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ON THE NON-SPECIAL PART OF THE WEIERSTRASS SEMIGROUPS OF n-POINTS OF A SMOOTH CURVE

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Abstract. Let X be a smooth curve of genus $g > 0$. For any $n \geq 2$ and any n distinct points $P_1, \dots, P_n \in X$ let $H(P_1, \dots, P_n)^+$ be the set of all $(a_1, \dots, a_n) \in \mathbb{N}^n$ such that $\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)$ is spanned and non-special. This set is closed by the operation $\preccurlyeq$, where $(a_1, \dots, a_n) \preccurlyeq (b_1, \dots, b_n)$ if and only if $a_i \leq b_i$ for all i. We raise some questions on the minimal cardinality of a subset of $H(P_1, \dots, P_n)^+$ generating it using $\preccurlyeq$ and compute this number in a few examples.

1. Introduction

Let X be a smooth curve of genus $g \geq 3$. Fix an integer $n > 0$ and $P_1, \dots, P_n \in X$ with $P_i \neq P_j$ for all $i \neq j$. Let $H(P_1, \dots, P_n) \subset \mathbb{N}^n$ be the set of all $(a_1, \dots, a_n) \in \mathbb{N}^n$ such that $\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)$ is spanned, i.e. such that there is rational function on X with $a_1P_1 + \cdots + a_nP_n$ as its divisors of poles. The set $H(P_1, \dots, P_n)$ is a semigroup for the componentwise addition $+ : \mathbb{N}^n \times \mathbb{N}^n \to \mathbb{N}^n$ and $G(P_1, \dots, P_n) := \mathbb{N}^n \setminus H(P_1, \dots, P_n)$ is a finite set. The semigroup $H(P_1, \dots, P_n)$ is called the Weierstrass semigroup of the points $P_1, \dots, P_n$, while the elements of $G(P_1, \dots, P_n)$ are called the gaps of $P_1, \dots, P_n$ ([1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [13], [14]). Define the following partial ordering $\preccurlyeq$ on $\mathbb{N}^n$. Write $(a_1, \dots, a_n) \preccurlyeq (b_1, \dots, b_n)$ if and only if $a_i \leq b_i$ for all i. Set $H(P_1, \dots, P_n)^+ := \{(a_1, \dots, a_n) \in H(P_1, \dots, P_n) : h^1(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)) = 0\}$. We have $a \in H(P_1, \dots, P_n)^+$ if and only if $b \in H(P_1, \dots, P_n)$ for all $b \in \mathbb{N}^n$ such that $b \succcurlyeq a$. Therefore $H(P_1, \dots, P_n)^+$ is a set closed for $\preccurlyeq$. The set $\mathbb{N}^n \setminus H(P_1, \dots, P_n)^+$ is finite, because every line bundle on g with degree $\geq 2g$ is spanned and non-special. If $n = 1$, then $H(P_1)^+$ is the set of all integers $\geq n_g + 1$, where n_g is the largest element of $G(P_1)$. A non-empty subset A is closed for $\preccurlyeq$ if and only $a + b \in A$ for every $a \in A$ and every $b \in \mathcal{N}^n$. Hence the union of $\{0\}$ and a set closed with respect to $\preccurlyeq$ is a semigroup. For each $S \subseteq \mathbb{N}^n$, $S \neq \emptyset$, there is a minimal subset $A[S]$ containing S and closed with respect to $\preccurlyeq$. Let $A \subseteq \mathbb{N}^n$ be a non-empty subset closed with respect to $\preccurlyeq$ and with $\mathbb{N}^n \setminus A$ finite. Let $\mathcal{C}(A)$ be a minimal set $S \subset A$ such that $A = A[S]$. The set $\mathcal{C}(A)$ exists, it is finite

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and it is unique, because the intersection of any family of subsets of $\mathbb{N}^n$ closed with respect to $\preccurlyeq$ is either empty or a set closed with respect to $\preccurlyeq$. We will say that $\mathcal{C}(A)$ is the set of all *corners* of A. If $A = H(P_1, \dots, P_n)^+$ for some curve X and some $P_1, \dots, P_n \in X$, then set $\mathcal{C}(P_1, \dots, P_n, ^+) := \mathcal{C}(H(P_1, \dots, P_n)^+)$. Let $\Delta(n, g)$ (resp. $\eta(n, g)$) denote the maximal (resp. minimal) cardinality of $\mathcal{C}(P_1, \dots, P_n, ^+)$ among all $X, P_1, \dots, P_n$ with X a smooth curve of genus g and $P_1, \dots, P_n$ distinct points of X.

Now assume that X is a hyperelliptic curve of genus $g \geq 2$ and that $P_1, \dots, P_n$ are distinct Weierstrass points of X. It is easy to check that the integer $\delta(n, g) := \sharp(\mathcal{C}(P_1, \dots, P_n, ^+))$ depends only from n and g, not even on the characteristic of the algebraically closed base field $\mathbb{K}$ (Remark 3.2). Of course, the integer $\delta(n, g)$ is defined only if a genus g hyperelliptic curve has at least n Weierstrass points, i.e. if $n \leq 2g + 2$ (case $\mathrm{char}(\mathbb{K}) \neq 2$) or $n \leq g + 1$ (case $\mathrm{char}(\mathbb{K}) = 2$).

Question 1. Is $\Delta(n, g) = \delta(n, g)$ if $g \gg n$?

Question 1 is true (for any $g \geq 2$) when $n = 2$. Indeed we prove the following easy result.

Theorem 1.1. *For any smooth curve X of genus $g \geq 2$ and n distinct points $P_1, \dots, P_n$ we have $\sharp(\mathcal{C}(P_1, \dots, P_n, +) \leq \binom{n+2g-1}{n-1}$ and equality holds if and only if $n = 2$, X is a hyperelliptic curve and P_1, P_2 are distinct Weierstrass points of X.*

Let X be an integral projective curve with arithmetic genus $g := p_a(X) \geq 2$. We may define without any modification $H(P_1, \dots, P_n)$ and $H(P_1, \dots, P_n)^+$ and hence $\mathcal{C}(P_1, \dots, P_n, ^+)$ if we only take distinct smooth points of X, i.e. we impose that $P_i \in X_{reg}$ for all i. Let $\Delta'(n, g)$ (resp. $\eta'(n, g)$) be the the maximal (minimal) integer $\sharp(\mathcal{C}(P_1, \dots, P_n)$ among all integra curves X of genus g and any n-ple of distinct point $P_1, \dots, P_n \in X_{reg}$. Obviously $\Delta'(n, g) \geq \Delta(n, g)$.

Question 2. Is $\Delta'(n, g) = \Delta(n, g)$? Is $\eta'(n, g) = \eta(n, g)$?

Remark 1.2. Theorem 1.1 works verbatim for smooth points of singular curves and hence $\Delta'(2, g) = \Delta(2, g) = 2g + 1$.

2. General results

For any $a = (a_1, \dots, a_n) \in \mathbb{N}^n$ set $\|a\| := a_1 + \cdots + a_n$. For any integer $r \geq 0$ set $S(n, r) := \{a \in \mathbb{N}^n : \|a\| = r\}$ and $D(n, r) := \{a \in \mathbb{N}^n : \|a\| \leq r\}$

Remark 2.1. Let $A \subset \mathbb{N}^n$ be a set closed with respect to $\preccurlyeq$ and such that $A \supseteq \mathbb{N}^n \setminus D(n, r)$ for some $r > 0$. Then $\mathcal{C}(A) \subseteq D(n, r + 1)$. In particular if $P_1, \dots, P_n$ are points on a smooth curve of genus g, then $\|a\| \leq 2g$ for all $a \in \mathcal{C}(P_1, \dots, P_n)$. Since $H(P_1, \dots, P_n)^+ \subseteq \mathbb{N}^n \setminus D(n, g)$, we also have $\|a\| \geq g + 1$.

Lemma 2.2. *Let $A \subset \mathbb{N}^n$ be a subset closed with respect to $\preccurlyeq$ and containing $\mathbb{N}^n \setminus D(n, d - 1)$. Then $\sharp(\mathcal{C}(A)) \leq \binom{n+d-1}{n-1}$ and $\|a\| \leq d$ for each $a \in \mathcal{C}(A)$.*

Proof. We have $\sharp(S(n, d)) = \binom{n+d-1}{n-1}$. Therefore it is sufficient to find an injective map $u : \mathcal{C}(A) \to S(n, d)$. We have $\mathcal{C}(A) \subset D(n, d) \setminus \{(0, \dots, 0)\}$. Fix $a =$

$(a_1, \dots, a_n) \in \mathcal{C}(H^+)$. Set $u(a) = (a_1, \dots, a_n + d - \|a\|)$. Take $b = (b_1, \dots, b_n)$ and assume $u(a) = u(b)$. We get $a_i = b_i$ for all $i \neq b$. Hence either $a \preccurlyeq b$ (case $a_n \leq b_n$) or $b \preccurlyeq a$ (case $a_n \geq b_n$). Since $a, b \in \mathcal{C}(A)$, we get $a = b$. I □

Proposition 2.3. *Fix a smooth curve X of genus $g \geq 2$ and n distinct points $P_1, \dots, P_n$ such that there is no $(a_1, \dots, a_n) \in \mathbb{N}^n$ with $\omega_X \cong \mathcal{O}_C(\sum a_i P_i)$. Then $\sharp(\mathcal{C}(P_1, \dots, P_n,^+)) \leq \binom{n+2g-2}{n-1}$*

Proof. Riemann-Roch gives $\mathbb{N}^n \setminus D(n, 2g-1) \subseteq H(P_1, \dots, P_n)^+$. Apply Lemma 2.2. □

Proof of Theorem 1.1. The inequality follows from Lemma 2.2, because every degree $2g$ line bundle on X is spanned and non-special. If X is hyperelliptic and P_1, P_2 are distinct Weierstrass points, then $\sharp(H(P_1, P_2)^+) = 2g + 1$ by part (i) of Section 3. Take arbitrary $X, P_1, \dots, P_n$ such that $\sharp(\mathcal{C}(P_1, \dots, P_n,^+)) = \binom{n+2g-1}{n-1}$. Let $u : \mathcal{C}(P_1, \dots, P_n,^+) \to R(n, 2g)$ be the map defined in the proof of Lemma 2.2. Since u is injective, we get that u is bijective. If $a = (a_1, \dots, a_n) \in D(r, 2g)$ and $a_n = 0$ and $a = u(b)$, then $a = b$, i.e. $H(P_1, \dots, P_n)^+ \cap \mathbb{N}^{n-1} \times \{0\} = \mathbb{N}^{n-1} \setminus D(n-1, 2g-1)$. In particular we get $2g - 1 \notin H(P_1)$, i.e. $\omega_X \cong \mathcal{O}_X((2g-2)P_1)$. We may exchange the role of the indices $1, \dots, n$ and get in this way that $\mathcal{O}_X((2g-2)P_i) \cong \omega_X$ for all i.

(a) Assume $n = 2$. Take $a = ((2g-2)P_1, 2P_2)$ and fix $b \in \mathcal{C}(P_1, \dots, P_n,^+)$ such that $a = u(b)$. By the definition of the map u either $b = (2g-2, 0)$ or $b = (2g-2, 1)$ or $b = (2g-2, 2)$. Since $\omega_X \cong \mathcal{O}_X((2g-2)P_1)$ we have $(2g-2, 0) \notin H(P_1, P_2)^+$. Riemann-Roch gives that $\omega_X(P_1)$ and $\omega_X(P_2)$ are not spanned and hence $(2g-1, 0) \notin H(P_1, P_2)$ and $(2g-2, 1) \notin H(P_1, P_2)^+$. Therefore $b = (2g-2, 2)$. Take $a' = ((2g-1)P_1, P_2)$ and fix $b' \in \mathcal{C}(P_1, \dots, P_n,^+)$ such that $a' = u(b')$. By the definition of the map u either $b' = (2g-1, 0)$ or $b = (2g-1, 1)$. Since $(2g-1, 0) \notin H(P_1, P_2)$, then $b' = (2g-1, 1)$. Since $(2g-2, 2) \in \mathcal{C}(P_1, P_2,^+)$, we have $(2g-3, 2) \notin H(P_1, P_2)^+$- Since $h^1(\mathcal{O}_X((2g-3)P_1 + 2P_2)) = 0$ we get that $\mathcal{O}_X((2g-3)P_1 + 2P_2)$ is not spanned. Therefore the line bundle $\mathcal{O}_X((2g-3)P_1 + 2P_2)$ is not spanned either at P_1 or at P_2, i.e. (Riemann-Roch) either $\mathcal{O}_X((2g-4)P_1 + 2P_2) \cong \omega_X$ or $\mathcal{O}_X((2g-3)P_1 + P_2) \cong \omega_X$. First assume $\mathcal{O}_X((2g-3)P_1 + P_2) \cong \omega_X$. Since $\omega_X \cong \mathcal{O}_X((2g-2)P_1)$, we get $\mathcal{O}_X(P_1) \cong \mathcal{O}_X(P_2)$, contradicting the assumptions $P_1 \neq P_2$ and $g > 0$. Now assume $\mathcal{O}_X((2g-4)P_1 + 2P_2) \cong \omega_X$, i.e. $\mathcal{O}_X((2g-4)P_1 + 2P_2) \cong \mathcal{O}_X((2g-2)P_1)$, i.e. $\mathcal{O}_X(2P_1) \cong \mathcal{O}_X(2P_2)$. Since $P_1 \neq P_2$, we get that X is hyperelliptic and that P_1, P_2 are Weierstrass points of X.

(b) Now assume $n > 2$. We use induction on n. Since $H(P_1, \dots, P_n)^+ \cap \mathbb{N}^{n-1} \times \{0\} = \mathbb{N}^{n-1} \setminus D(n-1, 2g-1)$, we get that X is hyperelliptic and that $P_1, \dots, P_{n-1}$ are Weierstrass points of X. Exchanging the role of n and $n-1$ we get that P_n is a Weierstrass point of X. If $n = 3$, then part (ii) of section 3 gives a contradiction. If $n \geq 4$, we get the contradiction taking $H(P_1, \dots, P_n)^+ \cap \mathbb{N}^3 \times \{(0, \dots, 0)\}$ and then quoting again part (ii) of section 3. □

The following example is called the *non-special* case, because it is the case $H(P_1, \dots, P_n)^+ = H(P_1, \dots, P_n)$.

Example 2.4. Let X be a smooth curve of genus g and $P_1, \dots, P_n \in X$ (we allow the case $P_i = P_j$ for some $i = j$). Assume that for all $a = (a_1, \dots, a_n) \in \mathbb{N}^n$ we have $h^0(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)) = \max\{\|a\| + 1 - g, 0\}$, i.e. $h^1(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)) = \max\{g - 1 - \|a\|, 0\}$. We have $H(P_1, \dots, P_n) = H(P_1, \dots, P_n)^+ = \mathbb{N}^n \setminus D(n, g)$ and hence $\mathcal{C}(P_1, \dots, P_n, ^+) = S(n, g+1)$. Therefore $\sharp(\mathcal{C}(P_1, \dots, P_n, ^+)) = \binom{g+n}{n-1}$.

Proposition 2.5. *We have $\eta(2, 2) = 3$ and the triples (X, P_1, P_2) such that $\sharp(\mathcal{C}(P_1, P_2, ^+)) = 3$ are the ones with neither P_1 nor P_2 a Weierstrass point of X and $\omega_X \cong \mathcal{O}_X(P_1 + P_2)$.*

Proof. Fix any X, P_1, P_2 with $P_1 \neq P_2$. The set $\mathcal{C}(P_1, P_2, ^+)$ contains exactly one element $(a, 0)$ and one element $(0, b)$. Since $g = 2$, we have $3 \leq a \leq b$ and $3 \leq b \leq 4$. Since $(2, 2)$ cannot be obtained from these two elements using $\preccurlyeq$, we get $\sharp(\mathcal{C}(P_1, P_2)) \geq 3$. Now assume $\sharp(\mathcal{C}(P_1, P_2)) \leq 3$. Since $\{(3, 1), (1, 3)\} \in H(P_1, P_2)^+$, we get $a = b = 3$ (i.e. neither P_1 nor P_2 is a Weierstrass point of X) and that $(1, 2) \notin H(P_1, P_2)^+$, i.e. $\mathcal{O}_X(2P_1 + P_2)$ is not spanned. Since P_1 is not a Weierstrass point, we have $h^0(\mathcal{O}_X(2P_1)) = 1$. Since $h^0(\mathcal{O}_X(2P_1 + P_2)) = 3$ and $\mathcal{O}_X(2P_1+P_2)$ is not spanned, Riemann-Roch gives $\omega_X \cong \mathcal{O}_X(P_1+P_2)$. Conversely, if neither P_1 nor P_2 a Weierstrass point of X and $\omega_X \cong \mathcal{O}_X(P_1 + P_2)$, then $(3, 0) \in H(P_1, P_2)^+$, $(0, 3) \in H(P_1, P_2)^+$, $(2, 1) \notin H(P_1, P_2)^+$, $(1, 2) \notin H(P_1, P_2)^+$ and hence $\mathcal{C}(P_1, P_2, ^+) = \{(3, 0), (2, 2), (0, 3)\}$. □

3. Hyperelliptic curves and their Weierstrass points

Let X be a smooth hyperelliptic curve of genus g and $P_1, \dots, P_n$ Weierstrass points of X with $P_i \neq P_j$ for all $i \neq j$. Set $\delta(g, n) := \sharp(\mathcal{C}(P_1, \dots, P_n, ^+))$. In this section we first show that the integer $\delta(n, g)$ does not depend from $X, P_1, \dots, P_n$ (not even from the characteristic of the base field), but only from g and n. But of course, we need to assume the existence of $X, P_1, \dots, P_n$. In characteristic $\neq 2$ for any fixed genus $g \geq 2$ we may take as n any integer $\leq 2g+2$. In characteristic 2 we need the assumption $n \leq g + 1$ and for some hyperelliptic curve even this bound is not achieved. We are in the set-up of [1, §3] with $a = 0$ and $b = c = 0$. In parts (i), (ii), (iii), (iv), (vii), (viii) we compute $\delta(n, g)$ when $2 \leq n \leq 4$. We also check $\delta(n, 2)$ for any $n \leq 6$ (parts (v) and (vi)). In each of these cases we also compute the integers $\sharp(C(P_1, \dots, P_n, ^+)) \cap S(n, x)$ for all x and describe the sets $H(P_1, \dots, P_n)^+ \cap S(n, x)$ and $\mathcal{C}(P_1, \dots, P_n, ^+) \cap S(n, x)$.

Remark 3.1. Let X be a hyperelliptic curve of genus g and $P_1, \dots, P_n$ Weierstrass points of X. Fix $(a_1, \dots, a_n) \in \mathbb{N}^n$. We have $h^1(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)) = \max\{0, g - \sum_i \lfloor (a_i + 1)/2 \rfloor\}$ and $h^0(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)) = a_1 + \cdots + a_n + 1 - g + h^1(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n))$. Therefore $h^1(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)) = 0$ if and only if $\sum_{\lfloor} (a_i + 1)/2 \rfloor \geq g$. Since $\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)$ is spanned outside $P_1, \dots, P_n$ we also get that $(a_1, \dots, a_n) \in H(P_1, \dots, P_n)^+$ if and only if either $\sum_i \lfloor (a_i + 1)/2 \rfloor \geq g + 1$ or $\sum_i \lfloor (a_i + 1)/2 \rfloor = g$ and each a_i is even. Note that in the latter case $\|a\|$ is even and $\|a\| \geq 2g$. By Remark 2.1 we need to test only $\sum_i \lfloor (a_i + 1)/2 \rfloor \geq g + 1$, except in the case $\|a\| 2g$ and all a_i even.

Claim : Assume that each a_i is even and that $\|a\| = 2g$. Then we have $a \in \mathcal{C}(P_1, \dots, P_n, ^+)$.

Proof of the Claim: Remark 2.1 gives $a \in H(P_1, \dots, P_n)^+$. Assume the existence of $b = (b_1, \dots, b_n) \in H(P_1, \dots, P_n)^+$ such that a is obtained from b using $\preccurlyeq$, i.e. $a_i \geq b_i$ for all i and there is $h \in \{1, \dots, n\}$ with $a_h > b_h$. Since $H(P_1, \dots, P_n)^+$ is closed for $\preccurlyeq$, using $\|a\| - \|b\| - 1$ intermediate steps (if necessary) we reduce to the case $\|b\| = 2g - 1$, i.e. $b_i = a_i$ for all $i \neq h$ and $b_h = a_h - 1$. Since $b_h \in \mathbb{N}$ and b_h is odd, then $b_h > 0$. Set $c_i := a_i$ if $i \neq h$ and $c_h = b_h - 1$. Set $c := (c_1, \dots, c_n)$. Since $\|c\| = 2g - 2$ and each c_j is even, then $c_1P_1 + \dots + c_nP_n \in |\omega_X|$. Riemann-Roch gives that P_h is a base point of the line bundle $\mathcal{O}_X(b_1P_1 + \dots + b_nP_n)$, a contradiction.

Remark 3.2. By induction on n using Remark 3.1 we first get the explicit description of the sets $H(P_1, \dots, P_n)^+ \cap S(n, r)$ for any r and then that the integer $\sharp(\mathcal{C}(P_1, \dots, P_n, ^+))$ depends only on n and g. We didn't found a closed formula for it, but we list it (with proofs) when $n \leq 4$.

(i) We have $\delta(2, g) = 2g + 1$.

Proof. By Remark 3.1 we have $(a_1, a_2) \in H(P_1, P_2)^+$ if and only if $a_1 + a_2 \geq 2g$. Hence $\mathcal{C}(P_1, P_2) = \{(c, 2g - c)\}_{0 \leq c \leq 2g}$. Therefore $\delta(g, 2) = 2g + 1$. □

(ii) We have $\delta(3, g) = g^2 + 4g + 1$ and $(a_1, a_2, a_3) \in \mathcal{C}(P_1, P_2, P_3)$ if and only if either each a_i is odd and $a_1 + a_2 + a_3 = 2g - 1$ or each a_i is even and $a_1 + a_2 + a_3 = 2g$ or $a_h = 0$ for one index, the other two a_i are odd and $a_1 + a_2 + a_3 = 2g$.

Proof. Fix $(a_1, a_2, a_3) \in \mathcal{C}(P_1, P_2, P_3)$. Remark 2.1 gives $a_1 + a_2 + a_3 \leq 2g$. Remark 3.1 gives $a_1 + a_2 + a_3 \geq 2g - 1$ and that equality holds if and only if each a_i is odd (note that this is true even in the case $g = 2$ in which $g + 1 = 2g - 1$). We have $a_i = 2b_i + 1$ and $a_1 + a_2 + a_3 = 2g - 1$ with $b_i \in \mathbb{N}$ if and only if $b_1 + b_2 + b_3 = g - 2$. Therefore $\sharp(\mathcal{C}(P_1, P_2, P_3) \cap S(3, 2g - 1)) = \binom{g}{2}$. Fix $a = (a_1, a_2, a_3) \in \mathcal{C}(P_1, P_2, P_3) \cap S(3, 2g - 1)$. Using $\preccurlyeq$ it gives the following elements $(a_1 + 1, a_2, a_3)$, $(a_1, a_2 + 1, a_3)$ and $(a_1, a_2, a_3 + 1)$. Any such element, say $(c_1, c_2, c_3) = (a_1, a_2, a_3 + 1)$, uniquely recovers a, because its odd elements (here a_1 and a_2) are coordinates of a, while two of the coordinates of a uniquely determine the third one, since $a_1 + a_2 + a_3 = 2g - 1$. Since $H(P_1, P_2, P_3)^+ \supset S(3, 2g)$ and $\sharp(S(3, 2g)) = \binom{2g+2}{2}$, we get $\sharp(\mathcal{C}(P_1, P_2, P_3) \cap S(3, 2g)) = \binom{2g+2}{2} - 3\binom{g}{2}$ and hence $\sharp(\mathcal{C}(P_1, P_2, P_3)) = \binom{2g+2}{2} - 2\binom{g}{2} = g^2 + 4g + 1$. □

(iii) We have $\delta(4, g) = (2g^3 + 3g^2 + 13g + 6)/6$ for all $g \geq 5$.

Proof. Since $g \geq 5$, we have $g + 1 \leq 2g - 4$. Fix any $b = (b_1, b_2, b_3, b_4) \in \mathbb{N}^4$. By Remark 2.2 we have $b \notin \mathcal{C}(P_1, P_2, P_3, P_4, ^+)$ if $\|b\| > 2g$. Now assume $\|b\| = 2g - 3$. Since $n = 4$ is even, at least one entry of b is even. Remark 3.1 gives $b \notin H(P_1, P_2, P_3, P_4)^+$. Therefore $H(P_1, P_2, P_3, P_4)^+ \cap S(4, x) = \emptyset$ for all $x < 2g - 2$. Now assume $\|b\| = 2g - 2$. Hence an even number of entries of b is odd. By Remark 3.1 we have $b \in H(P_1, P_2, P_3, P_4)^+$ if and only if all b_i's are odd. Writing $b_i = 2c_i + 1$ with $c_i \in \mathbb{N}$ and $\sum_i (2c_i + 1) = 2g - 2$ we get $\sharp(\mathcal{C}(P_1, P_2, P_3, P_4, ^+) \cap S(4, 2g - 2)) = \binom{g}{3}$. By Remark 3.1 $b \in H(P_1, P_2, P_3, P_4)^+ \cap$

$S(4, 2g-1)$ if and only if $b_1+b_2+b_3+b_4 = 2g-1$ and exactly one entry of b is even. Therefore a quadruple $b \in H(P_1, P_2, P_3, P_4)^+ \cap S(4, 2g-1)$ does not come from an element of $H(P_1, P_2, P_3, P_4)^+ \cap S(4, 2g-2)$ using $\preccurlyeq$, i.e. adding $+1$ to exactly one entry of an element of $H(P_1, P_2, P_3, P_4)^+ \cap S(4, 2g-2)$, if and only if its even entry is zero. Hence $\sharp(\mathcal{C}(P_1, P_2, P_3, P_4,^+) \cap S(4, 2g-2)) = 4\binom{g}{2} = 2g^2-2g$. Now assume $b \in S(4, 2g)$. We have $S(4, 2g) \subset H(P_1, P_2, P_3, P_4)^+$. The quadruple b has an even number of odd entries. The quadruple b does not come from an element of $H(P_1, P_2, P_3, P_4)^+ \cap S(4, 2g-1)$ using $\preccurlyeq$ if and only if either its 4 entries are even or if it has two entries odd and two entries zero. The first case happens for $\binom{g+3}{3}$ b's, while the second one happens for $6g$ b's. Therefore $\delta(4, g) = \binom{g}{3} + 2g^2 - 2g + \binom{g+3}{3} + 6g = (2g^3+3g^2+13g+6)/6$. □

(iv) We have $\delta(4, 2) = 26$ (this case does not exists in characteristic 2).

Proof. We have $g+1 = 3$, $2g = 4$, $(2, 1, 0, 0) \notin H(P_1, P_2, P_3, P_4)^+$, $(3, 0, 0, 0) \notin H(P_1, P_2, P_3, P_4)^+$ and $(1, 1, 1, 0) \in H(P_1, P_2, P_3, P_4)^+$. Permuting the indices we get $\sharp(\mathcal{C}(P_1, P_2, P_3, P_4,^+) \cap S(4, 3)) = 4$. Take $a = (a_1, a_2, a_3, a_4) \in S(4, 4)$. We have $a \in H(P_1, P_2, P_3, P_4)^+$. The quadruple a comes from $H(P_1, P_2, P_3, P_4)^+ \cap S(4, 3)$ using $\preccurlyeq$ if and only if at most one a_i is zero. We get 4 elements of $\mathcal{C}(P_1, P_2, P_3, P_4,^+) \cap S(4, 4)$ with exactly 1 non-zero entry and 6×3 elements of $\mathcal{C}(P_1, P_2, P_3, P_4,^+) \cap S(4, 4)$ with exactly 2 non-zero entries. Therefore we have $\sharp(\mathcal{C}(P_1, P_2, P_3, P_4,^+) \cap S(4, 4)) = 22$. Hence $\delta(4, 2) = 26$. □

(v) We have $\delta(5, 2) = 45$ (this case does not exists in characteristic 2).

Proof. The set $H(P_1, \dots, P_5)^+ \cap S(5, 3)$ is formed by all quintuples with exactly 3 ones and 2 zeroes as entries. Hence $a \in H(P_1, \dots, P_5)^+ \cap S(5, 4)$ does not come from $H(P_1, \dots, P_5)^+ \cap S(5, 3)$ if and only if at least 3 of its entries are zero; we have $3\binom{5}{3} = 30$ elements of $S(5, 4)$ with exactly 3 zeroes as entries and 5 elements of $S(5, 4)$ with 4 zeroes. Therefore $\delta(5, 2) = 10 + 30 + 5 = 45$. □

(vi) We have $\delta(6, 2) = 71$ (this case does not exists in characteristic 2).

Proof. The set $H(P_1, \dots, P_6)^+ \cap S(6, 3)$ is formed by all sextuples with exactly 3 ones and 3 zeroes as entries. Hence $a \in H(P_1, \dots, P_6)^+ \cap S(5, 4)$ does not come from $H(P_1, \dots, P_6)^+ \cap S(6, 3)$ if and only if at least 4 of its entries are zero; there are 6 elements of $S(6, 4)$ with 5 zero entries and $3\binom{6}{2}$ elements of $S(6, 4)$ with exactly 4 zero entries. Therefore $\delta(6, 2) = 20 + 6 + 45 = 71$. □

(vii) We have $\delta(4, 3) = 51$.

Proof. We have $g+1 = 4$ and $2g = 6$. By Remark 3.1 we have $(1, 1, 1, 1) \in H(P_1, P_2, P_3, P_4)^+$, $(4, 0, 0, 0) \notin H(P_1, P_2, P_3, P_4)^+$, $(3, 1, 0, 0) \notin H(P_1, P_2, P_3, P_4)^+$, $(2, 1, 1, 0) \notin H(P_1, P_2, P_3, P_4)^+$ and similar statements are true permuting the indices. Hence $H(P_1, P_2, P_3, P_4)^+ \cap S(4, 4) = \{(1, 1, 1, 1)\}$. By Remark 3.1 we have $a \in H(P_1, P_2, P_3, P_4)^+ \cap S(4, 5)$ if and only if a has 3 odd entries (since n is even, it cannot have all odd entries). If $a \in H(P_1, P_2, P_3, P_4)^+ \cap S(4, 5)$, then it comes from $(1, 1, 1, 1)$ using $\preccurlyeq$ if and only if it has no non non-zero entry. Therefore each element of $\mathcal{C}(P_1, P_2, P_3, P_4,^+) \cap S(4, 5)$ has as entries 1 zero, 2 ones and 1 three. Hence $\sharp(\mathcal{C}(P_1, P_2, P_3, P_4,^+) \cap S(4, 5)) = 12$. An element of $S(4, 6)$ comes

from an element of $H(P_1,P_2,P_3,P_4)^+ \cap S(4,5)$ using $\preccurlyeq$ if and only if it has either 4 odd entries or 2 odd entries and at least an even non-zero entry. Hence it is a corner if and only if either all its entries are even or it has two zero entries and two odd entries. Hence $\sharp(\mathcal{C}(P_1,P_2,P_3,P_4,^+) \cap S(4,6)) = \binom{6}{3} + 18$. Therefore $\sharp(\mathcal{C}(P_1,P_2,P_3,P_4,^+)) = 51$. □

(viii) We have $\delta(4,4) = 87$.

Proof. We have $g+1 = 5$ and $2g = 8$. Fix $a = (a_1,a_2,a_3,a_4) \in \mathbb{N}^4$. Since $n = 4$ is even, every element of $S(4,5)$ has at least one even entry. Remark 3.1 gives $H(P_1,P_2,P_3,P_4)^+ \cap S(4,5) = \emptyset$. Remark 3.1 give that $a \in H(P_1,P_2,P_3,P_4)^+ \cap S(4,6)$ if and only if every a_i is odd; writing $a_i = 2b_i + 1$ with $b_i \in \mathbb{N}$ we get that there are 4 such elements. $a \in H(P_1,P_2,P_3,P_4)^+ \cap S(4,7)$ if and only if exactly one entry of a is even; such a quadruple $a \in H(P_1,P_2,P_3,P_4)^+ \cap S(4,7)$ does not come from $H(P_1,P_2,P_3,P_4)^+ \cap S(4,6)$ using $\preccurlyeq$ if and only if the even entry is zero; therefore $\sharp(\mathcal{C}(P_1,P_2,P_3,P_4,^+) \cap S(4,7)) = 4\binom{4}{2} = 24$. An element of $S(4,8)$ comes from $H(P_1,P_2,P_3,P_4)^+ \cap S(4,7)$ using $\preccurlyeq$ if and only if either all its entries are even (there are $\binom{7}{3}$ such elements of $S(4,8)$) or it has at least two zero entries and two odd entries (there are $6 \cdot 4 = 24$ such elements). Therefore $\delta(4,4) = 4 + 24 + 35 + 24 = 87$. □

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