

BI-AMALGAMATIONS SUBJECT TO J-CLEAN AND WEAKLY J-CLEAN PROPERTIES

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ABSTRACT. This article investigates the transfer of commutative J-clean and weakly J-clean properties in several cases of bi-amalgamations. The main results recover previous results that concern with amalgamations and trivial extensions of these rings, also, they are used to provide new examples as well as counterexamples of J-clean and weakly J-clean rings arise as bi-amalgamations.

1. INTRODUCTION AND PRELIMINARIES

Throughout this article, A denotes a ring with unity, $Nil(A)$ is the nilradical of A ; $J(A)$ is the Jacobson radical of A ; $Id(A)$ is the set of all idempotents of A ; $U(A)$ is the set of all units of A .

Let $f : A \rightarrow B$ and $g : A \rightarrow C$ be ring homomorphisms, J an ideal of B , J' an ideal of C , and $f^{-1}(J) = g^{-1}(J')$. Then the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) is the subring of $B \times C$ defined by:

$$A \bowtie^{f,g} (J, J') := \{(f(a) + j, g(a) + j') \mid a \in A, (j, j') \in J \times J'\}.$$

The bi-amalgamation was introduced by S. Kabbaj, K. Louartiti, and M. Tamekkante in [18] as a generalization of the amalgamated duplication (that was introduced by D'Anna and Fontana [13, 14]) and the amalgamated algebras along an ideal (that was introduced by M. D'Anna, C. Finocchiaro and M. Fontana [10, 11, 12]). Some references for the bi-amalgamation are [6, 1, 18, 19, 4]. Let $f : A \rightarrow B$ be a ring homomorphism and J be an ideal of B . The subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}.$$

is called the amalgamation of A with B along J with respect to f . By [18, Example 2.1], $A \bowtie^f J = A \bowtie^{\iota, f} (I, J)$, where $I = f^{-1}(J)$ and $\iota = id_A$. For more about the amalgamation, one may refer to [10, 11, 12, 13, 14, 15].

Recall that an element x in a ring A is called clean if there is $u \in U(A)$ and $e \in Id(A)$ such that $x = u + e$. If this representation is unique, then x is called uniquely clean. If each element of A is (uniquely) clean, then it is called

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a (uniquely) clean ring. These rings were introduced by Nicholson [21]. A ring A is called nil-clean if each of its elements is nil clean, whereas $x \in A$ is called nil-clean if there is $n \in Nil(A)$ and $e \in Id(A)$ such that $x = n + e$. The nil-clean property was introduced by A. J. Diesel in [17], where he proved that the class of nil clean rings is strictly contained in the class of clean rings. Boolean rings are nil-clean rings. The homomorphic image of a (nil-)clean ring is again (nil-)clean.

A ring A is called a J-clean ring (were also termed semi-boolean rings) if every element a in A is J-clean, where $a \in A$ is called J-clean element if there is $e \in Id(A)$ and $j \in J(A)$ such that $a = e + j$. Homomorphic images, direct products, and direct sums of J-clean rings are again J-clean [5, 22]. In 2017, Danchev generalized J-clean rings into weakly J-clean rings (also termed weakly semi-boolean rings). He defined the weakly J-clean ring to be the ring in which each element is either the sum or the difference of an element from $J(A)$ and an element from $Id(A)$ [8].

The following diagram of implications displays the relations among the mentioned concepts, where all the implications are irreversible in general.

$$\begin{array}{c} \text{Nil clean ring} \Rightarrow \text{Weakly nil-clean rings} \\ \Downarrow \end{array}$$

$$\text{Uniquely clean rings} \Rightarrow \text{J-clean rings} \Rightarrow \text{Weakly J-clean rings} \Rightarrow \text{Clean rings}$$

The vertical and the middle implications are obvious conclusions from the definitions of the mentioned concepts and for a reference for the first and the last one, we appeal respectively to [22, 9]. Moreover, in the commutative setting, the uniquely clean and the J-clean concepts coincide [22, Proposition 26]; however J-clean and weakly J-clean are different even in the commutative case ($\mathbb{Z}_3$ is weakly J-clean by [8, Corollary 2.6] which is not J-clean by [22, Proposition 31]). Also, clean and weakly J-clean rings are different even in the commutative case; a good example of commutative clean which is not weakly J-clean is $\mathbb{Z}_9$ (see the remark before Proposition 26 of [22] and again [8, Corollary 2.6]). So in the commutative case, we have the following diagram of irreversible implications in general:

$$J \text{ (or Uniquely)-clean rings} \Rightarrow \text{Weakly J-clean rings} \Rightarrow \text{Clean rings}$$

In 2014, Mohamed Chhiti, Najib Mahdou, and Mohammed Tamekkante gave a characterization for the amalgamation $A \bowtie^f J$ to be (uniquely) clean [7]. Then, in 2018, Chahrazade Bakkari and Mohamed Es-Said gave a characterization for $A \bowtie^f J$ to be nil-clean [3]. Recently, K. Adarbeh and M. Adarbeh investigated some necessary and sufficient conditions for a bi-amalgamation $A \bowtie^{f,g} (J, J')$ to be clean as well as nil-clean; see [1, Theorem 2.3, Theorem 2.4, Proposition 2.6, Theorem 2.12]. They also investigated necessary and sufficient conditions for $A \bowtie^{f,g} (J, J')$ to be Boolean [1, Theorem 2.8]. Lastly, Farshad N., Safarisabet S. A., and Moussavi A. investigated the J-clean property and other related properties in the amalgamation. For example, they proved that if the amalgamation $A \bowtie^f J$ is J-clean, then A and $f(A) + J$ are J-clean. Also they proved that if $J \in Nil(B)$, and A is J-clean, then $A \bowtie^f J$ is J-clean [16].

This paper establishes necessary and sufficient conditions for a bi-amalgamation $A \bowtie^{f,g} (J, J')$ to be (uniquely) J-clean, as well as the weakly J-clean. All obtained

results are used to recover previous works carried on the amalgamations as well as provide examples of the involved ring theoretic properties.

2. TRANSFER RESULTS IN THE BI-AMALGAMATION

Throughout this section, all rings are commutative with unity. Recall that a homomorphic image of a (weakly) J-clean ring is again a (weakly) J-clean ring. The following result was proved by Kabbaj S. and Louartiti K. in [18]. We will refer to it frequently in proving this article results.

Lemma 2.1. [18, Proposition 4.1 parts (2) and (3)]. *Let $A \bowtie^{f,g} (J, J')$ be the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g) and let $I_o = f^{-1}(J) = g^{-1}(J')$. Then*

- (1) $\frac{A \bowtie^{f,g}(J, J')}{0 \times J'} \cong f(A) + J$ and $\frac{A \bowtie^{f,g}(J, J')}{J \times 0} \cong g(A) + J'$.
- (2) $\frac{A}{I_o} \cong \frac{A \bowtie^{f,g}(J, J')}{J \times J'} \cong \frac{f(A) + J}{J} \cong \frac{g(A) + J'}{J'}$.

From now until the end of this article, in all the results we will be using the notation of Lemma 2.1. The following proposition ensures that $f(A) + J$ and $g(A) + J'$ inherit the (weakly) J-clean property from $A \bowtie^{f,g} (J, J')$

Proposition 2.2. *If $A \bowtie^{f,g} (J, J')$ is (weakly) J-clean, then $f(A) + J$ and $g(A) + J'$ are (weakly) J-clean rings.*

Proof. Since $f(A) + J$ and $g(A) + J'$ are homomorphic images of $A \bowtie^{f,g} (J, J')$ (by Lemma 2.1) and $A \bowtie^{f,g} (J, J')$ is (weakly) J-clean, then they are (weakly) J-clean rings. $\square$

The following technical lemma is used to prove one of the main results of this article (Theorem 2.4).

Lemma 2.3. *If $a \in J(A)$, $f(a) + j \in J(f(A) + J)$, and $g(a) + j' \in J(g(A) + J')$, then $(f(a) + j, g(a) + j') \in J(A \bowtie^{f,g} (J, J'))$.*

Proof. Let $(f(x) + k, g(x) + k') \in A \bowtie^{f,g} (J, J')$. Since $a \in J(A)$, $f(a) + j \in J(f(A) + J)$, and $g(a) + j' \in J(g(A) + J')$, then $1 - ax$, $1 - (f(a) + j)(f(x) + k)$, and $1 - (g(a) + j')(g(x) + k')$ are units in A , $f(A) + J$, and $g(A) + J'$ respectively. Now,

$$(1, 1) - (f(a) + j, g(a) + j')(f(x) + k, g(x) + k') = (1 - (f(a) + j)(f(x) + k), 1 - (g(a) + j')(g(x) + k')) = (f(1 - ax) + \gamma, g(1 - ax) + \gamma'),$$

where $\gamma = -(f(a)k + jf(x) + jk) \in J$ and $\gamma' = -(g(a)k' + j'g(x) + j'k') \in J'$. By [1, Lemma 2.2], $(1, 1) - (f(a) + j, g(a) + j')(f(x) + k, g(x) + k')$ is a unit in $A \bowtie^{f,g} (J, J')$. Thus, $(f(a) + j, g(a) + j') \in J(A \bowtie^{f,g} (J, J'))$. $\square$

Assume that A is a J-clean ring and $\frac{A}{I_o}$ is a uniquely J-clean ring. The following theorem investigates necessary and sufficient conditions for $A \bowtie^{f,g} (J, J')$ to be J-clean.

Theorem 2.4. *Consider $A \bowtie^{f,g} (J, J')$, where A is J-clean. Suppose that $\frac{A}{I_o}$ is uniquely J-clean, then $A \bowtie^{f,g} (J, J')$ is J-clean if and only if $f(A) + J$ and $g(A) + J'$ are J-clean rings.*

Proof. Let $a \in A$ and $(j, j') \in J \times J'$. Then $a = b + e$, where $b \in J(A)$ and $e \in I(A)$ (since A is J-clean). Since $f(A) + J$ and $g(A) + J'$ are J-clean, then $f(a) + j = f(x_1) + j_1 + f(x_2) + j_2$ and $g(a) + j' = g(y_1) + j'_1 + g(y_2) + j'_2$, where $f(x_1) + j_1 \in J(f(A) + J)$, $f(x_2) + j_2 \in Id(f(A) + J)$ and $g(y_1) + j'_1 \in J(g(A) + J')$, $g(y_2) + j'_2 \in Id(g(A) + J')$. Clearly, $\overline{f(x_1)} = \overline{f(x_1) + j_1}$ and $\overline{f(b)} = \overline{g(y_1)} = \overline{g(y_1) + j'_1}$ and $\overline{g(b)} = \overline{g(y_2)} = \overline{g(y_2) + j'_2}$ are elements in $J(\frac{f(A)+J}{J})$ ($J(\frac{g(A)+J'}{J'})$). Also, $\overline{f(x_2)} = \overline{f(x_2) + j_2}$ and $\overline{f(e)}$ ($\overline{g(y_2)} = \overline{g(y_2) + j'_2}$ and $\overline{g(e)}$) are idempotents in $\frac{f(A)+J}{J}$ ($\frac{g(A)+J'}{J'}$). So we have

$$\overline{f(a)} = \overline{f(b)} + \overline{f(e)} = \overline{f(x_1)} + \overline{f(x_2)}, \text{ and}$$

$$\overline{g(a)} = \overline{g(b)} + \overline{g(e)} = \overline{g(y_1)} + \overline{g(y_2)}.$$

Since $\frac{A}{I_0}$ is uniquely J-clean, then by Lemma 2.1, $\frac{f(A)+J}{J}$ and $\frac{g(A)+J'}{J'}$ are uniquely J-clean, so

$$\overline{f(b)} = \overline{f(x_1)} \text{ and } \overline{f(e)} = \overline{f(x_2)}, \text{ and}$$

$$\overline{g(b)} = \overline{g(y_1)} \text{ and } \overline{g(e)} = \overline{g(y_2)}.$$

Then there exist $(j_3, j'_3) \in J \times J'$ and $(j_4, j'_4) \in J \times J'$ such that

$$f(x_1) = f(b) + j_3 \text{ and } g(y_1) = g(b) + j'_3, \text{ and}$$

$$f(x_2) = f(e) + j_4 \text{ and } g(y_2) = g(e) + j'_4.$$

Now,

$$\begin{aligned} (f(a) + j, g(a) + j') &= (f(x_1) + j_1 + f(x_2) + j_2, g(y_1) + j'_1 + g(y_2) + j'_2) \\ &= (f(b) + j_3 + j_1 + f(e) + j_4 + j_2, g(b) + j'_3 + j'_1 + g(e) + j'_4 + j'_2) \\ &= (f(b) + j_3 + j_1, g(b) + j'_3 + j'_1) + (f(e) + j_4 + j_2, g(e) + j'_4 + j'_2) \end{aligned}$$

Clearly, $(f(e) + j_4 + j_2, g(e) + j'_4 + j'_2) = (f(x_2) + j_2, g(y_2) + j'_2)$ is an idempotent in $A \bowtie^{f,g}(J, J')$. On the other hand, since b , $f(b) + j_3 + j_1$, and $g(b) + j'_3 + j'_1$ are elements in $J(A)$, $J(f(A) + J)$, and $J(g(A) + J')$ respectively, then by Lemma 2.3, $(f(b) + j_3 + j_1, g(b) + j'_3 + j'_1)$ is an element in $J(A \bowtie^{f,g}(J, J'))$. Therefore, $A \bowtie^{f,g}(J, J')$ is J-clean. The other implication follows from Proposition 2.2. $\square$

Recall that a ring A is J-clean if and only if A is clean and $\frac{A}{J(A)}$ is Boolean [23, Lemma 3.4]. Also, A is weakly J-clean if and only if $\frac{A}{Nil(A)}$ is weakly J-clean [8, Proposition 3.9]. The following lemmas are used to prove the second main result of this article (Theorem 2.7).

Lemma 2.5. *Let A be a ring and I be an ideal of A such that $I \subseteq Nil(A)$. Then A is J-clean if and only if $\frac{A}{I}$ is J-clean.*

Proof. If A is J-clean, then so is $\frac{A}{I}$ being a homomorphic image of a J-clean ring, namely A . Conversely, assume that $\frac{A}{I}$ is J-clean, then by [23, Lemma 3.4], $\frac{A}{I}$ is clean and $\frac{A/I}{J(A/I)}$ is Boolean. Since $I \subseteq Nil(A)$ and $\frac{A}{I}$ is clean, then by [20, Corollary 1.7], A is clean. Also, we have $\frac{A}{J(A)} \cong \frac{A/I}{J(A/I)} = \frac{A/I}{J(A/I)}$ is Boolean. Hence, again by [23, Lemma 3.4], A is J-clean. $\square$

Lemma 2.6. *Let A be a ring and I be an ideal of A such that $I \subseteq \text{Nil}(A)$. Then A is weakly J -clean if and only if $\frac{A}{I}$ is weakly J -clean.*

Proof. If A is weakly J -clean, then so is $\frac{A}{I}$ being a homomorphic image of a weakly J -clean ring, namely A . Conversely, assume that $\frac{A}{I}$ is weakly J -clean. By [8, Proposition 3.9], $\frac{A/I}{\text{Nil}(A/I)}$ is weakly J -clean. Since $I \subseteq \text{Nil}(A)$, $\text{Nil}(\frac{A}{I}) = \frac{\text{Nil}(A)}{I}$. Thus, $\frac{A}{\text{Nil}(A)} \cong \frac{A/I}{\text{Nil}(A/I)} = \frac{A/I}{\text{Nil}(A/I)}$ is weakly J -clean. Again by [8, Proposition 3.9], A is weakly J -clean $\square$

The following result establishes another necessary and sufficient conditions under which the bi-amalgamation $A \bowtie^{f,g} (J, J')$, where $I_o \subseteq \text{Nil}(A)$ and J and J' are nil ideals, will be (weakly) J -clean

Theorem 2.7. *Consider $A \bowtie^{f,g} (J, J')$*

- (1) *Suppose that $I_o \subseteq \text{Nil}(A)$. If $A \bowtie^{f,g} (J, J')$ is (weakly) J -clean, then A is (weakly) J -clean.*
- (2) *Suppose that $J \subseteq \text{Nil}(B)$ and $J' \subseteq \text{Nil}(C)$. If A is (weakly) J -clean, then $A \bowtie^{f,g} (J, J')$ is (weakly) J -clean.*

Proof. (1) Since $A \bowtie^{f,g} (J, J')$ is J -clean, $\frac{A}{I_o} \cong \frac{A \bowtie^{f,g} (J, J')}{J \times J'}$ is (weakly) J -clean. As $I_o \subseteq \text{Nil}(A)$, then by Lemma 2.5 (Lemma 2.6), A is (weakly) J -clean.
 (2) If $J \subseteq \text{Nil}(B)$ and $J' \subseteq \text{Nil}(C)$, then $J \times J' \subseteq \text{Nil}(A \bowtie^{f,g} (J, J'))$. Since A is (weakly) J -clean, $\frac{A}{I_o}$ is (weakly) J -clean but then $\frac{A \bowtie^{f,g} (J, J')}{J \times J'} \cong \frac{A}{I_o}$ is (weakly) J -clean. As $J \times J' \subseteq \text{Nil}(A \bowtie^{f,g} (J, J'))$, then again by Lemma 2.5 (Lemma 2.6), $A \bowtie^{f,g} (J, J')$ is (weakly) J -clean. $\square$

As a first application of Theorem 2.7; we provide a new counter-example of a weakly J -clean ring which is not J -clean that arises as a bi-amalgamation.

Example 2.8. Let $A = \mathbb{Z}_9$, $B = \mathbb{Z}_9 \times \mathbb{Z}_9$, and $C = \mathbb{Z}_{12} \times \mathbb{Z}_9$. Consider the two ring homomorphisms $f : A \rightarrow B$ ($x \mapsto (x, x)$) and $g : A \rightarrow C$ ($x \mapsto (4x, x)$). Let $J = 3\mathbb{Z}_3 \times 3\mathbb{Z}_3$ and $J' = 0 \times 3\mathbb{Z}_3$. Then $f^{-1}(J) = g^{-1}(J') = 3\mathbb{Z}_3$. Since $3 \in \text{Nil}(\mathbb{Z}_9)$, $I_o = 3\mathbb{Z}_3 \subseteq \text{Nil}(\mathbb{Z}_9)$. Also, $J \subseteq \text{Nil}(B)$ and $J' \subseteq \text{Nil}(C)$. Now, by [8, Example 2.1], A is weakly J -clean which is not J -clean. Hence, by Theorem 2.7 part (2), $A \bowtie^{f,g} (J, J')$ is weakly J -clean, but $A \bowtie^{f,g} (J, J')$ is not J -clean by Theorem 2.7 part (1).

A second application of Theorem 2.7 is the following proposition that provides two conditions under which A is (weakly) J -clean will be a necessary and sufficient condition for $A \bowtie^{f,g} (J, J')$ to be (weakly) J -clean.

Proposition 2.9. *Consider $A \bowtie^{f,g} (J, J')$. If one of the following conditions is satisfied:*

- (1) *f and g are surjective, and $I_o \subseteq \text{Nil}(A)$*
- (2) *f or g is injective, and $J \times J' \subseteq \text{Nil}(B \times C)$*

Then $A \bowtie^{f,g} (J, J')$ is (weakly) J -clean if and only if A is (weakly) J -clean.

Proof. Assume (1). Then by Theorem 2.7 part (1), $A \bowtie^{f,g} (J, J')$ is (weakly) J-clean implies A is (weakly) J-clean. Conversely, assume that A is (weakly) J-clean. Since $I_o = f^{-1}(J) = g^{-1}(J') \subseteq \text{Nil}(A)$, then $f(f^{-1}(J)) \subseteq f(\text{Nil}(A)) \subseteq \text{Nil}(B)$ and $g(g^{-1}(J')) \subseteq g(\text{Nil}(A)) \subseteq \text{Nil}(C)$. As f and g are surjective, then $J = f(f^{-1}(J)) \subseteq \text{Nil}(B)$ and $J' = g(g^{-1}(J')) \subseteq \text{Nil}(C)$. Therefore, by Theorem 2.7 part(2), $A \bowtie^{f,g} (J, J')$ is (weakly) J-clean.

Next, assume (2). Then by Theorem 2.7 part (2), A is (weakly) J-clean implies $A \bowtie^{f,g} (J, J')$ is (weakly) J-clean. Conversely, assume that $A \bowtie^{f,g} (J, J')$ is (weakly) J-clean. If f is injective, then since $J \subseteq \text{Nil}(B)$, we have $I_o = f^{-1}(J) \subseteq \text{Nil}(A)$. If g is injective, then since $J' \subseteq \text{Nil}(C)$, we have $I_o = g^{-1}(J') \subseteq \text{Nil}(A)$. So, by Theorem 2.7 part (1), A is (weakly) J-clean. $\square$

As a third application of Theorem 2.7, we construct new examples of J-clean rings. For that, we recall the definition of the trivial ring extension: Let A be a ring and M be an A -module. The trivial ring extension of A by M is the ring $A \ltimes M = A \times M$ with coordinate-wise addition and the multiplication is given by $(a, m)(a', m') = (aa', am' + a'm)$ [2]. The following is an example of a J-clean ring that arises as a bi-amalgamation.

Example 2.10. Let $A = \mathbb{Z}_{(2)} \ltimes \mathbb{Z}_{(2)}$, where $\mathbb{Z}_{(2)}$ is the localization of $\mathbb{Z}$ at the prime ideal (2). $\mathbb{Z}_{(2)}$ is J-clean by [23, Example 3.2] and hence A is also J-clean by [16, Example 2.19]. Now, consider the homomorphism $f : A \rightarrow A$ which maps (a, b) to $(a, 0)$. If $J = \text{Nil}(A) = 0 \ltimes \mathbb{Z}_{(2)}$, then by Theorem 2.7 part (2), $A \bowtie^{f,f} (J, J)$ is J-clean.

As we mentioned in the introduction, nil clean rings are J-clean. The authors of [23] provided an example of a J-clean ring that is not nil clean [23, Remark 3.9]. Below, we use their example and Theorem 2.7 to enrich the literature with a new counterexample of a J-clean ring that is not nil clean, which is a fourth application of Theorem 2.7.

Example 2.11. Let $A = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8 \cdots \times \mathbb{Z}_{2^n} \times \cdots$. By [23, Remark 3.9], A is J-clean which is not nil-clean, and hence $R = A \ltimes A$ is also J-clean ([16, Example 2.19]) which is not nil-clean ([17, Corollary 3.23 (1)]). Now, consider the homomorphism $f : R \rightarrow R$ which maps (a, b) to $(a, 0)$. If $J = \text{Nil}(A) = 0 \ltimes A$, then by Theorem 2.7 part (2), $R \bowtie^{f,f} (J, J)$ is J-clean. But since $I_o = f^{-1}(J) = 0 \ltimes A = \text{Nil}(A)$, and A is not nil-clean, then by referring to [1, Proposition 2.16], $R \bowtie^{f,f} (J, J)$ is not nil-clean.

A fifth application of Theorem 2.7 is the following counter-example of a commutative clean ring which is not J-clean.

Example 2.12. Let $A = \mathbb{Z}_3, B = C = A \ltimes A$, and $J = J' = 0 \ltimes A$. Let $f = g = \iota_A$, where ι_A is the canonical embedding of A into $A \ltimes A$ ($a \mapsto (a, 0)$). Then by [1, Example 2.7], $A \bowtie^{f,g} (J, J')$ is clean. But since $I_o = 0 = \text{Nil}(A)$, and A is not J-clean, then by Theorem 2.7 part (1), $A \bowtie^{f,g} (J, J')$ is not J-clean.

Next, we appeal to Lemmas 2.1, 2.5, 2.6 and Proposition 2.2 to provide a necessary and sufficient conditions for $A \bowtie^{f,g} (J, J')$ to be (weakly) J-clean in both of the cases $J' \subseteq \text{Nil}(C)$ and $J \subseteq \text{Nil}(B)$.

Proposition 2.13. *Consider $A \bowtie^{f,g} (J, J')$.*

- (1) *If $J' \subseteq \text{Nil}(C)$, then $A \bowtie^{f,g} (J, J')$ is (weakly) J-clean if and only if $f(A) + J$ is (weakly) J-clean.*
- (2) *If $J \subseteq \text{Nil}(B)$, then $A \bowtie^{f,g} (J, J')$ is (weakly) J-clean if and only if $g(A) + J'$ is (weakly) J-clean.*

Proof. (1) If $f(A) + J$ is (weakly) J-clean, then by Lemma 2.1, $\frac{A \bowtie^{f,g} (J, J')}{0 \times J'} \cong f(A) + J$ is (weakly) J-clean but $J' \subseteq \text{Nil}(C)$ gives that $0 \times J' \subseteq \text{Nil}(A \bowtie^{f,g} (J, J'))$, so by (Lemma 2.6) Lemma 2.5, $A \bowtie^{f,g} (J, J')$ is (weakly) J-clean. The converse follows from Proposition 2.2.

(2) Similar to the proof of part (1). □

As an application of Proposition 2.13, we provide the following corollary which investigates the transfer of the J-clean and the weakly J-clean properties in amalgamations along nil ideals; and hence we recover [16, Theorem 2.18 (1)] in the commutative setting.

Corollary 2.14. *Let $f : A \longrightarrow B$ be a ring homomorphism and J be an ideal of B such that $J \subseteq \text{Nil}(B)$. Then $A \bowtie^f J$ is (weakly) J-clean if and only if A is (weakly) J-clean.*

Proof. Note that $A \bowtie^f J = A \bowtie^{\iota_A, f} (I, J)$, where $I = f^{-1}(J)$ and $\iota_A = id_A$. Since $J \subseteq \text{Nil}(B)$, then by Proposition 2.13 part (1), $A \bowtie^f J = A \bowtie^{\iota_A, f} (I, J)$ is (weakly) J-clean if and only if $A = \iota_A(A) + I$ is (weakly) J-clean. □

Also, the transfer of the (weakly) J-clean property in the trivial ring extension can be recovered by the last corollary.

Corollary 2.15. *Let A be a ring and M an A -module. Then $A \ltimes M$ is (weakly) J-clean if and only if A is (weakly) J-clean.*

Proof. Since $A \ltimes M = A \bowtie^{\iota_A} (0 \ltimes M)$, where ι_A is the canonical embedding $\iota_A : A \hookrightarrow A \ltimes M$ defined by $a \mapsto (a, 0)$, for all $a \in A$, then this corollary follows from Corollary 2.14 and the fact that $0 \ltimes M \subseteq \text{Nil}(A \ltimes M)$. □

In the following proposition, we provide other conditions for a bi-amalgamation to be J-clean or the weakly J-clean ring.

Proposition 2.16. *Consider $A \bowtie^{f,g} (J, J')$. Suppose that $f(A)J = 0$, $g(A)J' = 0$ and $J \times J' \subseteq \text{Id}(B \times C)$. If A is (weakly) J-clean, then $A \bowtie^{f,g} (J, J')$ is (weakly) J-clean.*

Proof. Assume that A is J-clean. Let $(f(a) + j, g(a) + j') \in A \bowtie^{f,g} (J, J')$. Then $a = x + e$, where $x \in J(A)$ and $e \in \text{Id}(A)$. So, $(f(a) + j, g(a) + j') = (f(x), g(x)) + (f(e) + j, g(e) + j')$. Since $J \times J' \subseteq \text{Id}(B \times C)$ and $e \in \text{Id}(A)$, then by [1, Lemma 2.5], $(f(e) + j, g(e) + j') \in \text{Id}(A \bowtie^{f,g} (J, J'))$. Now, we show that $(f(x), g(x)) \in J(A \bowtie^{f,g} (J, J'))$. Let $f(y) + k \in f(A) + J$. Then $1 - xy$ is a unit in A since $x \in J(A)$. Since $f(A)J = 0$, $f(x)k = 0$, this implies $f(x)(f(y) + k) = f(x)f(y)$. Thus, $1 - f(x)(f(y) + k) = 1 - f(x)f(y) = f(1 - xy)$

is a unit in $f(A) + J$. It follows that $f(x) \in J(f(A) + J)$. Similarly, since $x \in J(A)$ and $g(A)J' = 0$, we have $g(x) \in J(g(A) + J')$. Hence, by Lemma 2.3, $(f(x), g(x)) \in J(A \bowtie^{f,g} (J, J'))$. Therefore, $A \bowtie^{f,g} (J, J')$ is J-clean. In a similar way, one can prove that if A is weakly J-clean, then $A \bowtie^{f,g} (J, J')$ is weakly J-clean. $\square$

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