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SPANNED VECTOR BUNDLES ON BLOWING UPS OF $\mathbb{P}^3$ AT FINITELY MANY POINTS AND WITH VERY LOW c_1

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ABSTRACT. We study spanned vector bundles on the blowing up of $\mathbb{P}^3$ at a few points. We give a full classification for very low c_1 and study some more general case.

1. INTRODUCTION

Fix an integer $s \geq 1$ and s distinct points $P_1, \dots, P_s$. Set $S := \{P_1, \dots, P_s\}$. Let $\pi_S : X_S \to \mathbb{P}^3$ be the blowing up of $\mathbb{P}^3$ along the points $P_1, \dots, P_s$. We often write π instead of π_S and X_s instead of X_S. We have $Pic(X_S) \cong \mathbb{Z}^{s+1}$ and we take $E := \pi^*(\mathcal{O}_{\mathbb{P}^3}(1))$ and $E_i := \pi^{-1}(P_i)$ as free generators of $Pic(X_S)$.

We study spanned vector bundles on X_S with very low c_1 (see Corollary 2.7 for $s = 1$, $c_1 = \mathcal{O}_{X_1}(E_0 - E_1)$, Theorem 3.2 and 3.3 for the case $s = 1$, $c_1 = \mathcal{O}_{X_1}(2E_0 - E_1 - E_2)$ and $s = 2$, $c_1 = \mathcal{O}_{X_2}(2E_0 - E_1 - E_2)$). It is sufficient to classify the spanned bundles without trivial factors. In section 2 we make some preliminary reduction step (e.g. it is sufficient to study all cases with $a_i < 0$ for all $i > 0$) and define the spanned bundles, for a fixed c_1, with largest rank (see Remark 2.2). There are many papers on spanned vector bundles ([1], [5], [6], [8], [10]) and the ones on $\mathbb{P}^2$ and $\mathbb{P}^3$ are used in this paper as a reduction step, i.e., a problem reduced to a similar problem on $\mathbb{P}^2$ or $\mathbb{P}^3$ is considered " solved " (see Corollaries 2.6 and 2.7 for $\mathbb{P}^2$, Lemma 2.8 for $\mathbb{P}^3$).

We work over an algebraically closed field $\mathbb{K}$ with characteristic zero.

2. PRELIMINARIES AND REDUCTION STEPS

Set $X_\emptyset = \mathbb{P}^3$ and let $\pi_\emptyset : \mathbb{P}^3 \to \mathbb{P}^3$ be the identity map. For any $i \in \{1, \dots, s\}$ set $S_i := S \setminus \{P_i\}$. We have $\pi_S = u_i \circ \pi_{S_i}$, where $u_i : X_S \to X_{S_i}$ is the blowing up of the point $\pi_{S_i}^{-1}(P_i)$. In the following X is a smooth rational 3-fold (e.g: $X = X_S$ for some S). We have $\omega_{X_S}^\vee \cong \mathcal{O}_{X_S}(4E_0 - 2E_1 - \cdots - 2E_s)$ ([7, Exercise II.8.5]).

Remark 2.1. Fix a globally generated rank $r \geq 2$ vector bundle $\mathcal{E}$ on a smooth 3-fold X and assume $h^1(\mathcal{O}_X) = 0$. Let C be the dependency locus of $r-1$ general sections of $\mathcal{E}$. We get an exact sequence

$$0 \to \mathcal{O}_X^{r-1} \to \mathcal{E} \to \mathcal{I}_C \otimes \mathcal{L} \to 0 \tag{2.1}$$

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with either $C = \emptyset$ or C a smooth curve ([3], [4]). Write $C = C_1 \sqcup \cdots \sqcup C_x$ with $x \geq 1$, where C_i are the connected components of C. Since $\mathcal{E}$ is spanned and $\mathcal{L}$ is the determinant of $\mathcal{E}$, then $\mathcal{L}$ is spanned. Since $\mathcal{E}$ is spanned, (2.1) gives that $\mathcal{I}_C \otimes \mathcal{L}$ is globally generated. Conversely, take a spanned line bundle $\mathcal{L}$ on X and a " nice " curve (if $r = 2$ assume that C is locally a complete intersection) with $\omega_C \otimes (\omega_X|C) \otimes \mathcal{L}^\vee$ globally generated; if $r = 2$ also assume $\omega_C \otimes (\omega_X|C) \otimes \mathcal{L}^\vee \cong \mathcal{O}_C$. Then $r-1$ general sections of $\omega_C \otimes (\omega_X|C) \otimes \mathcal{L}^\vee$ give an exact sequence (2.1) ([2]). Since $h^1(\mathcal{O}_X) = 0$, $\mathcal{E}$ is spanned if and only if $\mathcal{I}_C \otimes \mathcal{L}$ is spanned. Assume that $\mathcal{I}_C \otimes \mathcal{L}$ is spanned and take the intersection Y of two general elements of $|\mathcal{L}|$. It is easy to check that each component of C appears with multiplicity one in Y. Bertini's theorem gives $Y = C \cup D$ with either $D = \emptyset$ or D a reduced curve smooth outside the finite set $C \cap D$.

Remark 2.2. Let W be an integral projective variety. Set $n := \dim(W)$ and assume $n \geq 3$. Fix a spanned line bundle $\mathcal{L}$ on X, $\mathcal{L} \neq \mathcal{O}_W$. Fix a linear subspace $M \subseteq H^0(\mathcal{L})$, spanning $\mathcal{L}$ and let $\psi_M : M \otimes \mathcal{O}_W \to \mathcal{L}$ the evaluation map. Since ψ_M is surjective, the coherent sheaf $ker(\psi_M)$ is a vector bundle with rank $r := \dim(M) - 1$. The vector bundle $\mathcal{F} := ker(\psi_M)^\vee$ is a spanned vector bundle with determinant $\mathcal{L}$ and rank $\dim(M) - 1$ which fits in an exact sequence

$$0 \to \mathcal{L}^\vee \to \mathcal{O}_W^{\oplus(r+1)} \to \mathcal{F} \to 0.$$

which determines all Chern classes of $\mathcal{F}$. Since the natural map $M \to H^0(\mathcal{L})$ is injective, $\mathcal{F}$ has no trivial factor. In the case $W = X_s$ we may compute the multidegrees of the dependency locus of $r-1$ general sections of $\mathcal{F}$. Now assume $h^0(\mathcal{L}) \geq \dim(W)+1$. By a lemma of Serre for every integer r with $\dim(W) \leq r \leq h^0(\mathcal{L}) - 1$ there is an $(r+1)$-dimensional linear subspace $M \subseteq H^0(\mathcal{L})$ spanning $\mathcal{L}$. Therefore we cover all ranks r with $n+1 \leq r \leq h^0(\mathcal{L}) - 1$. These vector bundles may also be obtained in the following way. We assume $n = 3$, that W is smooth (isolated singularities would be OK with minimal modifications) and that $h^0(\mathcal{L}) \geq 4$. Let $Y \subset M$ be any locally Cohen-Macaulay scheme with pure dimension 1 and which is the complete intersection of 2 elements of $|\mathcal{L}|$. Assume $Y \neq \emptyset$ (e.g., assume $\mathcal{L}$ big and nef). We also assume $h^1(\mathcal{O}_W) = h^2(\mathcal{O}_W) = 0$ (e.g. M is rational) and $h^1(\mathcal{L}) = 0$. With these assumptions we have $h^1(\mathcal{I}_Y \otimes \mathcal{L}) = 0$ and hence $h^0(Y, \mathcal{L}|Y) = h^0(\mathcal{L}) - 2$. We have $\omega_Y \cong (\omega_W|Y) \otimes \mathcal{L}^{\otimes 2}$. Hence $\omega_Y \otimes (\omega_W^\vee|Y) \otimes \mathcal{L}^\vee) \cong \mathcal{L}|Y$ and so $h^0(\omega_Y \otimes (\omega_W^\vee|Y) \otimes \mathcal{L}^*) = h^0(\mathcal{L}) - 2$. Since $\dim(Y) = 1$, a general 2-dimensional linear section of $h^0(Y, \mathcal{L}|Y)$ spans $\mathcal{L}|Y$. Using the Serre correspondence ([2]) we get spanned vector bundles $\mathcal{E}$ on W fitting in (2.1) with $C = Y$ and as r any integer with $3 \leq r \leq h^0(\mathcal{L}) - 1$. Taking general sections we easily see that the two constructions give the same bundle. For the maximal r, i.e. when $r = h^0(\mathcal{L}) - 1$, we get a unique bundle (we need to take $M = H^0(\mathcal{L})$ and for the Serre correspondence we need to use $H^0(Y, \mathcal{L}|Y)$. Since this bundle (with the maximum rank possible) $\mathcal{F}$ is unique, we have $g^*(\mathcal{F}) \cong \mathcal{F}$ for every automorphism g of W such that $g^*(\mathcal{L}) \cong \mathcal{L}$. Now we also assume $h^1(\mathcal{L}^\vee) = 0$ (e.g. $\mathcal{L}$ ample). Since Y is a complete intersection of two elements of $|\mathcal{L}|$ we get $h^0(\mathcal{O}_Y) = 1$ and in particular Y is connected.

Lemma 2.3. *Let W be an n-dimensional projective variety, L, R line bundle on W and $\mathcal{E}$ a rank 2 vector bundle on W fitting in a non-trivial exact sequence*

$$0 \to L \xrightarrow{h} \mathcal{E} \xrightarrow{z} R \to 0$$

Assume $h^0(R \otimes L^\vee) = 0$ and that each endomorphism of E preserve (**??**). *Then $h^0(\mathcal{E}) = 1 + h^0(L \otimes R^\vee)$ and $\mathcal{E}$ is indecomposable.*

Proof. Let e be the extension of R by L giving $\mathcal{E}$. For each $t \in \mathbb{K} \setminus \{01\}$ the extension te has as its middle term a vector bundle isomorphic to $\mathcal{E}$. The extension $0e = 0$ has $L \oplus R$ as its middle term. We get a flat family of vector bundles $\{\mathcal{E}_t\}_{t \in \mathcal{E}}$ with $E_0 = L \oplus R$ and $\mathcal{E}_t \cong \mathcal{E}$ for all $t \neq 0$. By the semicontinuity theorem for cohomology gives $h^0(\mathcal{E} \otimes \mathcal{E}^\vee) \leq 2 + h^0(L \otimes R^\vee)$. Every $f : R \to L$ induces $g : \mathcal{E} \to \mathcal{E}$ with $g(L) = 0$ using $h \circ f$. Since the identity map $\mathcal{E} \to \mathcal{E}$ does not send L into 0, we get $h^0(\mathcal{E} \otimes \mathcal{E}^\vee) \leq 2 + h^0(L \otimes R^\vee)$. Every endomorphism $u : \mathcal{E} \to \mathcal{E}$ with $u(L) = 0$ induces a map $v : R \to R$. Since $e \neq 0$, we see that $z \circ u : R \to R$ is not an isomorphism. Since R is a line bundle, we get $z \circ u = 0$, i.e. $u = h \circ f \circ z$ for some $f : R \to L$. Now take any $u : \mathcal{E} \to \mathcal{E}$ with $u(L) \neq 0$. Since $h^0(R \otimes L^\vee) = 0$, we have $u(L) \subseteq L$. Hence $u|L = tId_L$ for some $t \in \mathbb{K}$. Therefore $u = tId_{\mathcal{E}} + f \circ z$ for some $f : R \to L$. The structure of the endomorphism ring of $\mathcal{E}$ shows that $\mathcal{E} \neq L_1 \oplus L_2$ for any two line bundles L_1, L_2 on W. □

Lemma 2.4. *Let W be a smooth 3-fold, $\mathcal{L}$ a line bundle on W and $C \subset W$ be a smooth curve with $x \geq 1$ connected components and $\omega_C \cong (\omega_M|C) \otimes \mathcal{L}$. Then the Serre corresponds gives rank r line bundle associated to C and with no trivial factor if and only if $2 \leq r \leq x + 1$.*

Proof. We have $h^0(\omega_C \otimes (\omega_M^\vee|C) \otimes \mathcal{L}^\vee) = h^0(\mathcal{O}_C) = x$. □

Lemma 2.5. *Let W be an integral projective variety and $\mathcal{L}$ a spanned vector bundle on W. Let $\eta : W \to Y$ be the Stein factorization of the morphism induced by $|\mathcal{L}|$. Assume that η is flat. Let $\mathcal{E}$ be a spanned vector bundle on W with $\mathcal{L}$ as its determinant. Then there is a unique vector bundle $\mathcal{F}$ on Y such that $\mathcal{E} \cong \eta^*(\mathcal{F})$. We have $\mathcal{F} \cong \eta_*(\mathcal{E})$ and $h^0(\mathcal{E}) = h^0(\mathcal{F})$.*

Proof. Let T be any scheme-theoretic fiber of η. Set $r = rk(\mathcal{E})$. Since a spanned vector bundle with trivial determinant is trivial, $\mathcal{E}_{|T}$ is trivial. In particular we have $h^0(T, \mathcal{E}_{|T}) = r$. Set $\mathcal{F} := \pi_*(\mathcal{E})$. Since Y is reduced, [9, Base change theorem, page 11] gives that $\mathcal{F}$ is locally free. We also get that the natural map $\eta^*(\eta_*(\mathcal{E})) \to \mathcal{E}$ is an isomorphism. The last statement follows from the projection formula, because $\eta_*(\mathcal{O}_W) = \mathcal{O}_Y$. □

Corollary 2.6. *Let $\tau : X_1 \to \mathbb{P}^2$ be the morphism associated at the linear projection $\mathbb{P}^3 \setminus \{P_1\} \to \mathbb{P}^2$. This morphism is a $\mathbb{P}^1$-bundle and in particular it is flat. Fix an integer $k > 0$. The map $\mathcal{F} \to \tau^*(\mathcal{F})$ induces a bijection between the spanned vector bundles on $\mathbb{P}^2$ with determinant $\mathcal{O}_{\mathbb{P}^2}(k)$ and the isomorphism classes of spanned vector bundles on X_1 with $\mathcal{O}_{X_1}(k, -k)$ as determinant. We have $h^0(\tau^*(\mathcal{F})) = \mathcal{F}$ and $\tau^*(\mathcal{F})$ has no trivial factor if and only if $\mathcal{F}$ has no trivial factor.*

By Corollary 2.6 we may use any result on the classification of spanned vector bundle on $\mathbb{P}^2$ with $c_1 = k$ (e.g. [5], [6]) to get a corresponding result for spanned vector bundle on X_1 with $c_1 = (k, -k)$. For instance we get the following result.

Corollary 2.7. *$\tau^*(T\mathbb{P}^2(-1))$ is the only rank $r \geq 2$ spanned vector bundle on X_1 with $c_1 = (1, -1)$ and no trivial factor.*

Lemma 2.8. *Let $\mathcal{E}$ be a spanned vector bundle on X_S with $c_1(\mathcal{E}) = \mathcal{O}_{X_S}(a_0E_0 + \cdots + a_sE_s)$ with $a_i = 0$ for some $i > 0$. Set $S_i := S \setminus \{P_i\}$. Let $u_i : X_S \to X_{S_i}$ be the blowing down of E_i. Set $\mathcal{F} := u_{i*}(\mathcal{E})$. Then $\mathcal{F}$ is a spanned vector bundle, $\mathcal{E} \cong u_i^*(\mathcal{F})$ and $h^k(\mathcal{E}) = h^k(\mathcal{F})$ for all $k \geq 0$.*

Proof. Set $r := rk(\mathcal{E})$. Since a spanned vector bundle with trivial determinant is trivial, we have $\mathcal{E}|E_i \cong \mathcal{O}_{E_i}^{\oplus r}$. For each $n \geq 2$ we have an exact sequence

$$0 \to (\mathcal{E}_{|E_i})^{\oplus(n+1)}(-n) \to \mathcal{E}_{|E_i} \to \mathcal{E}_{|(n-1)E_i} \to 0 \tag{2.2}$$

By induction on n from (2.2) we get $h^0(nE_i, \mathcal{E}_{|nE_i}) = r$. Since $\mathcal{E}_{|nE_i}$ is a rank r spanned vector bundle, $\mathcal{E}_{|E_i}$ is trivial and in particular $h^1(nE_i, \mathcal{E}_{|nE_i}) = 0$ for all $n > 0$. By the formal function theorem for cohomology we get that $\mathcal{F} := u_{i*}(\mathcal{F})$ is a rank r vector bundle on X_{S_i} and that $\mathcal{E} \cong u_i^*(\mathcal{F})$. Since $u_{i*}(\mathcal{O}_X) = \mathcal{O}_{S_i}$ and $R^j u_{i*}(\mathcal{O}_X) = 0$ for all $j > 0$, the projection formula and the Leray spectral sequence of u_i give $h^k(\mathcal{E}) = h^k(\mathcal{F})$ for all $k \geq 0$. □

Corollary 2.9. *Let $\mathcal{E}$ be a spanned vector bundle on X_S. Set $c_1(\mathcal{E}) := \mathcal{O}_{X_s}(a_0E_0 + \cdots a_sE_s)$. Assume the existence of $i \in \{1, \ldots, s\}$ such that $a_i = 0$. Then there is a unique vector bundle $\mathcal{F}$ on X_{S_i} such that $\mathcal{E} = u_i^*(\mathcal{F})$. Moreover $\mathcal{F}$ is spanned and $h^k(\mathcal{E}) = h^k(\mathcal{F})$ for all $k \geq 0$.*

Since we only need to check spanned determinants, by Corollary 2.9 it is sufficient (for a fixed $a_0 > 0$) to list all bundles for all s and all $c_1 = \mathcal{O}_{X_s}(a_0E_0 + \cdots a_sE_s)$ with $a_i < 0$ for all $i > 0$.

The cohomology ring $A(X_s)$ is freely generated over $\mathbb{Z}$ by the classes $t_0, \ldots, t_s$ of $E_0, \ldots E_s$ with the relations $t_it_j = 0$ for all $i \neq j$, $t_0^3 = 1$ and $t_i^3 = -1$ for $i = 1, \ldots, s$. For each locally Cohen-Macaulay scheme T with pure dimension 1 the multidegree $\underline{\deg}(T)$ of T is the element $(b_0, \ldots, b_s) \in \mathbb{Z}^{s+1}$ with $b_i := \deg(\mathcal{O}_T(E_i))$ for all i (degree of a line bundle). If, as usual, $C = C_1 \sqcup \cdots \sqcup C_x$ is a general dependency locus of a spanned bundle $\mathcal{E}$ set $e_y := \deg(\mathcal{O}_C(E_y))$ and $e(j)_y := \deg(\mathcal{O}_C(E_y))$, $1 \leq j \leq x$, $0 \leq y \leq s$.

The following lemmas explains why the case $c_1 = \mathcal{O}_{X_s}(kE_0 - E_1 - \cdots - E_s)$ is very particular.

Lemma 2.10. *Let $C = C_1 \cup \cdots \cup C_x \subset X_s$ be a smooth curve with each C_i connected and $\mathcal{I}_C(tE_0 - E_1 - \cdots - E_1)$ spanned for some $t > 1$.a, $t > 1$.*

(a) Fix $i \in \{1, \ldots, s\}$ and assume $C_h \subset E_i$ for some $h \in \{1, \ldots, x\}$. Then $e(h)_i = 1$, $e(k)_i = 0$ for all $k \in \{1, \ldots, x\} \setminus \{h\}$ and $C_k \cap E_h = \emptyset$ for all $k \in \{1, \ldots, x\} \setminus \{h\}$.

(b) Fix $i \in \{1, \ldots, s\}$ and assume $C_h \nsubseteq E_i$ for some $h \in \{1, \ldots, x\}$. Then $0 \leq e(h)_i \leq 1$ for all h and $e_i \leq 1$.

(c) If C is smooth dependency locus of a spanned bundle $\mathcal{E}$ with $c_1(\mathcal{E}) = (t, -1, \ldots, -1)$, then $C_h \nsubseteq E_i$ for all $h > 0$, $i > 0$.

Proof. Assume $C_h \cap E_i \neq \emptyset$. First assume $C_h \subset E_i$. Since $\det(\mathcal{E})|E_i \cong \mathcal{O}_{E_i}(1)$ and $\mathcal{I}_C(k,-1,\dots,-1)$ is spanned, $e(h)_i = 1$ (i.e. C_h is a line of the plane E_i) and $C_k \cap E_i = \emptyset$ for all $k \in \{1,\dots,x\}\setminus\{h\}$. We have $\mathcal{O}_{C_h}(4E_0 - 2E_1 - \cdots - 2E_s - c_1) = \mathcal{O}_{C_h}(-E_i)$ and hence $\deg(\mathcal{O}_{C_h}(4E_0 - 2E_1 - \cdots - 2E_s - c_1)) = 1$. Since $C_h \cong \mathbb{P}^1$, the sheaf $\omega_{C_h}(4E_0 - 2E_1 - \cdots - 2E_s - c_1)$ has no section. Hence by the Serre correspondence ([2]) C cannot be a dependency locus.

Now assume $C_h \nsubseteq E_i$. By the part just proved no component of C is contained in E_h, i.e. the scheme $C \cap E_i$ is a non-empty zero-dimensional scheme. Since $\det(\mathcal{E})|E_i \cong \mathcal{O}_{E_i}(1)$ and $\mathcal{I}_C(kE_0 - E_1 - \cdots - E_s)$ is spanned, we get $e_i = 1$. □

Remark 2.11. Let $C = C_1 \sqcup \cdots \sqcup C_x \subset X_s$ be a smooth dependency locus of a spanned bundle with $\mathcal{O}_{X_s}(kE_0 - E_1 - \cdots - E_s)$ as its determinant. Lemma 3.2 gives that $\pi|C$ is an embedding and in particular that $\pi(C)$ is a smooth curve with x connected components and $\deg(\pi(C_h)) = \sum_{i\geq 0} e(h)_i$ for each connected component C_h of C.

Lemma 2.12. *Fix a set $A \subseteq \{1,\dots,s\}$ with $A \neq \emptyset$. We have $h^1(\mathcal{O}_{X_s}(-\sum_{i\in A} E_i)) = \sharp(A) - 1$. We have $h^1(\mathcal{O}_{X_s}(E_i)) = 0$ and $h^0(\mathcal{O}_{X_s}(E_i)) = 1$ for all $i > 0$. If $s \geq 2$, then $h^1(\mathcal{O}_{X_s}(E_1 - E_2)) = 0$.*

Proof. The first statement follows from the exact sequence

$$0 \to \mathcal{O}_{X_s}(-\sum_{i\in A} E_i) \to \mathcal{O}_{X_S} \to \sqcup_{i\in A}\mathcal{O}_{E_i} \to 0,$$

because $E_i \cap E_j$ for all $i \neq j$ and $i,j > 0$. Fix $i \in \{1,\dots,s\}$. Since $h^0(\mathcal{O}_{X_s}(E_1)) = 1$ and $h^1(E_1, \mathcal{O}_{E_1}(E_1)) = 0$, the exact sequence

$$0 \to \mathcal{O}_{X_s} \to \mathcal{O}_{X_s}(E_1) \to \mathcal{O}_{E_1}(E_1) \to 0$$

gives $h^1(\mathcal{O}_{X_s}(E_1)) = 0$. Since $E_1 \cap E_2 = \emptyset$, the exact sequence

$$0 \to \mathcal{O}_{X_s}(E_1 - E_2) \to \mathcal{O}_{X_s}(E_1) \to \mathcal{O}_{X_s}(E_1 - E_2) \to 0$$

gives $h^1(\mathcal{O}_{X_s}(E_1 - E_2)) = 0$. □

Remark 2.13. $\mathcal{O}_{X_s}(E_0 - \sum_{i>0} E_i)$ is spanned if and only if $s = 1$. $\mathcal{O}_{X_s}(2E_0 - \sum_{i>0} E_i)$ is spanned if $s \leq 6$ and no 3 of the points of S are collinear, no 5 of the points of S are coplanar. For any $k \geq 2$ there is $S \subset \mathbb{P}^3$ with $\sharp(S) = k^3$ and $\mathcal{O}_{X_S}(kE_0 - \sum_{i>0} E_i)$ globally generated. In this case $\mathbb{P}^2$ is the image by the complete linear and the fiber are connected. We are in the set-up of Lemma 2.5 (the general fiber is reduced and connected) and we may apply [5] and [6] in this case. For any $k \geq 2$ for a general $S \subset X$ with $\sharp(S) = \binom{k+3}{3} - 3$, the line bundle $\mathcal{O}_{X_S}(kE_0 - \sum_{i>0} E_i)$ is not spanned.

3. The main results

Let $\mathcal{E}$ be a rank $r \geq 2$ spanned vector bundle on a smooth 3-fold W. Let $C \subset W$ be the dependency locus of $r-1$ general sections of $\mathcal{E}$. Then either $C = \emptyset$ or C is a smooth curve ([3][4]). Let $C_1,\dots,C_x$ denote the connected components of C.

Remark 3.1. Assume $C \neq \emptyset$. By the Serre correspondence ([2]) $\omega_C(4E_0 - 2E_1 - \cdots - 2E_s - c_1)$ is spanned and trivial if $r = 2$. First assume $\omega_C(4E_0 - 2E_1 - \cdots - 2E_s - c_1) \cong \mathcal{O}_C$. Since $h^0(\mathcal{O}_C) = x$, by the Serre correspondence we get a rank r spanned vector bundle for all r with $2 \leq r \leq x+1$. Now assume that $\omega_C(4E_0 - 2E_1 - \cdots - 2E_s - c_1)$ is spanned, but not trivial. The Serre correspondence gives a rank r spanned bundle with no trivial factor if and only if $3 \leq r \leq h^0(\omega_C(4E_0 - 2E_1 - \cdots - 2E_s - c_1)) + 1$ (just taking a suitable $(r-1)$-dimensional linear subspace of $H^0(\omega_C(4E_0 - 2E_1 - \cdots - 2E_s - c_1)))$. Now assume $C = \emptyset$. In this case $\mathcal{E}$ is an extension of $\det(\mathcal{E})$ by $\mathcal{O}_{X_s}^{\oplus(r-1)}$. If $h^1(\det(\mathcal{E})^\vee) = 0$ (e.g. if $\det(\mathcal{E})$ is ample), the $\mathcal{E}$ has $r-1$ trivial factors.

Theorem 3.2. *Assume $s = 1$. The rank $r \geq 2$ spanned vector bundles on X_1 with $c_1 = \mathcal{O}_{X_1}(2E_0 - E_1)$ and no trivial factor are the following ones(in each case we also list the multidegrees $e(i)_j$ of the connected components of a general dependency locus C and the number x of connected components of C):*

1. *$3 \leq r \leq 8$, $\mathcal{E}$ as in Remark 2.1, $C = Y$, $x = 1$, and multidegree $(3,1)$;*
2. *$\mathcal{O}_{X_1}(E_0) \oplus \mathcal{O}_{X_1}(E_0 - E_1)$; each curve C with multidegree $(1,0)$ gives this bundle;*
3. *$r = 3, 4$, $x = 1$, $(e_0, e_1) = (2, 0)$;*
4. *$r = 3$, $x = 1$, $(e_0, e_1) = (2, 1)$.*

Proof. We have $\omega_{X_1}(E_1 - 2E_0) \cong \mathcal{O}_{X_1}(2E_0 - E_1)$. Therefore we look at C's with $\omega_C(2E_0 - E_1)$ spanned. Since $(2t_0 - t_1)(2t_0 - t_1) = 4t_0^2 + t_1^2$, the complete intersection curve Y has multidegree $(3,1)$, $h^0(\mathcal{O}_Y) = 1$ (Remark 2.1), arithmetic genus 1 and $\omega_Y \cong \mathcal{O}_Y$. Therefore $\deg(\omega_Y(2E_0 - E_1)) = \deg(\mathcal{O}_Y(2E_0 - E_1)) = 8-1$ and $\omega_Y(2E_0 - E_1)$ is spanned, but not trivial. Riemann-Roch gives $h^0(\omega_Y(2E_0 - E_1)) = 7$. Hence this case gives all ranks r with $3 \leq r \leq 8$ (Remark 2.1). Now assume $C \neq Y$. Since $Y = C \cup D$, $p_a(Y) = 1$ and Y is connected, at most one connected component of C is not rational and this component (if it exists) is elliptic.

Assume the existence of some $\mathcal{E}$ of rank $r \geq 2$ and call C a general dependency locus of $r-1$ sections of $\mathcal{E}$. The line bundle $\omega_C(2E_0 - E_1)$ is spanned (and trivial if $r = 2$). Part (c) of Lemma 3.2 gives that no component of C is contained in E_1 and hence $\deg(\mathcal{O}_{C_h}(E_1)) \geq 0$ and $e(h)_0 - e(h)_1 \geq 0$. Part (b) of Lemma 3.2 gives $e_1 \leq 1$ and that $C \cap E_1$ is at most one point. Since $e_1 \leq 1$, the case $(e_0, e_1, x) = (2, 2, 2)$ is not allowed.

Claim 1: The vector bundle $\mathcal{O}_{X_1}(E_0) \oplus \mathcal{O}_{X_1}(E_0 - E_1)$ is the only vector bundle associated to a curve C with multidegree $(1,0)$.

Proof of Claim 1: We have $c_2(\mathcal{O}_{X_1}(E_0) \oplus \mathcal{O}_{X_1}(E_0 - E_1)) = t_0^2$ and hence the dependency locus C has multidegree $(1,0)$. Conversely, take any smooth curve $C \subset X_1$ with multidegree $(1,0)$, i.e. corresponding to a line of $\mathbb{P}^3$ not containing P_1. Since $C \cong \mathbb{P}^1$ and $C \cap E_1 = \emptyset$, then $\omega_C(2E_0 - E_1) \cong \mathcal{O}_C$. Since $x = 1$, C only corresponds to a rank 2 spanned bundle $\mathcal{E}$ (Remark 3.1). Since $h^0(\mathcal{O}_C(E_0 - E_1)) = 2 < h^0(\mathcal{O}_{X_1}(E_0 - E_1))$, we have $h^0(\mathcal{I}_C(E_0 - E_1)) > 0$ and so there is a non-zero map $m : \mathcal{O}_{X_1}(E_0) \to \mathcal{E}$. Since $h^0(\mathcal{I}_C(E_0 - 2E_1)) = h^0(\mathcal{I}_C(-E_1)) = 0$, *coker*$(m)$ is torsion free. Hence *coker*$(m) \cong \mathcal{I}_T(E_0 - E_1)$ with either $T = \emptyset$ or T a locally complete intersection curve. Since $c_2(\mathcal{O}_{X_1}(E_0)\oplus\mathcal{O}_{X_1}(E_0-E_1))=(1,0)$, we first get $T=\emptyset$ and then $\mathcal{E}\cong\mathcal{O}_{X_1}(E_0)\oplus\mathcal{O}_{X_1}(E_0-E_1)$ by Lemma 2.12.

Assume $C\neq Y$ and C with multidegree $(e_0,e_1)\neq(1,0)$. Since $C\subsetneq Y$, we have $e_0-e_1\le 3$ and $0\le e_1\le 1$ and $(e_0,e_1)\neq(3,1)$. We know that $e(i)_0-e(i)_1\ge 0$.

First assume $(e_0,e_1,x)=(2,0,2)$. Hence $\pi(C)$ is the union of two disjoint lines of $\mathbb{P}^3$ not through P_1. The linear projection of $\pi(C)$ from P_1 shows that there is a unique line $L\subset\mathbb{P}^3$ through P_1 with $\sharp(L\cap\pi(C))=2$. Let $T\subset X_1$ be any curve with multidegree $(1,1)$ and $L=\pi(T)$. Since L is contained in the base locus of $\mathcal{I}_{\pi(C)\cup\{P_1\},\mathbb{P}^3}(2)$, T is contained in the base locus of $\mathcal{I}_C(2E_0-E_1)$.

Now assume $(e_0,e_1,x)=(2,0,1)$. In this case $\pi(C)$ is a smooth conic not containing P_1. Taking the linear projection from P_1 we see that $\mathcal{I}_C(2E_0-E_1)$ is globally generated if and only if P_1 is not contained in the plane spanned by $\pi(C)$. Since $C\cong\mathbb{P}^1$ and $\deg(\omega_C(2E_0-E_1))=2$, we have $h^0(\omega_C(2E_0-E_1))=3$. Hence this case gives globally generated bundles if and only if $3\le r\le 4$.

Now assume $(e_0,e_1,x)=(2,1,1)$. In this case $\pi(C)$ is a degree 3 rational normal curve containing P_1 and $\omega_C(2E_0-E_1)$ has degree 1. Hence $\omega_C(2E_0-E_1)$ is spanned, non-trivial and $h^0(\omega_C(2E_0-E_1))=2$. Let $D\subset X_1$ be any smooth and connected curve such that $D\cap E_1=\emptyset$ and $\pi(D)$ is a general secant line of $\pi(C)$. The curve $C\cup D$ is the complete intersection of two elements of $\mathcal{I}_C(2E_0-E_1)$ and hence $\mathcal{I}_{C\cup D}(2E_0-E_1)$ is spanned. Hence $\mathcal{I}_C(2E_0-E_1)$ is spanned outside D. Taking another D' as D with $D'\cap D=\emptyset$ we get that $\mathcal{I}_C(2E_0-E_1)$ is spanned. Therefore this case gives a rank r spanned bundle if and only if $r=3$.

Now assume $(e_0,e_1,x)=(2,1,2)$. In this case either $\pi(C)$ is the disjoint union of a smooth conic through P_1 and a line not through P_1 or the disjoint union of a smooth conic K not containing P_1 and a line L through P_1 (Remark 2.11). In both cases we have $h^0(\mathcal{I}_C(2E_0-E_1))=h^0(\mathbb{P}^3,\mathcal{I}_{\pi(C)}(2))=2$. Since C is not the complete intersection of to elements of $|\mathcal{I}_C(2E_0-E_1)|$, $\mathcal{I}_C(2E_0-E_1)$ is not spanned.

Now assume $(e_0,e_1,x)=(2,1,3)$. Since $\pi(C)$ is the disjoint union of 3 lines, we have $h^0(\mathcal{I}_C(2E_0-E_1))\le 1$ and hence $\mathcal{I}_C(2E_0-E_1)$ is not spanned.

If $e_1=0$ and $e_0=3$, then $\mathcal{I}_C(2E_0-E_1))$ is not spanned, because in all cases with smooth $\pi(C)$ we have $h^0(\mathcal{I}_{\pi(C)}(2))\le 2$. □

Theorem 3.3. *Assume $s=2$. The rank $r\ge 2$ spanned vector bundles on X_2 with $c_1=\mathcal{O}_{X_2}(2E_0-E_1-E_2)$ and no trivial factor are as the following ones:*

1. *$3\le r\le 7$, $\mathcal{E}$ as in Remark 2.1, $C=Y$, $x=1$, and multidegree $(2,1,1)$;*
2. *$\mathcal{O}_{X_2}(E_0-E_2)\oplus\mathcal{O}_{X_2}(E_0-E_1)$; C has $x=1$ and multidegree $(1,0,0)$.*
3. *$\mathcal{O}_{X_2}(E_0)\oplus\mathcal{O}_{X_2}(E_0-E_1-E_2)$ and the only non-trivial extension $\mathcal{G}$ of $\mathcal{O}_{X_2}(E_0)$ by $\mathcal{O}_{X_2}(E_0-E_1-E_2)$; $\mathcal{G}$ is indecomposable and $h^0(\mathcal{G})=2$;*
4. *$r=3,4$, $x=1$ and $(e_0,e_1,e_2)=(2,0,0)$;*

Proof. Since $\omega_{X_2}^\vee\cong\mathcal{O}_{X_2}(4E_0-2E_1-2E_2)$, we need to check the smooth curves C with $\omega_C(2E_0-E_1-E_2)$ globally generated. By Lemma 3.1 no connected component of C is contained in $E_1\cup E_2$, $e_1\le 1$, $e_2\le 1$, and $e(h)_i\ge 0$ and $e(h)_0-e(h)_i$ for all $h=1,\dots,x$ and all $i\ge 0$.

We have $(2t_0-t_1-t_2)(2t_0-t_1-t_2)=4t_0^2+t_1^2+t_2^2$ and hence the complete intersection curve Y has multidegree $(2,1,1)$, $p_a(Y)=1$, $\omega_Y\cong\mathcal{O}_Y$ and $\omega_Y(2E_0-$

$E_1 - E_2)$ is spanned, but not trivial. By Remark 2.1 this case gives a rank r spanned bundle if and only if $3 \le r \le 7$.

Now assume $C \subsetneq Y$ and hence $e_0 + e_1 + e_2 \le 4$, $0 \le e_1 \le 1$, $0 \le e_2 \le 1$ and $(e_0, e_1, e_2) \ne (2, 1, 1)$.

Claim 1: The vector bundles $\mathcal{O}_{X_2}(E_0 - E_2) \oplus \mathcal{O}_{X_2}(E_0 - E_1)$ and the extensions of $\mathcal{O}_{X_1}(E_0 - E_1 - E_2)$ by $\mathcal{O}_{X_2}(E_0)$ are the only vector bundles associated to a curve C with multidegree $(1, 0, 0)$.

Proof of Claim 1: We have $c_2(\mathcal{O}_{X_2}(E_0) \oplus \mathcal{O}_{X_1}(E_0 - E_1 - E_2)) = c_2(\mathcal{O}_{X_2}(E_0) \oplus \mathcal{O}_{X_2}(E_0 - E_1 - E_2)) = t_0^2$, i.e. the associated curve of these bundles has multidegree $(1, 0, 0)$. Take any curve C with multidegree $(1, 0, 0)$, i.e. with $\pi(C)$ a line not intersecting $\{P_1, P_2\}$. Since $\omega_C(2E_0 - E_1 - E_2) \cong \mathcal{O}_C$ and C is connected, it only corresponds to a rank 2 vector bundle (Lemma 2.3). Let $\mathcal{E}$ be a rank 2 vector bundle associated to C. Since $h^0(\mathcal{O}_C(E_0 - E_1)) = 2 < 3 = h^0(\mathcal{O}_{X_2}(E_0 - E_1))$, there is a non-zero map $m : \mathcal{O}_{X_2}(E_0) \to \mathcal{E}$. Since $h^0(\mathcal{I}_C(-E_1)) = h^0(\mathcal{I}_C(E_0 - 2E_1)) = 0$, either $coker(m)$ is torsion free (and as in the proof of Theorem 3.2 we get $\mathcal{E} \cong \mathcal{O}_{X_2}(E_0 - E_2) \oplus \mathcal{O}_{X_2}(E_0 - E_1))$, or m drops rank on E_2 and hence it induces a non-zero map $m' : \mathcal{O}_{X_2}(E_0 - E_2) \to \mathcal{E}$. Since $h^0(\mathcal{I}_C(E_0 - 2E_2)) = 0$, we get that $coker(m')$ is torsion free. Lemma 2.3 conclude the proof, because two bundles associated to proportional extensions are isomorphic.

Take C with $x = 1$ and multidegree either $(1, 1, 0)$ or $(1, 0, 1)$ or $(1, 1, 1)$. We have $C \cong \mathbb{P}^1$ and $\deg(\mathcal{O}_C(2E_0 - E_1 - E_2) \le 1$ and hence $h^0(\omega_C(2E_0 - E_1 - E_2)) = 0$. Therefore this case gives no bundle. No bundle comes from the same multidegrees, but with $x \ge 2$, because at least one component would be a curve $C_h \cong \mathbb{P}^1$ with multidegree $(1, 1, 0)$ or $(1, 0, 1)$ or $(1, 1, 1)$.

Take C with multidegree $(2, 0, 0)$ and $x = 1$, i.e. with $\pi(C)$ a smooth conic containing neither P_1 nor P_2. If at least one of the points P_1, P_2 is contained in the plane $\langle \pi(C) \rangle$ spanned by $\pi(C)$, then $\mathcal{I}_C(2E_0 - E_1)$ has the associated element of $|\mathcal{O}_{X_2}(E_0 - E_1 - E_2)|$ in its base locus. Hence we assume $\langle \pi(C) \rangle \cap \{P_1, P_2\} = \emptyset$. Let $H \subset \mathbb{P}^3$ be a general plane containing the line $\langle \{P_1, P_2\} \rangle$ spanned by $\{P_1, P_2\}$. Set $\{O_1, O_2\} := \pi(C) \cap H$ (they are just two points because $\langle \pi(C) \rangle \cap \{P_1, P_2\} = \emptyset$. Take $Q_i \in X_2$ with $\pi(Q_i) = O_i$. Let $D \subset X_2$ be a smooth connected curve of multidegree $(2, 1, 1)$ containing $\{Q_1, Q_2\}$ (it corresponds to a smooth conic of the plane H containing $\{P_1, P_2, O_1, O_2\}$. The curve $C \cup D$ is a complete intersection of two elements of $|\mathcal{O}_{X_2}(2E_0 - E_1 - E_2)|$ and hence $\mathcal{I}_{C \cup D}(2E_0 - E_1 - E_2)$ is spanned. Therefore $\mathcal{I}_C(2E_0 - E_1 - E_2)$ is spanned outside D. Varying H and hence D we get that $\mathcal{I}_C(2E_0 - E_1 - E_2)$ is spanned. Since $C \cong \mathbb{P}^1$ and $\deg(\mathcal{O}_C(2E_0 - E_1 - E_2)) = 4$, we have $h^0(\omega_C(2E_0 - E_1 - E_2)) = 3$. Hence this case gives rank r spanned bundles if and only if $3 \le r \le 4$.

Take C with multidegree $(2, 0, 0)$ and $x = 2$, i.e. with $\pi(C)$ the union of two disjoint lines, say $\pi(C) = L_1 \sqcup L_2$, none of them containing either P_1 or P_2. It is easy to check that $h^0(\mathcal{I}_C(2E_0 - E_1 - E_2)) = h^0(\mathbb{P}^3, \mathcal{I}_{\pi(C) \cup \{P_1, P_2\}}(2)) = 2$. Since $C \ne Y$, we get that $\mathcal{I}_C(2E_0 - E_1 - E_2)$ is not spanned.

Take C with multidegree $(2, 1, 0)$ and $x = 1$, i.e. with $\pi(C)$ a degree 3 rational normal curve, $P_1 \in \pi(C)$ and $P_2 \notin \pi(C)$. We have $h^0(\mathcal{I}_C(2E_0 - E_1 - E_2)) = h^0(\mathbb{P}^3, \mathcal{I}_{\pi(C) \cup \{P_2\}}(2)) = 2$. Since $C \ne Y$, this case does not give spanned bundles.

The same proof works if C has multidegree $(2, 0, 1)$ and $x = 1$.

Take C with multidegree $(2,1,0)$ and $x=2$. Either $\pi(C)$ is the disjoint union of a smooth conic through P_1, but not P_2 and a line not intersecting $\{P_1,P_2\}$ or the disjoint union of a line through P_1 and a smooth conic not intersecting $\{P_1,P_2\}$. In both cases we have $h^0(\mathcal{I}_C(2E_0-E_1-E_2)) = h^0(\mathbb{P}^3,\mathcal{I}_{\pi(C)\cup\{P_2\}}(2)) \le 2$. Since $C\ne Y$, this case does not give a spanned bundle.

The same proof works with C of multidegree $(2,0,1)$ and $x=2$.

When C has multidegree either $(2,1,0)$ or $(2,0,1)$ and $x=3$ (i.e. by $\pi(C)$ is a disjoint union of 3 lines) we have $h^0(\mathcal{I}_C(2E_0-E_1)) \le 1$.

Assume $e_0=3$ and $e_1=e_2=0$. By Remark 2.11 for an arbitrary x we have $h^0(\mathcal{I}_C(2E_0-E_1-E_2)) = h^0(\mathbb{P}^3,\mathcal{I}_{\pi(C)\cup\{P_1,P_2\},\mathbb{P}^3}(2)) \le 2$ and hence (since $C\ne Y$) $\mathcal{I}_C(2E_0-E_1-E_2)$ is not spanned.

Take $C \ne Y$ with multidegree (e_0,e_1,e_2) with $e_0+e_1+e_2 \ge 4$. Since $h^0(\mathcal{I}_C(2E_0-E_1-E_2)) \le 2$ (unless $\mathcal{I}_C(2E_0-E_1-E_2)$ has a 2-dimensional base locus), $\mathcal{I}_C(2E_0-E_1-E_2)$ is not spanned. □

Remark 3.4. In each cases of Theorem 3.2 and 3.3 the cohomology of the bundles (and a parameter space of the bundles) are easily obtained from the cohomology of the curve C listed in the proof. Therefore we are not just gives the possible Chern numbers (or other discrete invariants) of them.

Remark 3.5. Take $s=3$, $c_1=\mathcal{O}_{X_3}(2E_0-E_1-E_2-E_3)$, but P_1,P_2,P_3 collinear. In this case $\mathcal{O}_{X_3}(2E_0-E_1-E_2-E_3)$ is not spanned.

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