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THE GROUP IDEMPOTENTS IN A PARTIAL GALOIS EXTENSION

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Abstract. Let (R, α_G) be a partial Galois extension with a partial action α_G of a finite group G, B the Boolean ring generated by $\{1_g | g \in G\}$ where 1_g is the central idempotent associated with $g \in G$. Let $e \neq 0 \in B$ and $G(e) = \{g \in G | e1_g \neq 0\}$. We call e a group idempotent if $G(e)$ is a subgroup of G. It is shown that if e is a group idempotent, then $(Re, \alpha_{G(e)})$ is a partial Galois extension induced by e. Thus the set of these partial Galois extensions in (R, α_G) is computed, and a structure theorem for (R, α_G) is obtained.

1. Introduction and preliminaries

Galois theory for rings has been intensively investigated in [1, 3, 4, 7], and recently generalized to partial Galois extensions for rings due to many applications of a partial action of a group on a ring ([2, 5, 6, 7, 8, 9]). In [8], let $B(R)$ be the Boolean semigroup generated by $\{1_g | g \in G\}$ under the multiplication of R where 1_g is the central idempotent associated with $g \in G, e \in B(R)$, and $G(e) = \{g \in G | e1_g \neq 0\}$. It is shown that if e is a minimal element in $B(R)$ and invariant under the partial action α_g for each $g \in G(e)$, then $G(e)$ is a group such that Re is a Galois extension of $(Re)^{G(e)}$ with Galois group $G(e)$ (see *Proposition* 6 in [8]). The group $G(e)$ plays an important role for the structure of a partial Galois extension (R, α_G) as given by *Theorem* 3.8 in [8]. In the present paper, let (R, α_G) be a partial Galois extension with a partial action α_G and B the Boolean ring generated by $\{1_g | g \in G\}$. For any non-zero element $e \in B$, we call e a group idempotent if $G(e)$ is a subgroup of G. We shall show some properties of a group idempotent and that if e is a group idempotent, then $(Re, \alpha_{G(e)})$ is a partial Galois extension in (R, α_G) such that $G(e)$ is the group $K \subset G$ maximal with (Re, α_K) as a partial Galois extension. Thus the set of these partial Galois extensions induced by an idempotent in (R, α_G) is computed and a structure theorem of (R, α_G) is obtained.

Let R be a ring with 1 and G a finite group with identity 1_G. As defined in [5], (R, α_G) is a ring with a partial action α_G if $\alpha_g : D_{g^{-1}} \to D_g$ is a ring isomorphism where $D_{g^{-1}}$ and D_g are ideals of R for all $g \in G$ such that (1) $D_{1_G} = R$ where

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α_{1_G} is the identity automorphism of R; (2) $\alpha_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$ for all $g, h \in G$; (3) $(\alpha_g\alpha_h)(r) = \alpha_{gh}(r)$ for every $r \in (D_{h^{-1}} \cap D_{(gh)^{-1}})$. Assume that $D_g = R1_g$ where 1_g is a central idempotent in R for each $g \in G$. Denote $\{r \in R | \alpha_g(r1_{g^{-1}}) = r1_g \text{ for all } g \in G\}$ by R^{α_G}. Then (R, α_G) is called a partial Galois extension of R^{α_G} if there exist $\{x_i; y_i \in R | i = 1, \ldots, n\}$ for some integer n such that $\sum_{i=1}^{n} x_i\alpha_g(y_i1_{g^{-1}}) = \delta_{1,g}1_R$ for $g \in G$, where $\{x_i; y_i\}$ is called a partial Galois system for R. We shall employ the following identity $\alpha_g(1_h1_{g^{-1}}) = 1_{gh}1_g$ for all $g, h \in G$ ([5], page 79).

2. Group Idempotents

By keeping the notations in section 2, (R, α_G) denotes a partial Galois extension. Let B be the Boolean ring generated by $\{1_g | g \in G\}$ with the usual partial order $e \leq f$ when $e = fe$. Since G is finite, B is finite with minimal elements $\{E_i | i = 1, \ldots, m \text{ for some integer } m\}$ denoted by $\mathcal{M}$. Recall that for an $e \neq 0 \in B, G(e) = \{g \in G | e1_g \neq 0\}$ and e is called a group idempotent if $G(e)$ is a subgroup of G. By the canonical form of an element $e \in B, e = \sum_{i \in I_e} E_i$ where $I_e \subset \{1, \ldots, m\}$. In this section, we shall show some properties of $\mathcal{M}$.

Lemma 2.1. *Let $\mathcal{M} = \{E_i | i = 1, \ldots, m$ for some integer $m\}$ be the set of minimal elements in B. Then $E_i = \prod_{g \in G} 1_g \neq 0$ with a maximal number of factors 1_g.*

Proof. For any $e \neq 0 \in B$, e is a sum of products of 1_g for some $g \in G$ and a product of 1_g in the sum is less than e, so $E_i = \prod_{g \in G} 1_g \neq 0$ with a maximal number of factors 1_g.

The following is a characterization of a group idempotent in $\mathcal{M}$.

Lemma 2.2. *For any $E \in \mathcal{M}, E$ is a group idempotent if and only if $E \in R^{\alpha_{G(E)}}$.*

Proof. See *Proposition* 6 in [8].

Theorem 2.3. (1) *Let $e \neq 0 \in R^{\alpha_{G(e)}}$. If $e1_g1_h \neq 0$ for all $g, h \in G(e)$, then $G(e)$ is a subgroup of G.* (2) *For any $e \neq 0 \in B, e \in \mathcal{M}$ if and only if $\prod_{g \in G} 1_g \neq 0$ with a maximal number of factors 1_g for all $g \in G(e)$.*

Proof. (1) Let $g \in G(e)$. Since $\alpha_g(e1_{g^{-1}}) = e1_g \neq 0, g^{-1} \in G(e)$. Also for any $g, h \in G(e), \alpha_g(e1_h1_{g^{-1}}) = \alpha_g(e1_h1_{g^{-1}}) = \alpha_g(e1_{g^{-1}})\alpha_g(1_h1_{g^{-1}}) = (e1_g)(1_{gh}1_g) = e1_{gh}1_g \neq 0$. By hypothesis, $e1_h1_{g^{-1}} \neq 0$, so $e1_{gh} \neq 0$; and so $gh \in G(e)$. Thus $G(e)$ is a group. (2) Since for any minimal element E of B, $E = \prod_{g \in G} 1_g \neq 0$ with a maximal number of factors 1_g for $g \in G$ by *Lemma* 2.1, $E = \prod_{g \in G} 1_g \neq 0$ for all $g \in G(e)$. Thus For any $e \neq 0 \in B, e \in \mathcal{M}$ if and only if $\prod_{g \in G} 1_g \neq 0$ with a maximal number of factors 1_g for all $g \in G(e)$.

Next we show that $G(e)$ can be computed from $G(E_i)$ where $e = \sum_{i \in I} E_i$ for some $I \subset \{1, \ldots, m\}$.

Theorem 2.4. *Let $e \in B, e = \sum_{i\in I} E_i$ where $I \subset \{1, \ldots, m\}$. Then, $G(e)$ is a group if and only if $\cup_{i\in I} G(E_i)$ is a group.*

Proof. It suffices to show that $G(e) = \cup_{i\in I} G(E_i)$. In fact, for any $g \in G(e), e1_g \neq 0$, so $(\sum_{i\in I} E_i)1_g \neq 0$. Hence $E_i 1_g \neq 0$ for some $i \in I$. Thus $g \in G(E_i)$; and so $G(e) \subset \cup_{i\in I} G(E_i)$. Conversely, for any $g \in \cup_{i\in I} G(E_i), g \in G(E_i)$ for some $i \in I$, so $E_i 1_g \neq 0$. But $G(e) = \sum_{i\in I} G(E_i)$ and $E_i E_j = E_i \delta_{ij}$ for $i, j \in I$, so $e1_g \neq 0$. Thus $g \in G(e)$. This completes the proof.

Recall that $\mathcal{M} = \{E_1, \ldots, E_m\}$ is the set of minimal elements in B. We want to show that any α_h for an $h^{-1} \in G(E_i)$ maps E_i in $\mathcal{M}$ to an element in $\mathcal{M}$.

Theorem 2.5. *Let $E \in \mathcal{M}$ and $h^{-1} \in G(E)$, then $\alpha_h(E1_{h^{-1}}) \in \mathcal{M}$.*

Proof. By *Lemma* 2.1, $E = \prod_{g\in G} 1_g \neq 0$ with a maximal number of factors 1_g, so for any $h^{-1} \in G(E), 1_{h^{-1}}$ is a factor of E. Hence $E1_{h^{-1}} = E \neq 0$; and so $\alpha_h(E1_{h^{-1}}) = \alpha_h(\prod_{g\in G(E)} 1_g 1_{h^{-1}}) = (\prod_{g\in G(E)})1_{hg}1_h \neq 0$. We claim that $(\prod_{g\in G(E)})1_{hg}1_h$ is in $\mathcal{M}$. It suffices to show that for any $q \notin hG(E), (\prod_{g\in G(E)})1_{hg}1_h1_q = 0$. Applying $\alpha_{h^{-1}}$ to $(\prod_{g\in G(E)})1_{hg}1_h1_q$ we have $\alpha_{h^{-1}}(\prod_{g\in G(E)} 1_{hg}1_h1_q)1_h) = (\prod_{g\in G(E)} 1_g)1_{h^{-1}q}1_{h^{-1}} = E1_{h^{-1}}1_{h^{-1}q} = E1_{h^{-1}q}$. But $q \notin hG(E)$, so $h^{-1}q \notin G(E)$. Thus $E1_{h^{-1}q} = 0$. Noting that $\alpha_{h^{-1}}$ is an isomorphism, we conclude that $(\prod_{g\in G(E)})1_{hg}1_h1_q = 0$; that is, $\alpha_h(E1_{h^{-1}})$ is in $\mathcal{M}$. This completes the proof.

3. Partial Galois Extensions

Keeping the notations and definitions in section 2, we shall show that any group idempotent e induces a partial Galois extension $(Re, \alpha_{G(e)})$. Thus we can show that the map $e \to (Re, \alpha_{G(e)})$ is a one-to-one correspondence between the set of group idempotents and the set of the partial Galois extensions induced by a group idempotent with a partial action of a subgroup of G maximal for the partial Galois extension Re. Also we obtain a structure theorem for (R, α_G) as a direct sum of these partial Galois extensions.

Theorem 3.1. *Let e be a group idempotent. Then $e \in R^{\alpha_{G(e)}}$ and $(Re, \alpha_{G(e)})$ is a partial Galois extension such that $G(e)$ is the subgroup K of G maximal for the partial Galois extension (Re, α_K).*

Proof. Let $e = \sum_{i\in I} E_i$ for some $I \subset \{1, \ldots, m\}$. Then $G(e) = \cup_{i\in I} G(E_i)$ by *Theorem* 2.4. For any $h \in G(e)$, since $G(e)$ is a group, h^{-1} is in $G(e)$. Hence there exists an $i \in I$ such that $E_i 1_{h^{-1}} \neq 0$. By *Theorem* 2.5, $\alpha_h(E_i 1_{h^{-1}}) = (\prod_{g\in G(E_i)})1_{hg}1_h \in \mathcal{M}$. Noting that $\{hg, h | g \in G(E_i)\} \subset G(e)$ such that $(\prod_{g\in G(E_i)})1_{hg}1_h \neq 0$, we have that the minimal idempotent $(\prod_{g\in G(E_i)})1_{hg}1_h$ is a term of e. Moreover, for each $i \in I$ such that $E_i 1_{h^{-1}} = 0, \alpha_h(E_i 1_{h^{-1}}) = 0$. Thus $\alpha_h(e1_{h^{-1}}) = \alpha_h(\sum_{i\in I}(E_i 1_{h^{-1}})) = \sum_{i\in I} \alpha_h(E_i 1_{h^{-1}}) = e1_h$; and so e is in $R^{\alpha_{G(e)}}$. By hypothesis,

(R, α_G) is a partial Galois extension, so there exist $\{x_i; y_i \in R | i =, \ldots, n\}$ for some integer n such that $\sum_i x_i \alpha_g(y_i 1_{g^{-1}}) = \delta_{1g} 1_R$. Thus $\sum_i e x_i \alpha_g(e y_i 1_{g^{-1}}) = e \sum_i x_i \alpha_g(y_i 1_{g^{-1}}) = e\delta_{1g} 1_R$ for each $g \in G(e)$. This implies that $(Re, \alpha_{G(e)})$ is a partial Galois extension. Moreover, let (Re, α_K) be a partial Galois extension with a partial action of a subgroup K. Then $e1_k \neq 0$ for each $k \in K$. Thus $K \subset G(e)$; and so $G(e)$ is the subgroup K of G maximal for the partial Galois extension (Re, α_K).

Corollary 3.2. *Let $\lambda : e \to (Re, \alpha_{G(e)})$ for a group idempotent $e \in B$. Then λ is a one-to-one correspondence between the set of group idempotents in B and the set of partial Galois extensions induced by a non-zero idempotent (Re, α_K) with a partial action of a subgroup K of G maximal with the partial action.*

Proof. This is an immediate consequence of *Theorem* 3.1.

Recall that for any non-zero $e \in B, e = \sum_{i \in I_e} E_i$ where $E_i \in \mathcal{M}$ and $I_e \subset \{1, \ldots, m\}$. We shall give an equivalent condition for (R, α_G) as a direct sum of the partial Galois extensions as given in *Theorem* 3.1.

Lemma 3.3. *Let $e, f \in B$ be distinct group idempotents such that $e = \sum_{i \in I_e} E_i$ and $f = \sum_{j \in I_f} E_j$. Then, $ef = 0$ if and only if $G(E_i) \neq G(E_j)$ for any $i \in I_e$ and $j \in I_f$.*

Proof. Noting that E_i and E_j are minimal elements in B, we have that $E_i = E_j$ if and only if $G(E_i) = G(E_j)$. Also $E_i E_j = 0$ for different E_i and E_j, so $ef = 0$ if and only if $E_i E_j = 0$ for any $i \in I_e$ and $j \in I_f$. This is equivalent to $G(E_i) \neq G(E_j)$ for any $i \in I_e$ and $j \in I_f$.

Theorem 3.4. *Let (R, α_G) be a partial Galois extension. Then, $(R, \alpha_G) = \oplus \sum_{i=1}^{k} (Re_i, \alpha_{G(e_i)})$, a direct sum of partial Galois extensions as given in Theorem* 3.1 *if and only if $G(E_s) \neq G(E_t)$ for any $s \in I_{e_i}$ and $t \in I_{e_j}$ for any $i \neq j$ where $\{e_i, |i = 1, \ldots, k\}$ for some integer k are group idempotents summing to 1_R.*

Proof. $(\to)$. Since (R, α_G) is a direct sum of partial Galois extensions as given in *Theorem* 3.1, $e_i e_j = 0$ for any $i \neq j$. Thus $G(E_s) \neq G(E_t)$ for any $s \in I_{e_i}$ and $t \in I_{e_j}$ for any $i \neq j$ by *Lemma* 3.3 and $\{e_i, |i = 1, \ldots, k\}$ for some integer k are group idempotents summing to 1_R.

$(\leftarrow)$. By hypothesis, $\{e_i, |i = 1, \ldots, k\}$ for some integer k are group idempotents, so $(Re_i, \alpha_{G(e_i)})$ is a partial Galois extension for each i as given in *Theorem* 3.1. Moreover, $G(E_s) \neq G(E_t)$ for any $s \in I_{e_i}$ and $t \in I_{e_j}$ for any $i \neq j$ where $\{e_i, |i = 1, \ldots, k\}$, so $e_i e_j = 0$ for any $i \neq j$ by *Lemma* 3.3 again. Noting that $\sum_i e_i = 1_R$, we have $(R, \alpha_G) = \oplus \sum_{i=1}^{k} (Re_i, \alpha_{G(e_i)})$, a direct sum of partial Galois extensions as given in *Theorem* 3.1. Thus the proof is complete.

Remark To compute the Boolean ring B generated by $\{1_g | g \in G\}$, it suffices to compute the set of all minimal elements $\{E_i | i = 1, \ldots, m\}$ by the canonical form of an element in B. Then $G(e)$ can be computed from $G(E_i)$ by *Theorem* 2.4. Thus the set of all group idempotents can be computed also by *Theorem* 2.4.

We conclude the present paper with an example to demonstrate the computation.

Example Let (R, α_G) be the partial Galois extension as given by Example 6.3 in [5]: $R = \sum_{i=1}^{6} Ae_i$ where A is a commutative Galois algebra with Galois cyclic group G with a generator g of order 6, and $\{e_i | i = 1, \ldots, 6\}$ are orthogonal non-zero idempotents summing to 1_R. The partial action α_G on R is defined by: $\alpha_{g^i} : A(e^{6-i}) \to A(e^i)$ for each i. Then every non-identity element e_i is a minimal element in B so that $M = \{1_{g^6}1_g, 1_{g^6}1_{g^2}, 1_{g^6}1_{g^3}, 1_{g^6}1_{g^4}, 1_{g^6}1_{g^5}\}$. Thus there are three proper group idempotents: $E_1 = \{1_{g^6}1_g + 1_{g^6}1_{g^5}, E_2 = 1_{g^6}1_{g^2} + 1_{g^6}1_{g^4}, E_3 = 1_{g^6}1_{g^3}\}$; and so there are three proper partial Galois extensions of (R, α_G) by *Theorem* 3.1.

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References

1. R. Alfaro, G. Szeto, Skew Group Rings Which Are Azumaya, *Comm. Algebra,* **22** 1995, 2255-2261.
2. J. Avila, M. Ferrero, J. Lazzarin, Partial Actions and Partial Fixed Rings, *Comm. Algebra,* **38** 2010, 2079-2091.
3. S.U. Chase, D.K. Harrison, A. Rosenberg, Galois Theory and Galois Cohomology of Commutative Rings, *Mem. Amer. Math. Soc.,* 1965, 15-33.
4. F.R. DeMeyer, Galois Theory in Separable Algebras over Commutative Rings, *Illinois J. Math.,* **10** 1966, 287-295.
5. M. Dokuchaev, M. Ferrero, A. Paques, Partial Actions and Galois Theory, *J. Pure Appl. Algebra,* **208** 2007, 77-87.
6. R. Exel, M. Laca, J. Quigg, Partial Dynamical System and C*-Algebras Generated by Partial Isometries, *J. Operator Theory,* **47** 2002, 169-186.
7. X.L. Jiang, G. Szeto, Galois Extensions Induced by a Central Idempotent in a Partial Galois Extension, *International J. Algebra* **8** 2014, 505-510.
8. J.M. Kuo, G. Szeto, The Structure of a Partial Galois Extension, *Monatsh Math.,* **DOI** 2013, 10.1007/s00605-013-0591-1.
9. A. Paques, V. Rodriques, A. Sant'Ana, Galois Correspondences for Partial Galois Azumaya Extensions, *J. Algebra and Its Applications,* **10** 2011, 835-847.

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