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COMMON PROPERTIES OF THE OPERATOR EQUATIONS IN ULTRAMETRIC SPECRTAL THEORY

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Abstract. Let E and F be two ultrametric Banach spaces over $\mathbb{K}$. Let $A, D \in B(E,F)$ and $B, S \in B(F,E)$ such that $ABA = ASA$ (resp. $ASD = DBD$ and $DBA = ASA$). In this paper, the operator equation $ABA = ASA$ is studied, and the joint properties of $AS - I_F$ and $BA - I_E$ are described. In particular, it is proved that $N(I_F - AS)$ is complemented in F if and only if $N(I_E - BA)$ is complemented in E. Moreover, the approach is generalized (i.e., $ASD = DBD$ and $DBA = ASA$) for considering relationships between the properties of $I_F - AS$ and $I_E - BD$. Finally, several illustrative examples are provided.

1. Introduction

In classical operator theory, Barnes [1] studied many common spectral properties of $\lambda - AB$ and $\lambda - BA$ in a complex Banach space. Recently, Corach et al. [2] gave an extensions of Jacobson's lemma, they established some common properties of $AS - I_F$ and $BA - I_E$ when $ABA = ASA$. In [12], Zeng et Zhong continued to examine the joint properties of AS and BA from the attitude of classical spectral theory considering $A \in B(E,F)$ and $B, S \in B(F,E)$ such that $ABA = ASA$, where E and F were assumed to be Banach spaces over $\mathbb{C}$. In particular, they gave a positif answer to one question raised by the authors [2], by showing that $AS - I_F$ is of closed range if and only if $BA - I_E$ is of closed range. Yan and Fang [11] examined the joint properties of BD and AS in terms of regularity when $A, D \in B(E,F)$ and $B, S \in B(F,E)$ satisfy the operator equations $DBD = ASD$ and $DBA = ASA$.

In ultrametric operator theory, Ettayb [4] extended and studied some properties of the operator equation $ABA = ASA$ on ultrametric Banach spaces and many properties in common of $I_F - AS$ and $I_E - BA$ are given. In particular, if E and F are ultrametric Banach spaces a spherically complete field $\mathbb{K}$, $A \in B(E,F)$ and $B, S \in B(F,E)$ such that $ABA = ASA$, hence $N(I_F - AS)$ is a complemented subspace of F if and only if $N(I_E - BA)$ is a complemented subspace of E.

Throughout this paper, $\mathbb{K}$ is a complete ultrametric valued field with a non-trivial valuation $|\cdot|$, E and F are ultrametric Banach spaces over $\mathbb{K}$, $(\mathbb{Q}_p, |.|_p)$

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denotes the field of p-adic numbers and $B(E,F)$ is the collection of all continuous linear operators from E into F. If $E=F$, we set $B(E,E)=B(E)$. Let $A\in B(E)$, $N(A)$, $R(A)$, $\sigma(A)$ and A^* denote the kernel, the range, the spectrum and the adjoint of A respectively. I_E is the identity operator on E, I_F denotes the identity operator on F and $\mathcal{M}_n(\mathbb{K})$ is the space of all $n\times n$ square matrices over $\mathbb{K}$. For more details, see [3] and [9].

In this paper, an operator equation $ABA=ASA$ is studied, and the common properties of $AS-I_F$ and $BA-I_E$ are described. In particular, it is proved that $N(I_F-AS)$ is complemented in F if and only if $N(I_E-BA)$ is complemented in E. Moreover, the approach is generalized for considering relationships between the properties of I_F-AS and I_E-BD. We extend and develop some results of [1, 2, 11, 12] in ultrametric spectral theory. Finally, several descriptive examples are furnished.

2. Preliminaries

We continue by remembering some preliminaries.

Definition 2.1. [3] A field $\mathbb{K}$ is said to be ultrametric if it is endowed with an absolute value $|\cdot|:\mathbb{K}\to\mathbb{R}^+$ such that

(i) $|\alpha|=0$ if, and only if, $\alpha=0$;
(ii) For all $\alpha,\mu\in\mathbb{K}$, $|\alpha\mu|=|\alpha||\mu|$;
(iii) For each $\alpha,\mu\in\mathbb{K}$, $|\alpha+\mu|\leq\max\{|\alpha|,|\mu|\}$.

From now, we assume that $\mathbb{K}=(\mathbb{K},|\cdot|)$ is a complete ultrametric valued field.

Definition 2.2. [3] Let E be a vector space over $\mathbb{K}$. A function $\|\cdot\|:E\to\mathbb{R}_+$ is called an ultrametric norm if:

(i) For any $x\in E$, $\|x\|=0$ if and only if $x=0$;
(ii) For all $x\in E$ and $\alpha\in\mathbb{K}$, $\|\alpha x\|=|\alpha|\|x\|$;
(iii) For each $x,y\in E$, $\|x+y\|\leq\max(\|x\|,\|y\|)$.

Definition 2.3. [3] An ultrametric vector space $(E,\|\cdot\|)$ satisfying the conditions of Definition 2.2 is called an ultrametric normed space.

Definition 2.4. [3] A complete ultrametric normed space is called an ultrametric Banach space.

Example 2.5. [3] The space $c_0(\mathbb{K})$ is the set of all sequences $(v_i)_{i\in\mathbb{N}}$ in $\mathbb{K}$ such that $\lim_{i\to\infty}v_i=0$. Then $(c_0(\mathbb{K}),\|\cdot\|)$ is an ultrametric Banach space over $\mathbb{K}$ where for each $(v_i)_{i\in\mathbb{N}}\in c_0(\mathbb{K})$, $\|(v_i)_{i\in\mathbb{N}}\|=\sup_{i\in\mathbb{N}}|v_i|$.

For additional details on ultrametric Banach spaces, see [3] and [9].

Definition 2.6. [9] Let $P\in B(E)$. P is called a projection if $P^2=P$.

Remark 2.7. [9] If P is a projection on E, hence $R(P)$ is the kernel of I_E-P, thus $R(P)$ is a closed subspace of E.

Lemma 2.8. [9] *Let $P\in B(E)$ be a projection, we have:*

(i) *I_E-P is a projection;*

(ii) *If* $P \neq 0$, *hence* $\|P\| \geq 1$;
(iii) *If* $P \neq 0$ *and* $P \neq I_E$, *then* $\|P\| = \|I_E - P\|$;
(iv) *If* Q *is a projection such that* $PQ = QP$, *hence* PQ *is a projection.*

We have the following definition.

Definition 2.9. [7] A subspace D of E is said to be complemented if there is a continuous projection $P \in B(E)$ such that $R(P) = D$. In such case, $D = R(P)$ and $D_1 = N(P)$ are closed subspaces of E and $E = D \oplus D_1$.

Remark 2.10. [7] If D and D_1 are closed subspaces of E such that $E = D \oplus D_1$, hence D is complemented and D_1 is a complement of D.

Definition 2.11. [7] $\mathbb{K}$ is said to be spherically complete if all nested sequence of balls $(B_n)_{n\geq 1}$ in $\mathbb{K}$ such that $B_{n+1} \subset B_n$ has a non-empty intersection.

For additional details on ultrametric operator theory, we refer to [3] and [9].

Definition 2.12. [3] Let $S \in B(E)$. The spectrum $\sigma(S)$ of S is

$$\sigma(S) = \mathbb{K} \backslash \rho(S),$$

where $\rho(S) = \{\lambda \in \mathbb{K} : \lambda I_E - S \text{ is invertible}\}$.

Definition 2.13. Let $S \in B(E)$, we have:

(i) S is said to be left invertible if there is $T \in B(E)$ such that $TS = I_E$;
(ii) S is said to be right invertible if there is $A \in B(E)$ such that $SA = I_E$.

Definition 2.14. Let $S \in B(E)$. The left spectrum $\sigma_l(S)$ of S is

$$\sigma_l(S) = \mathbb{K} \backslash \rho_l(S),$$

where $\rho_l(S) = \{\lambda \in \mathbb{K} : \lambda I_E - S \text{ is left invertible}\}$.

Definition 2.15. Let $S \in B(E)$. The right spectrum $\sigma_r(S)$ of S is

$$\sigma_r(S) = \mathbb{K} \backslash \rho_r(S),$$

where $\rho_r(S) = \{\lambda \in \mathbb{K} : \lambda I_E - S \text{ is right invertible}\}$.

Definition 2.16. [8] Let $S \in B(E, F)$, we have:

(i) S is upper semi-Fredholm if $R(S)$ is closed and $\dim N(S)$ is finite;
(ii) S is lower semi-Fredholm if $\dim(F/R(S))$ is finite;
(iii) S is Fredholm if it is lower and upper semi-Fredholm.

Definition 2.17. Let $S \in B(E)$. The upper semi-Fredholm spectrum $\sigma_{uF}(S)$ of S is

$$\sigma_{uF}(S) = \mathbb{K} \backslash \rho_{uF}(S),$$

where $\rho_{uF}(S) = \{\lambda \in \mathbb{K} : \lambda I_E - S \text{ is upper semi-Fredholm}\}$.

Definition 2.18. Let $S \in B(E)$. The lower semi-Fredholm spectrum $\sigma_{lF}(S)$ of S is

$$\sigma_{lF}(S) = \mathbb{K} \backslash \rho_{lF}(S),$$

where $\rho_{lF}(S) = \{\lambda \in \mathbb{K} : \lambda I_E - S \text{ is lower semi-Fredholm}\}$.

Definition 2.19. Let $S \in B(E)$. The Fredholm spectrum $\sigma_F(S)$ of S is

$$\sigma_F(S) = \mathbb{K}\backslash\rho_F(S),$$

where $\rho_F(S) = \{\lambda \in \mathbb{K} : \lambda I_E - S \text{ is } \text{ Fredholm}\}$.

Proposition 2.20. *If $A \in B(E, F)$ and $S \in B(F, E)$, hence*

(i) $N(S) \cap N(I_F - AS) = \{0\}$;
(ii) $S(N(I_F - AS)) = N(I_E - SA)$.

Proof.
(i) Let $x \in N(S) \cap N(I_F - AS)$, hence $Sx = 0$ and $ASx = x$, thus $x = 0$.
(ii) Let $x \in N(I_E - SA)$, then

$$SAx = x, \tag{2.1}$$

hence $ASAx = Ax$, thus $(I_F - AS)Ax = 0$. Consequently $Ax \in N(I_F - AS)$. Then $SAx \in S(N(I_F - AS))$. By (2.1), we get $x \in S(N(I_F - AS))$. Thus $N(I_E - SA) \subseteq S(N(I_F - AS))$. Conversely, let $x \in N(I_F - AS)$, then $ASx = x$, hence $SASx = Sx$, then $Sx \in N(I_E - SA)$. Thus $S(N(I_F - AS)) \subseteq N(I_E - SA)$. $\square$

Proposition 2.21. [6] *If $A \in B(E, F)$ is surjective, then the induced bijection $A_1 : E/N(A) \to F$ is an isomorphism of topological vector spaces.*

3. Main Results

We continue with the following theorem.

Theorem 3.1. *If $A \in B(E, F)$ and $S \in B(F, E)$, hence $R(I_F - AS)$ is closed in F if and only if $R(I_E - SA)$ is closed in E.*

Proof. Assume that $R(I_F - AS)$ is closed in F. Set $\widehat{I_F - AS} : F/N(I_F - AS) \longrightarrow R(I_F - AS)$ where $(\widehat{I_F - AS})(x + N(I_F - AS)) = (I_F - AS)x$. By Proposition 2.21, there is a continuous linear operator $C : R(I_F - AS) \longrightarrow F/N(I_F - AS)$ such that for each $x \in F$,

$$C(\widehat{I_F - AS})(x + N(I_F - AS)) = x + N(I_F - AS).$$

Define $\widehat{S} : F/N(I_F - AS) \longrightarrow E/N(I_E - SA)$ by

$$\widehat{S}(x + N(I_F - AS)) = Sx + N(I_E - SA).$$

By Proposition 2.20, $\widehat{S}$ is well-defined and continuous. Now, we define $D : R(I_E - SA) \longrightarrow E/N(I_E - SA)$ by

$$Dz = z + N(I_E - SA) + \widehat{S}CAz.$$

Here note that $Az = (I_F - AS)Ay \in R(I_F - AS)$ where $z = y - SAy$ for some $y \in E$. Then CAz is well-defined. By definition of D, it is continuous. Thus

$$\begin{aligned} D(y - SAy) &= y - SAy + N(I_E - SA) + \widehat{S}CA(y - SAy) \\ &= y - SAy + N(I_E - SA) + \widehat{S}Ay = y + N(I_E - SA). \end{aligned}$$

Hence $R(I_E - SA)$ is closed. Similarly, by symmetry we obtain that if $R(I_E - SA)$ is closed in E. Then $R(I_F - AS)$ is closed in F. $\square$

Let $m \in \mathbb{N}$, one can see that

$$(I_F - AS)^{m+1} = I_F - U_m S \text{ and } (I_E - SA)^{m+1} = I_E - SU_m,$$

where $U_m = \sum_{k=1}^{m+1} (-1)^{k-1} \binom{m+1}{k} A(SA)^{k-1}$.

Lemma 3.2. *If $A \in B(E,F)$ and $S \in B(F,E)$, hence*

(i) *For each $m \in \mathbb{N}$, $S(N((I_F - AS)^m)) = N((I_E - SA)^m)$;*

(ii) *For any $m \in \mathbb{N}$, $R((I_F - AS)^m)$ is closed in F if and only if $R((I_E - SA)^m)$ is closed in E.*

Proof.

(i) From $(I_F - AS)^{m+1} = I_F - U_m S$ and $(I_E - SA)^{m+1} = I_E - SU_m$ and (ii) of Proposition 2.20, we get for each $m \in \mathbb{N}$, $S(N((I_F - AS)^m)) = N((I_E - SA)^m)$.

(ii) By $(I_F - AS)^{m+1} = I_F - U_m S$ and $(I_E - SA)^{m+1} = I_E - SU_m$ and Theorem 3.1, we conclude that for each $m \in \mathbb{N}$, $R((I_F - AS)^m)$ is closed in F if and only if for each $m \in \mathbb{N}$, $R((I_E - SA)^m)$ is closed in E. □

Lemma 3.3. *Let $A \in B(E,F)$ and $S \in B(F,E)$, then for each $m \in \mathbb{N}$, $R(AS - I_F) + N((AS - I_F)^m)$ is closed in F if and only if $R(SA - I_E) + N((SA - I_E)^m)$ is closed in E.*

Proof. Assume that $R(AS - I_F) + N((AS - I_F)^m)$ is closed in F. Let $(x_n)_{n\in\mathbb{N}} \subseteq R(SA - I_E) + N((SA - I_E)^m)$ such that $x_n \to x$ for some $x \in E$ as $n \to \infty$. Hence there exist $(z_n)_{n\in\mathbb{N}} \subset R(SA - I_E)$ and $(w_n)_{n\in\mathbb{N}} \subset N((SA - I_F)^m)$ such that for all $n \in \mathbb{N}$, $x_n = z_n + w_n$. Thus

$$\begin{aligned} Ax &= \lim_{n\to\infty} Ax_n \\ &= \lim_{n\to\infty} A(z_n + w_n). \end{aligned}$$

Then $Az_n \in A(R(SA - I_E)) \subseteq R(AS - I_F)$ and $Aw_n \in A(N((SA - I_E)^m)) \subseteq N((AS - I_F)^m)$. Since $R(AS - I_F) + N((AS - I_F)^m)$ is closed in F, we have $Ax \in R(AS - I_F) + N((AS - I_F)^m)$. Hence there are $a \in R(AS - I_F)$ and $b \in N((AS - I_F)^m)$ such that $Ax = a + b$. Thus $Sa \in S(R(AS - I_F)) \subseteq R(SA - I_E)$ and $Sb \in S(N((AS - I_F)^m)) \subseteq N((SA - I_E)^m)$. Hence $SAx \in R(SA - I_E) + N((SA - I_E)^m)$. Thus $x = (I_E - SA)x + SAx \in R(SA - I_E) + N((SA - I_E)^m)$. By symmetry, if $R(SA - I_E) + N((SA - I_E)^m)$ is closed in E, hence $R(AS - I_F) + N((AS - I_F)^m)$ is closed in F. □

Theorem 3.4. *Let E and F be two ultrametric Banach spaces over a spherically complete field $\mathbb{K}$. Let $A \in B(E,F)$ and $S \in B(F,E)$. Hence, $I_F - AS$ is upper semi-Fredholm if and only if $I_E - SA$ is upper semi-Fredholm.*

Proof. It follows from Proposition 2.20 and Theorem 3.1. □

Similar to the proof of Theorem 3.4, we have.

Theorem 3.5. *Let E and F be two ultrametric Banach spaces over a spherically complete field $\mathbb{K}$. If $A \in B(E,F)$ and $S \in B(F,E)$, then*

$$\sigma_{uF}(AS)\backslash\{0\} = \sigma_{uF}(SA)\backslash\{0\}.$$

Theorem 3.6. *Let $A \in B(E,F)$ and $S \in B(F,E)$. Hence*

(i) *$I_F - AS$ is left invertible if and only if $I_E - SA$ is left invertible;*
(ii) *$I_F - AS$ is right invertible if and only if $I_E - SA$ is right invertible.*

Proof. (i) Assume that $I_F - AS$ is left invertible, thus there is $D \in B(F)$ such that $D(I_F - AS) = I_F$. On the other hand

$$\begin{aligned}(I_E + SDA)(I_E - SA) &= I_E - SA + SDA - SDASA \\ &= I_E - SA + SD(I_F - AS)A \\ &= I_E - SA + SA \\ &= I_E.\end{aligned}$$

Thus $I_E - SA$ is left invertible. Conversely, assume that $I_E - SA$ is left invertible. Hence there is $B \in B(E)$ such that $B(I_E - SA) = I_E$. Then

$$\begin{aligned}(I_F + ABS)(I_F - AS) &= I_F - AS + ABS - ABSAS \\ &= I_F - AS + AB(I_E - SA)S \\ &= I_F - AS + AS \\ &= I_F.\end{aligned}$$

Thus $I_F - AS$ is left invertible.

(ii) Assume that $I_F - AS$ is right invertible, hence there is $D \in B(F)$ such that $(I_F - AS)D = I_F$. On the other hand

$$\begin{aligned}(I_E - SA)(I_E + SDA) &= I_E - SA + SDA - SASDA \\ &= I_E - SA + S(I_F - AS)DA \\ &= I_E - SA + SA \\ &= I_E.\end{aligned}$$

Thus $I_E - SA$ is right invertible. Conversely, assume that $I_E - SA$ is right invertible. Hence there exists $B \in B(E)$ such that $(I_E - SA)B = I_E$. Then

$$\begin{aligned}(I_F - AS)(I_F + ABS) &= I_F - AS + ABS - ASABS \\ &= I_F - AS + A(I_E - SA)BS \\ &= I_F - AS + AS \\ &= I_F.\end{aligned}$$

Thus $I_F - AS$ is right invertible.

□

Theorem 3.7. *Let $A \in B(E,F)$ and $B, S \in B(F,E)$ such that $ABA = ASA$. Then*

(i) *If $I_F - AS$ is left invertible, then $I_E - BA$ is left invertible;*
(ii) *If $I_E - BA$ is right invertible, hence $I_F - AS$ is right invertible.*

Proof. (i) Assume that $I_F - AS$ is left invertible, hence there is $D \in B(F)$ such that $D(I_F - AS) = I_F$. Thus $A = D(I_F - AS)A = DA(I_E - BA)$.

Then $BA = BDA(I_E - BA)$. Thus $BA - I_E = BDA(I_E - BA) - I_E$. Hence $I_E = (BDA + I_E)(I_E - BA)$. Thus $I_E - BA$ is left invertible.

(ii) Assume that $I_E - BA$ is right invertible, hence there is $D \in B(E)$ such that $(I_E - BA)D = I_E$. Consequently $A = A(I_E - BA)D = (I_F - AS)AD$. Then $AS = (I_F - AS)ADS$. Thus $I_F - AS = I_Y - (I_F - AS)ADS$. Hence $I_F = (I_F - AS)(I_F + ADS)$. Thus $I_F - AS$ is right invertible.

□

By Theorem 3.7, we conclude the following lemma.

Lemma 3.8. *Let $A \in B(E, F)$ and $B, S \in B(F, E)$ such that $ABA = ASA$. Then*

(i) *If $I_F - AB$ is left invertible, then $I_E - SA$ is left invertible;*
(ii) *If $I_E - SA$ is right invertible, hence $I_F - AB$ is right invertible.*

Theorem 3.9. *Let $A \in B(E, F)$ and $B, S \in B(F, E)$ such that $ABA = ASA$, hence*

(i) *$I_F - AS$ is left invertible if and only if $I_E - BA$ is left invertible;*
(ii) *$I_E - BA$ is right invertible if and only if $I_F - AS$ is right invertible;*
(iii) *$I_E - BA$ is invertible if and only if $I_F - AS$ is invertible.*

Proof. (i) From (i) of Theorem 3.7, it suffices to prove that if $I_E - BA$ is left invertible, thus $I_F - AS$ is left invertible. Suppose that $I_E - BA$ is left invertible, hence from (i) of Theorem 3.6, $I_F - AB$ is left invertible. By (i) of Lemma 3.8, we get $I_E - SA$ is left invertible. From (i) of Theorem 3.6, we get $I_F - AS$ is left invertible.

(ii) By (ii) of Theorem 3.7, it suffices to prove that if $I_E - AS$ is right invertible, hence $I_F - BA$ is right invertible. Assume that $I_E - AS$ is right invertible, then by (ii) of Theorem 3.6, $I_F - SA$ is right invertible. From (ii) of Lemma 3.8, we have $I_E - AB$ is right invertible. By (ii) of Theorem 3.6, we get $I_F - BA$ is right invertible.

(iii) It follows by (i) and (ii).

□

Similar to the proof of Theorem 3.9, we conclude the following:

Theorem 3.10. *If $A \in B(E, F)$ and $B, S \in B(F, E)$ such that $ABA = ASA$, hence*

(i) $\sigma_l(AS)\backslash\{0\} = \sigma_l(BA)\backslash\{0\}$;
(ii) $\sigma_r(AS)\backslash\{0\} = \sigma_r(BA)\backslash\{0\}$;
(iii) $\sigma(AS)\backslash\{0\} = \sigma(BA)\backslash\{0\}$.

Proposition 3.11. *If $A \in B(E, F)$ is left invertible, hence A is bounded below.*

Proof. Assume that A is left invertible, hence there is $D \in B(F, E)$ such that $DA = I_E$. Then for each $x \in E$, $\|x\| \leq \|D\|\|Ax\|$. Thus for each $x \in E$, $\|x\|\|D\|^{-1} \leq \|Ax\|$. Consequently, A is bounded below. □

Proposition 3.12. *If $A \in B(E, F)$ is left invertible, then A is injective.*

Proof. Assume that A is left invertible, by Proposition 3.11, A is bounded below. Hence there is $C > 0$ such that for each $u \in E$,

$$C\|u\| \leq \|Au\|. \tag{3.1}$$

Then if $Au = 0$ and from (3.1), we get $u = 0$. □

Theorem 3.13. *If $A \in B(E,F)$ and $B, S \in B(F,E)$ such that $ABA = ASA$, then*

$$I_F - AS \quad \text{is bounded below if and only if} \quad I_E - BA \quad \text{is bounded below.}$$

Proof. Suppose that $I_F - AS$ is bounded below, hence there is $C > 0 : C\|v\| \leq \|(I_F - AS)v\|$ for all $v \in F$. From $ABA = ASA$, we get $(AS - I_F)Au = A(BA - I_E)u$. Hence for each $u \in E$,

$$C\|Au\| \leq \|(I_F - AS)Au\| = \|A(I_E - BA)u\| \leq \|A\|\|(I_E - BA)u\|. \tag{3.2}$$

From $u = (I_E - BA)u + BAu$ and (3.2), we get

$$\begin{aligned} \|u\| &\leq \max\{\|(I_E - BA)u\|, \|BAu\|\} \\ &\leq \max\{\|(I_E - BA)u\|, C^{-1}\|B\|\|A\|\|(I_E - BA)u\|\} \\ &= \max\{1, C^{-1}\|B\|\|A\|\}\|(I_E - BA)u\|. \end{aligned}$$

Thus $I_E - BA$ is bounded below. Similarly, we obtain that if $I_E - BA$ is bounded below, then $I_E - AS$ is bounded below. □

Lemma 3.14. *Let $A \in B(E,F)$ and $B, S \in B(F,E)$ such that $ABA = ASA$, then $R(AS - I_F)$ is closed in F if and only if $R(BA - I_E)$ is closed in E.*

Proof. Assume that $R(AS - I_F)$ is closed in F. Let $(x_n)_{n\in\mathbb{N}} \subset R(BA - I_E) : x_n \to x$ for some $x \in E$ as $n \to \infty$. Hence there is $(z_n)_{n\in\mathbb{N}} \subset E$ such that for each $n \in \mathbb{N}$, $x_n = (BA - I_E)z_n$. Thus

$$\begin{aligned} Ax &= \lim_{n\to\infty} Ax_n \\ &= \lim_{n\to\infty} A(BA - I_E)z_n \\ &= \lim_{n\to\infty} (ABA - A)z_n \\ &= \lim_{n\to\infty} (ASA - A)z_n \\ &= \lim_{n\to\infty} (AS - I_F)Az_n. \end{aligned}$$

Since $R(AS - I_F)$ is closed in F, there is $y \in F$ such that $(AS - I_F)y = Ax$. Thus $y = ASy - Ax$. Hence

$$\begin{aligned} x &= BAx - (BA - I_E)x \\ &= B(AS - I_F)y - (BA - I_E)x \\ &= (BAS - B)(ASy - Ax) - (BA - I_E)x \\ &= BASASy - BASAx - BASy + BAx - (BA - I_E)x \\ &= BABASy - BABAx - BASy + BAx - (BA - I)x \\ &= (BA - I_E)(BASy - BAx - x). \end{aligned}$$

Consequently, $x \in R(BA - I_E)$. Then $R(BA - I_E)$ is closed in E. Conversely, suppose that $R(BA - I_E)$ is closed in E. From Theorem 3.1, $R(AB - I_F)$ is closed in F. Thus $R(SA - I_E)$ is closed in E. From Theorem 3.1, $R(AS - I_F)$ is closed in F. Consequently, $R(AS - I_F)$ is closed in F if and only if $R(BA - I_E)$ is closed in E. □

Theorem 3.15. *Let E and F be two ultrametric Banach spaces over a spherically complete field $\mathbb{K}$. Let $A \in B(E, F)$ and $B, S \in B(F, E)$ such that $ABA = ASA$. Then*

(i) *$N(I_F - AS)$ is complemented in F if and only if $N(I_E - BA)$ is complemented in E;*
(ii) *$R(I_F - AS)$ is complemented in F if and only if $R(I_E - BA)$ is complemented in E.*

Proof.
(i) Assume that $N(I_F - AS)$ is complemented in F. Hence there is a projection $P \in B(F)$ such that $R(P) = N(I_F - AS)$. Then $(I_F - AS)P = 0$, thus $P = ASP$. Set $Q = 2BPA - BPASA$. From $P = ASP$, it follows that $ABP = ABASP = ASASP = P$. Then

$$\begin{aligned} Q^2 &= (2BPA - BPASA)(2BPA - BPASA) \\ &= 4BPABPA - 2BPABPASA - 2BPASABPA + BPASABPASA \\ &= 4BPA - 2BPASA - 2BPA + BPASA \\ &= Q. \end{aligned}$$

On the other hand

$$\begin{aligned} (I_E - BA)Q &= (I_E - BA)(2BPA - BPASA) \\ &= 2BPA - BPASA - 2BABPA + BABPASA \\ &= 2BPA - BPABA - 2BPA + BPABA \\ &= 0. \end{aligned}$$

Thus $R(Q) \subseteq N(I_E - BA)$. Conversely, let $x \in N(I_E - BA)$, thus $BAx = x$. Then $ABAx = Ax$. Hence $ASAx = Ax$. Consequently $Ax \in N(I_F - AS) = R(P)$. Then $PAx = Ax$. On the other hand

$$\begin{aligned} Qx &= (2BPA - BPASA)x \\ &= 2BPAx - BPASAx \\ &= 2BAx - BPAx \\ &= 2BAx - BAx \\ &= BAx \\ &= x. \end{aligned}$$

Thus $x \in R(Q)$. Consequently, $N(I_E - BA) \subseteq R(Q)$. Hence $N(I_E - BA) = R(Q)$. Conversely, if P' is the projection onto $N(I_E - BA)$, hence one can prove that $Q' = 2AP'BAS - AP'BABAS$ is a projection such that $R(Q') = N(I_F - AS)$.

(ii) Suppose that $R(I_F - AS)$ is complemented in F. Hence there is a projection $P \in B(F)$ such that $R(P) = R(I_F - AS)(= N(I_F - P))$. Then $(I_F - P)(I_F - AS) = 0$. Thus $(I_F - P)AS = I_F - P$. Put $Q = I_E - BAS(I_F - P)A$. From $R(P) = R(I_F - AS)$ and $R(BASPA) \subseteq R(I_E - BA)$, we get

$$\begin{aligned} Q^2 &= (I_E - BAS(I_F - P)A)(I_E - BAS(I_F - P)A) \\ &= I_E - 2BAS(I_F - P)A + BAS(I_F - P)ASAS(I_F - P)A \\ &= I_E - 2BAS(I_F - P)A + BAS(I_F - P)(I_F - P)A \\ &= I_E - 2BAS(I_F - P)A + BAS(I_F - P)A \\ &= Q \end{aligned}$$

and

$$\begin{aligned} Q &= I_E - BAS(I_F - P)A \\ &= I_E - BASA + BASPA \\ &= (I_E - BA)(I_E + BA) + BASPA, \end{aligned}$$

hence $R(Q) \subseteq R(I_E - BA)$. Let $x \in R(I_E - BA)$, hence there is $u \in E$ such that $x = (I_E - BA)u$. Then $Ax = A(I_E - BA)u = (I_F - AS)Au \in R(I_F - AS) = R(P)$. Thus

$$Qx = [I_E - BAS(I_F - P)A]x = x.$$

Hence $x \in R(Q)$. Consequently $R(I_E - BA) \subseteq R(Q)$. Conversely, if P' is a projection such that $R(P') = R(I_E - BA)$, hence one can see that $Q' = I_F - ABA(I_E - P')S$ is a projection such that $R(Q') = R(I_F - AS)$. □

Lemma 3.16. *If $A \in B(E, F)$ and $B, S \in B(F, E)$ such that $ABA = ASA$, hence for each $n \in \mathbb{N}$,*
(i) $AR((BA - I_E)^n) \subseteq R((AS - I_F)^n)$;
(ii) $AN((BA - I_E)^n) \subseteq N((AS - I_F)^n)$;
(iii) $BASN((AS - I_F)^n) \subseteq N((BA - I_E)^n)$;
(iv) $BASR((AS - I_F)^n) \subseteq R((BA - I_E)^n)$.

Proof.
(i) Let $x \in R((BA - I_E)^n)$, hence there is $u \in E$ such that $x = (BA - I_E)^n u$. Then

$$Ax = A(BA - I_E)^n u = (AS - I_F)^n Au \in R((AS - I_F)^n).$$

Thus $AR((BA - I_E)^n) \subseteq R((AS - I_F)^n)$.
(ii) Let $x \in N((BA - I_E)^n)$, then $(BA - I_E)^n x = 0$. Hence

$$(AS - I_F)^n Ax = A(BA - I_E)^n x = 0.$$

Consequently, $x \in N((AS - I_F)^n)$. Thus $A(N((BA - I_E)^n)) \subseteq N((AS - I_F)^n)$.
(iii) If $u \in N((AS - I_F)^n)$, then $(AS - I_F)^n u = 0$. Hence

$$(BA - I_E)^n BASu = BAS(AS - I_F)^n u = 0.$$

Consequently, $BASu \in N((BA - I_E)^n)$. Then

$$BASN((AS - I_F)^n) \subseteq N((BA - I_E)^n).$$

(iv) Let $z \in R((AS - I_F)^n)$, hence there is $y \in F$ such that $z = (AS - I_F)^n y$. Then

$$BASz = BAS(AS - I_F)^n y = (BA - I_E)^n BASy \in R((BA - I_E)^n).$$

Consequently, $BASR((AS - I_F)^n) \subseteq R((BA - I_E)^n)$. □

Theorem 3.17. *Let E and F be two ultrametric Banach spaces over a spherically complete field $\mathbb{K}$. If $A \in B(E, F)$ and $B, S \in B(F, E)$ such that $ABA = ASA$, hence*

$I_F - AS$ is upper semi-Fredholm if and only if $I_E - BA$ is upper semi-Fredholm.

Proof. Suppose that $I_F - AS$ is upper semi-Fredholm, hence $R(I_F - AS)$ is closed and $\alpha(I_F - AS)$ is finite. By Lemma 3.14, we get $R(I_E - BA)$ is closed. Now we prove that $\alpha(I_E - BA)$ is finite. Define $\widehat{A} : N(I_E - BA) \longrightarrow N(I_F - AS)$ induced by $A \in B(E, F)$. From (ii) of Lemma 3.16, $\widehat{A}$ is well-defined. In fact, we check that $\widehat{A}$ is one-to-one. If $u \in N(I_E - BA)$ and $\widehat{A}u = 0$, hence $u = BAu = 0$. Thus A is one-to-one, then $\alpha(I_E - BA) \leq \alpha(I_F - AS)$. Hence $I_E - BA$ is upper semi-Fredholm. Conversely, by Lemma 3.14, we get $R(I_F - AS)$ is closed. From Lemma 2.20 and by symmetry, we get

$$\alpha(I_F - AS) = \alpha(I_E - SA) \leq \alpha(I_F - AB) = \alpha(I_E - BA).$$

This completes the proof. □

Similar to the proof of Theorem 3.17, we get.

Theorem 3.18. *If $A \in B(E, F)$ and $B, S \in B(F, E)$ such that $ABA = ASA$, hence*

$$\sigma_{uF}(AS)\backslash\{0\} = \sigma_{uF}(BA)\backslash\{0\}.$$

Lemma 3.19. *Let $A, D \in B(E, F)$ and $B, S \in B(F, E)$ such that $ASD = DBD$ and $ASA = DBA$. Lef f be a polynomial. Then*
(i) $DR(f(BD - I_E)) \subseteq R(f(AS - I_F))$;
(ii) $DN(f(BD - I_E)) \subseteq N(f(AS - I_F))$;
(iii) $BASN(f(AS - I_F)) \subseteq N(f(BD - I_E))$;
(iv) $BASR(f(AS - I_F)) \subseteq R(f(BD - I_E))$.

Proof.
(i) Let $x \in R(f(BD - I_E))$, hence there is $u \in E$ such that $x = f(BD - I_E)u$. Then

$$Dx = D(f(BD - I_E))u = f(AS - I_F)Du \in R(f(AS - I_F)).$$

Thus $DR((BD - I_E)^n)) \subseteq R(f(AS - I_F))$.
(ii) Let $x \in N(f(BD - I_E))$, then $f(BD - I_E)x = 0$. From $ASD = DBD$. Hence

$$f(AS - I_F)Dx = Df(BD - I_E)x = 0.$$

Consequently, $Dx \in N(f(AS - I_F))$. Thus $DN(f(BD - I_E)) \subseteq N(f(AS - I_F))$.

(iii) Let $y \in N(f(AS-I_F))$, then $f(AS-I_F)y = 0$. From $DBA = ASA$. Hence

$$f(BD-I_E)BASy = BASf(AS-I_F)y = 0.$$

Consequently $BASy \in N(f(BD-I_E))$. Then

$$BASN(f(AS-I_F)) \subseteq N(f(BD-I_E)).$$

(iv) Let $z \in R(f(AS-I_F))$, hence there is $y \in F$ such that $z = f(AS-I_F)y$. By $DBA = ASA$, then

$$BASz = BASf(AS-I_F)y = f(BD-I_E)BASy \in R(f(BD-I_E)).$$

Consequently $BASR(f(AS-I_F)) \subseteq R(f(BD-I_E))$. □

Lemma 3.20. *If $A, D \in B(E,F)$ and $B, S \in B(F,E)$ such that $ASD = DBD$ and $ASA = DBA$. Hence for each $m \in \mathbb{N}$, $R(AS-I_F)+N((AS-I_F)^m)$ is closed in F if and only if $R(BD-I_E)+N((BD-I_E)^m)$ is closed in E.*

Proof. Assume that $R(AS-I_F)+N((AS-I_F)^m)$ is closed in F. Let $(x_n)_{n\in\mathbb{N}} \subseteq R(BD-I_E)+N((BD-I_E)^m)$ such that $x_n \to x$ for some $x \in E$ as $n \to \infty$. Hence there are sequences $(z_n)_{n\in\mathbb{N}} \subseteq R(BD-I_E)$ and $(w_n)_{n\in\mathbb{N}} \subseteq N((BD-I_E)^m)$ such that for each $n \in \mathbb{N}$, $x_n = z_n + w_n$. Thus

$$\begin{aligned} Dx &= \lim_{n\to\infty} Dx_n \\ &= \lim_{n\to\infty} D(z_n + w_n). \end{aligned}$$

From (i) and (ii) of Lemma 3.19, we have $Dz_n \in R(AS-I_F)$ and $Dw_n \in N((AS-I_F)^m)$. Since $R(AS-I_F)+N((AS-I_F)^m)$ is closed in F, there exist $z \in F$ and $w \in N((AS-I_F)^m)$ such that $(AS-I_F)z+w = Dx$. Thus $z = ASz-Dx+w$. Consequently,

$$\begin{aligned} x &= BDx-(BD-I_E)x \\ &= B((AS-I_F)z+w)-(BD-I_E)x \\ &= B(AS-I_F)z+Bw-(BD-I_E)x \\ &= B(AS-I_F)(ASz-Dx+w)+Bw-(BD-I_E)x \\ &= BASASz-BASDx+BASw-BASz+BDx-Bw+Bw-(BD-I_E)x \\ &= BDBASz-BDBDx+BASw-BASz-(BD-I_E)x \\ &= (BD-I_E)(BASz-BDx-x)+BASw. \end{aligned}$$

Since $w \in N((AS-I_F)^m)$ and from (iii) of Lemma 3.19, we get $x \in N((BD-I_E)^m)+R(BD-I_E)$. Then $R(BD-I_E)+N((BD-I_E)^m)$ is closed in E. Conversely, assume that $N((BD-I_E)^m)+R(BD-I_E)$ is closed in E. By Lemma 3.3, $N((DB-I_F)^m)+R(DB-I_F)$ is closed in F. Then, by symmetry, $R(SA-I_E)+N((SA-I_E)^m)$ is closed in E. From Lemma 3.3, $R(AS-I_F)+N((AS-I_F)^m)$ is closed in F. This completes the proof. □

From Lemma 3.20 and for $m = 0$, we conclude that.

Corollary 3.21. *If $A, D \in B(E,F)$ and $B, C \in B(F,E)$ such that $ASD = DBD$ and $ASA = DBA$. Hence $R(AS - I_F)$ is closed in F if and only if $R(BD - I_E)$ is closed in E.*

Lemma 3.22. *If $A, D \in B(E,F)$ and $B, S \in B(F,E)$ such that $ASD = DBD$ and $ASA = DBA$. Hence for each $m \in \mathbb{N}$, $R((AS - I_F)^m)$ is closed in F if and only if $R((BD - I_E)^m)$ is closed in E.*

Proof. Set

$$(\forall m \in \mathbb{N})\ B_m = \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} (BD)^{k-1} D$$

and

$$(\forall m \in \mathbb{N})\ C_m = \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} S(AS)^{k-1}.$$

Since $ASD = DBD$ and $ASA = DBA$, for all $m \in \mathbb{N}$, $AC_mD = DB_mD$ and $DB_mA = AC_mA$. Moreover, for each $m \in \mathbb{N}$,

$$\begin{aligned} I_F - AC_m &= I_F - \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} AS(AS)^{k-1} \\ &= I_F + \sum_{k=1}^{m} (-1)^{k} \binom{m}{k} (AS)(AS)^{k-1} \\ &= I_F + \sum_{k=1}^{m} \binom{m}{k} (-AS)^{k} \\ &= \sum_{k=0}^{m} \binom{m}{k} (-AS)^{k} \\ &= (I_F - AS)^m. \end{aligned}$$

On the other hand, we get for all $m \in \mathbb{N}$, $(I_E - BD)^m = I_E - B_mD$. By Corollary 3.21, for each $m \in \mathbb{N}$, $R((AS - I_F)^m)$ is closed in F if and only if $R((BD - I_E)^m)$ is closed in E. □

The next theorem is a generalization of Theorem 3.9.

Theorem 3.23. *If $A, D \in B(E,F)$ and $B, S \in B(F,E)$ such that $ASD = DBD$ and $ASA = DBA$, hence*

$$I_E - BD \quad \textit{is invertible if and only if} \quad I_F - AS \quad \textit{is invertible.}$$

Proof. Similarly to the proof of Theorem 3.9 by using $ASD = DBD$ and $ASA = DBA$. □

The following theorem is a generalization of Theorem 3.10.

Theorem 3.24. *If $A, D \in B(E,F)$ and $B, S \in B(F,E)$ such that $ASD = DBD$ and $ASA = DBA$, hence*

(i) $\sigma_l(AS)\backslash\{0\} = \sigma_l(BD)\backslash\{0\}$;
(ii) $\sigma_r(AS)\backslash\{0\} = \sigma_r(BD)\backslash\{0\}$;

(iii) $\sigma(AS)\backslash\{0\} = \sigma(BD)\backslash\{0\}$.

Proof. Similar to the proof of Theorem 3.10 by using $ASD = DBD$ and $ASA = DBA$. □

We generalize the Theorem 3.17 as follows.

Theorem 3.25. *Let E and F be two ultrametric Banach spaces over a spherically complete field $\mathbb{K}$. Let $A, D \in B(E,F)$ and $B, S \in B(F,E)$ such that $ASD = DBD$ and $ASA = DBA$. Hence*

$$I_F - AS \text{ is upper semi-Fredholm if and only if } I_E - BD \text{ is upper semi-Fredholm.}$$

Proof. Suppose that $I_F - AS$ is upper semi-Fredholm, hence $R(I_F - AS)$ is closed and $\alpha(I_F - AS)$ is finite. By (iii) of Lemma 3.19, we get $R(I_E - BD)$ is closed. Now, we prove that $\alpha(I_E - BD)$ is finite. Define $\widehat{D} : N(I_E - BD) \longrightarrow N(I_F - AS)$ induced by $D \in B(E,F)$. From (ii) of Lemma 3.19, $\widehat{D}$ is well-defined. In fact, we check that $\widehat{D}$ is one-to-one. Let $x \in N(I_E - BD)$ and $\widehat{D}x = 0$. Hence $Dx = 0$. Since $I = (I_E + BD)(I_E - BD) + BDBD$. From $x \in N(I_E - BD)$ and $Dx = 0$, we get $x = 0$. Hence $\widehat{D}$ is one-to-one, then $\alpha(I_E - BD) \leq \alpha(I_F - AC)$. Hence $I_E - BD$ is upper semi-Fredholm. Conversely, by Lemma 3.19, we get $R(I_F - AS)$ is closed. From Lemma 2.20 and by symmetry, we get

$$\alpha(I_F - AS) = \alpha(I_E - SA) \leq \alpha(I_F - DB) = \alpha(I_E - BD).$$

This completes the proof. □

Similar to the proof of Theorem 3.18, we conclude the following:

Theorem 3.26. *If $A, D \in B(E,F)$ and $B, S \in B(F,E)$ such that $ASD = DBD$ and $ASA = DBA$, hence*

$$\sigma_{uF}(AS)\backslash\{0\} = \sigma_{uF}(BD)\backslash\{0\}.$$

We generalize the Theorem 3.13 as follows.

Theorem 3.27. *Let $A, D \in B(E,F)$ and $B, S \in B(F,E)$ such that $ASD = DBD$ and $ASA = DBA$, hence*

$$I_E - BD \text{ is bounded below if and only if } I_F - AS \text{ is bounded below.}$$

Proof. Similar to the proof of Theorem 3.13. □

Recall that $I_F - AS$ is relatively regular operator if, there is $D \in B(F)$ such that $(I_F - AS)D(I_F - AS) = I_F - AS$. For more details on classical relatively regular operators, see [10]. We deduce the next theorem.

Theorem 3.28. *Let $A \in B(E,F)$ and $B, S \in B(F,E)$ such that $ABA = ASA$. Hence*

$$I_F - AS \text{ is relatively regular if and only if } I_E - BA \text{ is relatively regular.}$$

Proof. Since $I_F - AS$ is relatively regular, there is $D \in B(F)$ such that $(I_F - AS)D(I_F - AS) = I_F$. On the other hand

$$\begin{array}{rcl}(I_E - BA)BASDA(I_E - BA) & = & B(I_F - AB)ASD(I_F - AS)A \\ & = & BAS(I_F - AS)D(I_F - AS)A \\ & = & BAS(I_F - AS)A \\ & = & (BA)^2(I_E - BA),\end{array}$$

we get

$$\begin{array}{rcl}(I_E - BA)(I_E + BA + BASDA)(I_E - BA) & = & (I_E - BA)(I_E + BA)(I_E - BA) \\ & & +(I_E - BA)BASDA(I_E - BA) \\ & = & (I_E - (BA)^2)(I_E - BA) \\ & & +(BA)^2(I_E - BA) \\ & = & I_E - BA.\end{array}$$

Conversely, suppose that $I_E - BA$ is relatively regular, hence there is $U \in B(E)$ such that $(I_E - BA)U(I_E - BA) = I_E$. Then by using the same argument as above, we conclude that $I_F - AS$ is relatively regular and $(I_F + AS + AUBAS)$ is inner inverse of $I_F - AS$ (i.e., $(I_F - AS)(I_F + AS + AUBAS)(I_F - AS) = I_F - AS$). □

We finish with the following examples.

Example 3.29. Let $A, B, S \in B(c_0(\mathbb{K}))$ be given respectively by

$$A(x_1, x_2, x_3, x_4, \cdots) = (0, x_2, 0, x_4, \cdots),$$
$$B(x_1, x_2, x_3, x_4, \cdots) = (0, x_1, x_2, x_4, \cdots)$$

and

$$S(x_1, x_2, x_3, x_4, \cdots) = (0, 0, x_1, x_4, \cdots).$$

It is easy to see that $ABA = ASA$.

Example 3.30.

(i) Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p).$$

It is easy to see that $ABA = ASA = 0_{\mathcal{M}_2(\mathbb{Q}_p)}$.

(ii) Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \text{ and } S = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p).$$

Then $ABA = ASA$.

(iii) Let $a, b, c \in \mathbb{Q}_p$. Let

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \text{ and } S = \begin{pmatrix} a & b & c \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \in \mathcal{M}_3(\mathbb{Q}_p).$$

One can see that if $a = c = 0$ and $b = 1$, then $B = S$ and $ABA = ASA$. Moreover in the case $B \neq S$, we have $ABA = ASA$.

Example 3.31. Let $A, B, S \in B(c_0(\mathbb{K}))$ be given respectively by

$$A(x_1, x_2, x_3, x_4, \cdots) = (x_2, x_3, 0, x_4, x_5, x_6, \cdots),$$

$$B(x_1, x_2, x_3, x_4, \cdots) = (x_4, x_5, x_6, x_7, x_8, x_9 \cdots),$$

$$D(x_1, x_2, x_3, x_4, \cdots) = (0, 0, 0, x_1, x_2, x_3, \cdots)$$

and

$$S(x_1, x_2, x_3, x_4, \cdots) = (x_2, x_3, 0, x_4, x_5, x_6, \cdots).$$

It is easy to see that $ASD = DBD$ and $ASA = DBA$.

Example 3.32. Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = S = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p).$$

One can see that $ASD = DBD$ and $ASA = DBA$.

Example 3.33. Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p).$$

One can see that $ASD = DBD$ and $ASA = DBA$.

Example 3.34. Let $A, B, D, S \in B(c_0(\mathbb{K}))$ be given by respectively

$$A(x_1, x_2, x_3, x_4, \cdots) = (x_2, 0, 0, 0, \cdots),$$

$$B(x_1, x_2, x_3, x_4, \cdots) = (x_1 + x_2, 2x_1 + x_2, 0, 0, 0, \cdots),$$

$$D(x_1, x_2, x_3, x_4, \cdots) = (2x_1, 0, 0, \cdots)$$

and

$$S(x_1, x_2, x_3, x_4, \cdots) = (x_1, 2x_1 + x_2, 0, 0, 0, \cdots).$$

It is easy to see that $ASD = DBD$ and $ASA = DBA$.

4. Conclusion

We studied the operator equation $ABA = ASA$ where $A \in B(E, F)$ and $B, S \in B(F, E)$ on ultrametric Banach spaces. We proved many common spectral properties of AB and SA on ultrametric Banach spaces. In particular, we showed that the operators $I_E - BA$ and $I_F - AS$ share many spectral properties in common such as closedness, left (right) invertibilily, boundedness below and complementability of kernels and ranges of operators. On the other hand, the situation is generalized to study some common spectral properties of operator equations $ASD = DBD$ and $ASA = DBA$ where $A, D \in B(E, F)$ and $B, S \in B(F, E)$. In addition, we provided a few examples to illustrate our work.

In the future, we will establish other results on common spectral properties of the operator equation $ABA = ASA$.

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