ using $\leq$ if and only if either all its entries are even (there are $\binom{7}{3}$ such elements of $S(4, 8)$) or it has at least two zero entries and two odd entries (there are $6 \cdot 4 = 24$ such elements). Therefore $\delta(4, 4) = 4 + 24 + 35 + 24 = 87$. □

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**References**


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